AN OVERVIEW OF INFINITE ERGODIC THEORY

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Abstract. We review the basic ergodic theory of non-singular transformations placing special emphasis on those transformations admitting σ-finite, infinite invariant measures. The topics to be discussed include invariant measures, recurrence, ergodic theorems, pointwise dual ergodicity, distributional limits, structure and intrinsic normalizing constants.

Introduction

Infinite ergodic theory is the study of measure preserving transformations of infinite measure spaces. It is part of the more general study of non-singular transformations (since a measure preserving transformation is also a non-singular transformation).

This paper is an attempt at an introductory overview of the subject, and is necessarily incomplete. More information on most topics discussed here can be found in [1]. Other references are also given in the text.

Before discussing the special properties of infinite measure preserving transformations, we need to review some basic non-singular ergodic theory first.

Let $(X, \mathcal{B}, m)$ be a standard σ-finite measure space. A *non-singular transformation* of X is only defined modulo nullsets, and is a map $T : X_0 \to X_0$ (where $X_0 \subset X$ has full measure), which is measurable and has the *non-singularity property* that for $A \in \mathcal{B},\ m(T^{-1}A) = 0$ if and only if $m(A) = 0$. A *measure preserving transformation* of X is a non-singular transformation T with the additional property that $m(T^{-1}A) = m(A)\ \forall\ A \in \mathcal{B}$.

If T is a non-singular transformation of a σ-finite measure space $(X, \mathcal{B}, m)$, and p is another measure on $(X, \mathcal{B})$ *equivalent* to m (denoted $p \sim m$ and meaning that p and m have the same nullsets), then T is a non-singular transformation of $(X, \mathcal{B}, p)$.

Thus, a non-singular transformation of a σ-finite measure space is actually a non-singular transformation of a probability space.

Considering a non-singular transformation $(X, \mathcal{B}, m, T)$ of a probability space as a *dynamical system* (see [23], [29]), the measure space X represents the set of *"configurations"* of the system, and T represents the change under *"passage of time"*. The non-singularity of T reflects the assumed property of the system that configuration sets that are impossible sometimes are always impossible. A probability preserving transformation would describe a system in a "steady state", where configuration sets occur with the same likelihood at all times.

Invariant Measures

Given a particular non-singular transformation, one of the first tasks is to ascertain whether it could have been obtained by starting with a measure preserving transformation, and then "passing" to some equivalent measure.

If $T : X \to X$ is non-singular then $f \to f \circ T$ defines a linear isometry of $L^\infty(m)$. There is a predual called the *Frobenius-Perron* or *transfer* operator, $\widehat{T} : L^1(m) \to L^1(m)$, which is defined by

$$f \mapsto \nu_f(\cdot) = \int_\cdot f dm \mapsto \widehat{T} f = \frac{d\nu_f \circ T^{-1}}{dm}$$

and satisfies

$$\int_X \widehat{T} f.g dm = \int_X f.g \circ T dm \quad f \in L^1(m),\ g \in L^\infty(m).$$

Note that the domain of definition of $\widehat{T}$ can be extended to all non-negative measurable functions. This definition can be made when m is infinite, but σ-finite.

Evidently the density $h \geq 0$ of an absolutely continuous invariant measure μ satisfies $\widehat{T} h = h$, since for any $g \geq 0$ measurable,

$$\int_X \widehat{T} h g dm = \int_X h g \circ T dm = \int_X g \circ T d\mu = \int_X g d\mu = \int_X h g dm.$$

Clearly, if T is invertible, then

$$\widehat{T} f = \frac{dm \circ T^{-1}}{dm} f \circ T^{-1},$$

and if the non-singular transformation T of X is *locally invertible* in the sense that there are disjoint measurable sets $\{A_j : j \in J\}$ (J finite or countable) such that $m(X \setminus \bigcup_{j\in J} A_j) = 0$, and T is invertible on each A_j, then

$$\widehat{T} f = \sum_{j \in J} 1_{TA_j} \frac{dm \circ v_j}{dm} f \circ v_j$$

where $v_j : TA_j \to A_j$ is measurable and satisfies $T \circ v_j \equiv \mathrm{Id}$.

Boole's transformations I

For some locally invertible non-singular transformations T and measurable functions f, $\widehat{T} f$ can be computed explicitly. For example, consider the transformations $T : \mathbb{R} \to \mathbb{R}$ defined by

$$Tx = \alpha x + \beta + \sum_{k=1}^{n} \frac{p_k}{t_k - x} \tag{1.1}$$

where $n \geq 1,\ \alpha,\ p_1, \ldots, p_n \geq 0$ (not all zero) and $\beta, t_1, ..., t_n \in \mathbb{R}$.

These transformations (called *Boole's transformations*) were considered by G. Boole in [15]. They are non-singular transformations of $\mathbb{R}$ equipped with Lebesgue measure $m_\mathbb{R}$, and for $h : \mathbb{R} \to \mathbb{R}$ a non-negative measurable function,

$$\widehat{T} h(x) = \sum_{y \in \mathbb{R},\ Ty = x} \frac{h(y)}{T'(y)}.$$

Note that the $1-1$ Boole's transformations are the real Möbius transformations.

1.1 Boole's Formula [15]. *For T as above, $x \in \mathbb{R}$ and $\omega \in \mathbb{C}$, $T\omega \neq x$,*

$$\sum_{y\in\mathbb{R},\ Ty=x} \frac{1}{(y-\omega)T'(y)} = \frac{1}{x-T(\omega)}.$$

If $\omega \in \mathbb{R}^{2+}$, the upper half plane, and $\omega = a+ib,\ a,b\in\mathbb{R},\ b>0$ then

$$\operatorname{Im}\frac{1}{x-\omega} = \frac{b}{(x-a)^2+b^2} = \pi\varphi_\omega(x)$$

where φ_ω is the well known *Cauchy density* and Boole's formula has the immediate corollary:

$$\widehat{T}\varphi_\omega = \varphi_{T(\omega)},\ \text{ or } P_\omega\circ T^{-1} = P_{T\omega} \tag{1.2}$$

where $dP_\omega := \varphi_\omega dm_{\mathbb{R}}$.

The original proof of Boole's formula in [15] uses the

1.2 Proposition [15]. *Suppose that $F:\mathbb{C}\to\overline{\mathbb{C}}$ is rational, and $E:\mathbb{C}\to\mathbb{C}$ is a polynomial. Then*

$$\sum_{x\in\mathbb{C}:\ E(x)=0} F(x) = -\sum_{a \text{ a pole of } F} \operatorname{Res}(F(\log E)';a) + \operatorname{Res}(F(\log E)';0).$$

Modern proofs of (1.2) use the fact that Boole's transformations are $\mathbb{R}$-restrictions of analytic endomorphisms of $\mathbb{R}^{2+}$ (*inner functions*). Many of the results given here for Boole's transformations remain valid for arbitrary inner functions ([4], [38]).

Bounded analytic functions on $\mathbb{R}^{2+}$ are Cauchy integrals of their restrictions to $\mathbb{R}$ and writing $dP_\omega := \varphi_\omega dm_{\mathbb{R}}$, one sees ([36]) that for $t>0$,

$$\widehat{P_\omega\circ T^{-1}}(t) = \int_{\mathbb{R}} e^{itT(x)}dP_\omega(x) = e^{itT(\omega)} = \widehat{P_{T(\omega)}}(t),$$

whence (1.2).

As a consequence of (1.2), we see that the Boole transformation T has an absolutely continuous invariant probability if $\exists\ \omega\in\mathbb{R}^{2+}$ with $T(\omega)=\omega$ (in which case P_ω is T-invariant). It turns out that this is the only way it can admit an absolutely continuous invariant probability ([38]).

If $\alpha>0$ in (1.1), then ([36]) using the fact that $\pi b\varphi_{a+ib}\to 1$ as $b\to\infty$, $\frac{a}{b}\to 0$ we see that $m_{\mathbb{R}}\circ T^{-1} = \alpha^{-1}m_{\mathbb{R}}$, whence T preserves $m_{\mathbb{R}}$ if $\alpha=1$ in (1.1). By considering other analogous limits, one also sees that:

$Tx = \tan x$ preserves the measure $d\mu_0(x) := \frac{dx}{x^2}$ (see [4]);

and

if $f(\omega) = \int_{\mathbb{R}}\frac{d\nu(t)}{t-\omega}$ (the complex Hilbert transform of ν) where $\nu\perp m_{\mathbb{R}}$, and $T(x) := \lim_{y\downarrow 0} f(x+iy)$ a.e. (the Hilbert transform of ν), then $m_{\mathbb{R}}\circ T^{-1} = \nu(\mathbb{R})\mu_0$ (see [22]).

CONDITIONS FOR EXISTENCE OF INVARIANT MEASURES.

Unfortunately, one cannot expect always to be able to identify absolutely continuous invariant measures by explicit computation. To help remedy this situation, there are many conditions for existence of such (see [35]). Some conditions for existence of absolutely continuous invariant probabilities depend on the following:

1.3 Proposition. *Let T be a non-singular transformation of $(X, \mathcal{B}, m)$. If $\exists\ f \in L^1(m),\ f \geq 0,\ \int_X f dm > 0$ such that $\{\frac{1}{n}\sum_{k=1}^n \hat{T}^k f :\ n \geq 1\}$ is a uniformly integrable family, then $\exists$ a T-invariant probability $P << m$.*

The invariant probability's density $h \in L^1(m),\ h \geq 0$ is found as a weak limit point of $\{\frac{1}{n}\sum_{k=1}^n \hat{T}^k f :\ n \geq 1\}$ which is weakly sequentially precompact in $L^1(m)$ owing to the assumed uniform integrability.

Expanding interval maps I

Let $I = [0, 1]$, m_I be Lebesgue measure on I, and α be a collection of disjoint open subintervals of I such that

$$m(I \setminus U_\alpha) = 0 \text{ where } U_\alpha = \bigcup_{a \in \alpha} a.$$

A piecewise onto, C^2 *interval map* with *basic partition* α is a map $T : I \to I$ is such that

$$\text{For each } a \in \alpha,\ T|_a \text{ extends to a } C^2 \text{ diffeomorphism } T : \overline{a} \to I. \tag{1.3}$$

Note that if $a_1, ..., a_n \in \alpha$ then $a = \bigcap_{k=1}^n T^{-(k-1)} a_k$ is an open interval, and $T^n : \overline{a} \to I$ is a C^2 diffeomorphism. Hence, if T is a piecewise onto, C^2 interval map with basic partition α then, for $n \geq 1$, T^n is an interval map with basic partition

$$\alpha_0^{n-1} := \bigvee_{k=0}^{n-1} T^{-k}\alpha = \{\bigcap_{k=1}^{n} T^{-(k-1)} a_k :\ a_1, ..., a_n \in \alpha\}.$$

The piecewise onto, C^2 interval map T is called *expanding* if

$$\exists \lambda > 1 \ \ni\ |T'x| \geq \lambda\ \forall\ x \in I. \tag{1.4}$$

The following condition limits the multiplicative variation (or *distortion*) of v_a' $(a \in \alpha)$. It is known as *Adler's condition* or *bounded distortion.* It follows from (1.3) and (1.4) in case α is finite.

$$\exists M > 1 \ \ni\ \frac{|T''x|}{|T'x|^2} \leq M\ \forall\ x \in I.$$

Given a piecewise onto, C^2 interval map T with basic partition α, and $a \in \alpha_0^{n-1}$; denote by v_a the C^2 diffeomorphism $v_a : I \to \overline{a}$ satisfying $T^n \circ v_a(x) = x$.

The basic result concerning piecewise onto, C^2 expanding interval maps satisfying Adler's condition is that Adler's condition holds uniformly for all powers of T ([12]):

$$\frac{|T^{n\prime\prime}x|}{|T^{n\prime}x|^2} \leq K = \frac{\lambda M}{\lambda - 1}\ \forall\ x \in I,\ n \geq 1,$$

whence Renyi's distortion property (see [41]):

$$|v_a'(x)| = e^{\pm K} m_I(a) \ \forall\ x \in I, \quad a \in \bigcup_{n=1}^{\infty} \alpha_0^{n-1}. \tag{1.5}$$

As a consequence of Renyi's distortion property, we have that

$$\widehat{T}^n 1 = \sum_{a \in \alpha_0^{n-1}} |v_a'| = e^{\pm K},$$

and so by proposition 1.3 (a uniformly bounded sequence being uniformly integrable) $\exists$ an absolutely continuous, invariant probability with density h satisfying $h = \widehat{T} h = e^{\pm K}$.

It is shown in [27] that h is Lipschitz continuous. To see this using [21] note that for f Lipschitz continuous (hence differentiable a.e.),

$$(\widehat{T}^n f)' = \sum_{a \in \alpha_0^{n-1}} v_a'' f \circ v_a + \sum_{a \in \alpha_0^{n-1}} v_a'^2 f' \circ v_a$$

whence

$$\|(\widehat{T}^n f)'\|_\infty \le \frac{e^K}{\lambda^n} \|f'\|_\infty + e^{2K} \|f\|_\infty$$

and it follows from [21] that $\exists$ h_0 Lipschitz continuous and $M > 0$, $r \in (0,1)$ such that $\widehat{T} h_0 = h_0$ and $\|\widehat{T}^n f - h_0 \int_I f dm\|_L \le M r^n \|f\|_L \quad \forall\ f$ Lipschitz continuous where $\|f\|_L := \|f\|_\infty + \|f'\|_\infty$. In particular, $h = h_0$ mod m.

The assumption that the interval map is expanding is not crucial. Gauss' continued fraction map $Tx = \{\frac{1}{x}\}$ is not expanding, but $(T^2)' \ge 4$ and the above applies.

However the piecewise onto, C^2 interval map $Tx = \{\frac{1}{1-x}\}$ satisfies Adler's condition, and no power is expanding, as $T(0) = 0$, $T'(0) = 1$. In fact T admits no absolutely continuous, invariant probability, the infinite measure $d\nu(x) := \frac{dx}{x}$ being T-invariant.

Conditions for the existence of absolutely continuous, infinite, invariant measures depend on recurrence properties.

Recurrence and Conservativity

There are non-singular transformations T of $(X, \mathcal{B}, m)$ which are *recurrent* in the sense that

$$\liminf_{n\to\infty} |h \circ T^n - h| = 0 \text{ a.e. } \forall\ h : X \to \mathbb{R} \text{ measurable}.$$

One extreme form of non-recurrent (or *transient*) behaviour is exhibited by *wandering sets.*

Let T be a non-singular transformation of the standard measure space $(X, \mathcal{B}, m)$.

A set $W \subset X$ is called a *wandering* set (for T) if the sets $\{T^{-n}W\}_{n=0}^{\infty}$ are disjoint. Let $\mathcal{W} = \mathcal{W}(T)$ denote the collection of measurable wandering sets.

Evidently, the collection of measurable wandering sets is a hereditary collection (any subset of a wandering set is also wandering), and T-invariant (W a wandering set $\implies$ $T^{-1}W$ a wandering set).

Using a standard exhaustion argument it can be shown that $\exists$ a countable union of wandering sets $\mathfrak{D}(T) \in \mathcal{B}$ with the property that any wandering set $W \in \mathcal{B}$ is contained in $\mathfrak{D}(T) \mod m$ (i.e. $m(W \setminus \mathfrak{D}(T)) = 0$). Evidently $\mathfrak{D}(T)$ is unique mod m and $T^{-1}\mathfrak{D} \subseteq \mathfrak{D}$ mod m. It is called the *dissipative part* of the non-singular transformation T. In case T is invertible, it can be shown that $\exists$ a wandering set $W \in \mathcal{B}$ such that $\mathfrak{D}(T) = \bigcup_{n\in\mathbb{Z}} T^n W$, whence $T^{-1}\mathfrak{D} = \mathfrak{D}$.

The *conservative part* of T is defined to be $\mathfrak{C}(T) := X \setminus \mathfrak{D}(T)$ and the partition $\{\mathfrak{C}(T),\ \mathfrak{D}(T)\}$ is called the *Hopf decomposition* of T.

The non-singular transformation T is called *conservative* if $\mathfrak{C}(T) = X$ mod m, and (totally) *dissipative* if $\mathfrak{D}(T) = X$ mod m.

Using

Halmos' recurrence theorem [26]. *If T is conservative, then*

$$\sum_{n=1}^{\infty} 1_B \circ T^n = \infty \ \text{a.e. on } B, \quad \forall B \in \mathcal{B}.$$

one can show that a non-singular transformation is recurrent if, and only if it is conservative.

CONDITIONS FOR CONSERVATIVITY.

If there exists a finite, T-invariant measure $q << m$, then clearly there can be no wandering sets with positive q-measure, whence $q(\mathfrak{D}) = 0$ and $[\frac{dq}{dm} > 0] \subseteq \mathfrak{C}$ mod m. In particular ([40]) any probability preserving transformation is conservative.

A measure preserving transformation of a σ-finite, infinite measure space is not necessarily conservative. For example $x \mapsto x + 1$ is a measure preserving transformation of $\mathbb{R}$ equipped with Borel sets, and Lebesgue measure, which is totally dissipative.

2.1 Proposition.

1) If $T : X \to X$ is non-singular, then

$$\mathfrak{C}(T) = [\sum_{n=1}^{\infty} \widehat{T}^k f = \infty] \qquad \text{mod } m, \ \forall\ f \in L^1(m), f > 0.$$

2) If $T : X \to X$ is a measure preserving transformation, then $T^{-1}\mathfrak{C}(T) = \mathfrak{C}(T)$ mod m. Indeed

$$\mathfrak{C}(T) = \left[\sum_{n=1}^{\infty} f \circ T^n = \infty\right] \qquad \text{mod } m, \quad \forall\ f \in L^1(m), f > 0 \ \text{a.e.}.$$

Boole's transformations II

If

$$Tx = \alpha x + \beta + \sum_{k=1}^{n} \frac{p_k}{t_k - x}$$

where $n \geq 1$, $\alpha > 0$, $p_1, \ldots, p_n \geq 0$ and β, $t_1, ..., t_n \in \mathbb{R}$ then (as deduced from Boole's formula) $m_{\mathbb{R}} \circ T^{-1} = \alpha^{-1} m_{\mathbb{R}}$.

When $\alpha > 1$,

$$\sum_{n=1}^{\infty} \widehat{T}^n f < \infty \text{ a.e. } \forall \;\; f \in L^1(m) \cap L^\infty(m), \; f > 0 \text{ a.e.},$$

whence by proposition 2.1(1), T is totally dissipative.

Now let

$$Tx = x + \beta + \sum_{k=1}^{n} \frac{p_k}{t_k - x} \tag{2.1}$$

where $n \geq 1$, $p_1, \ldots, p_n \geq 0$ (not all zero) and β, $t_1, ..., t_n \in \mathbb{R}$. As above, $m_{\mathbb{R}} \circ T^{-1} = m_{\mathbb{R}}$.

To see when T is conservative, we use proposition 2.1. For T defined by (2.1) and $\omega \in \mathbb{R}^{2+}$, $T^n(\omega) \to \infty$. Write $T^n(\omega) := u_n + iv_n \to \infty$, whence $\pi \widehat{T}^n \varphi_\omega(x) = \operatorname{Im} \frac{1}{x - T^n(\omega)} = \frac{v_n}{(x-u_n)^2 + v_n^2} \sim \frac{v_n}{u_n^2 + v_n^2}$ and T is conservative iff

$$\sum_{n=1}^{\infty} \frac{v_n}{u_n^2 + v_n^2} = \infty.$$

We see (as in [4], [5]) that when $\beta \neq 0$,

$$v_n \uparrow v_\infty < \infty, \quad u_n = \beta n - \frac{\nu}{\beta} \log n + O(1) \text{ as } n \to \infty; \tag{2.2}$$

and when $\beta = 0$,

$$\sup_{n \geq 1} |u_n| < \infty, \; v_n \sim \sqrt{2\nu n} \text{ as } n \to \infty \tag{2.3}$$

where $\nu := \sum_{k=1}^{n} p_k$.

It follows that T is conservative when $\beta = 0$ ($\sum_{n=1}^{\infty} \frac{v_n}{u_n^2 + v_n^2} = \infty$); and totally dissipative when $\beta \neq 0$ ($\sum_{n=1}^{\infty} \frac{v_n}{u_n^2 + v_n^2} < \infty$).

INDUCED TRANSFORMATIONS, CONSERVATIVITY AND INVARIANT MEASURES.

Suppose T is conservative and non–singular, and let $A \in \mathcal{B}_+$, then m–a.e. point of A returns infinitely often to A under iterations of T, and in particular the *return time* function to A, defined for $x \in A$ by $\varphi_A(x) := \min\{n \geq 1 : T^n x \in A\}$ is finite m–a.e. on A.

The *induced transformation* ([33]) on A is defined by $T_A x = T^{\varphi_A(x)} x$, and can be defined whenever the return time function is finite m–a.e. on A (whether T is conservative, or not).

The first key observation is that $m|_A \circ T_A^{-1} << m|_A$. This is because

$$T_A^{-1} B = \bigcup_{n=1}^{\infty} [\varphi = n] \cap T^{-n} B.$$

It follows that $\varphi_A \circ T_A$ is defined a.e. on A and an induction now shows that all powers $\{T_A^k\}_{k \in \mathbb{N}}$ are defined a.e. on A, and satisfy

$$T_A^k x = T^{(\varphi_A)_k(x)} x \text{ where } (\varphi_A)_1 = \varphi_A, \; (\varphi_A)_k = \sum_{j=0}^{k-1} \varphi_A \circ T_A^j.$$

2.2 Proposition (c.f. [33]). *Let T be a non–singular transformation of $(X,\mathcal{B},m)$,* $\blacksquare$ *and suppose that $A\in\mathcal{B},\ m(A)>0$ satisfies $\varphi_A<\infty$ a.e. on A.*

1) If T is conservative, then the induced transformation T_A is a conservative, non-singular transformation of $(A,\mathcal{B}\cap A,m|_A)$.

2) If T is a measure preserving transformation, then T_A is a measure preserving transformation of $(A,\mathcal{B}\cap A,m|_A)$, and in case $\bigcup_{n=0}^\infty T^{-n}A=X$ mod m, T_A a conservative iff T is conservative.

2.3 Proposition (c.f. [32]). *Let T be a non-singular transformation of $(X,\mathcal{B},m)$, and suppose that the return time function to $A\in\mathcal{B},\ m(A)>0$ is finite m–a.e. on A.*

Let $A\in\mathcal{B}_+$, and suppose that $q<<m|_A$ is a T_A-invariant measure. Set, for $B\in\mathcal{B}$,

$$\mu(B)=\sum_{k=0}^{\infty}q(A\cap T^{-k}B\setminus\bigcup_{j=1}^{k}T^{-j}A).$$

Then $\mu<<m$ is a T-invariant measure.

Non-expanding interval maps I

Let T be a piecewise onto, C^2 interval map with basic partition $\alpha=\{(0,u)\}\cup\alpha_0$ satisfying Adler's condition. Suppose that

$$Tx=x+cx^{1+p}+o(x^{1+p})\quad\text{as } x\to 0$$

where $c>0,\ p\ge 1,\ T(u)=1,\ T''\ge 0$ on $[0,u]$ and $\exists\ \kappa>1$ such that

$$T'x\ge\kappa\ \ \forall\ x\in\ \ a\in\alpha_0$$

(e.g. $Tx=\{\frac{1}{1-x}\}$ where $p=c=1$).

Evidently the return time function to $[u,1]$ is finite on $[u,1]$, and $T_{[u,1]}$ is an expanding, piecewise onto, C^2 interval map of $[u,1]$.

It turns out that $T_{[u,1]}$ also satisfies Adler's condition. We sketch a way to see this (the full proof is in [45]). Let $x\in[u,1]$, then $T_{[u,1]}x=v_0^{-n}\circ(x)$ for some $n\ge 0$, where $v_0:I\to[0,u],\ T\circ v_0=\mathrm{Id}$. It follows that

$$\frac{|T''_{[u,1]}x|}{(T'_{[u,1]}x)^2}\le\frac{|v_0^{-n\prime\prime}(Tx)|}{v_0^{-n\prime}(Tx)^2}+1.$$

Adler's condition for $T_{[u,1]}$ will now follow from

$$\sup_{y\in\ (u,1),\ n\ge 1}\frac{|v_0^{n\prime\prime}(y)|}{v_0^{n\prime}(y)}<\infty.$$

To show this, calculate first that

$$v_0^{n\prime\prime}(y)=v_0^{n\prime}(y)\sum_{k=0}^{n-1}\frac{v_0''(v_0^ky)}{v_0'(v_0^ky)}v_0^{k\prime}(y),$$

whence

$$\frac{|v_0^{n''}(y)|}{v_0^{n'}(y)} \le M \sum_{k=0}^{n-1} v_0^{k'}(y) \le M \sum_{k=0}^{\infty} \frac{v_0^k(y) - v_0^{k+1}(y)}{y - v_0(y)}$$
$$= M \frac{y}{y - v_0(y)} \le M' < \infty \;\; \forall\, y \in \;(u,1).$$

As in section 1, there is a $T_{[u,1]}$-invariant probability on $[u,1]$ with Lipschitz continuous density bounded away from 0.

By proposition 2.3, there is an absolutely continuous T-invariant σ-finite measure and μ on I, and by proposition 2.2 T is a conservative measure preserving transformation of $(I, \mathcal{B}, \mu)$.

It can be shown (see [46]) that the density h of μ has the form

$$h(x) = \frac{\varkappa(x)}{x^p}$$

where $\varkappa$ is Lipschitz continuous, and bounded away from zero. Therefore μ is an infinite measure.

This method for proving conservativity and existence of invariant measure is difficult to apply when no suitable set to induce on presents itself. More widely applicable developments of the method use a ”jump transformation” to replace induction (see [43], [45]).

Ergodicity and Exactness

Ergodicity is an irreducibility property. A non-singular transformation T of the measure space $(X, \mathcal{B}, m)$ is called *ergodic* if

$$A \in \mathcal{B},\; T^{-1}A = A \quad \bmod\ m \;\Rightarrow\; m(A) = 0, \text{ or } m(A^c) = 0.$$

An invertible ergodic non-singular transformation T of a non-atomic measure space is necessarily conservative.

3.1 Proposition. *Let T be a non-singular transformation. The following are equivalent:*

$$T \text{ is conservative and ergodic;}$$

$$\sum_{n=1}^{\infty} 1_A \circ T^n = \infty \;\; a.e. \quad \forall A \in \mathcal{B}_+;$$

$$\sum_{n=0}^{\infty} \widehat{T}^n f = \infty \quad a.e.\ \forall f \in L^1(m), \quad f \ge 0 \ a.e.\ , \ \int_X f dm > 0.$$

3.2 Proposition. *Suppose that T is conservative, and $A \in \mathcal{B}_+$. Then*

$$T \;\; ergodic \;\Rightarrow\;\; T_A \;ergodic,$$

$$T_A \;\; is\ ergodic,\ and\ \bigcup_{n=1}^{\infty} T^{-n}A = X \ mod\ m \Rightarrow \;\; T \ is\ ergodic.$$

3.3 Theorem (Unicity of invariant measure). *Let T be a conservative, ergodic, non-singular transformation of $(X, \mathcal{B}, m)$. Then, up to multiplication by constants, there is at most one m-absolutely continuous, σ-finite T-invariant measure μ and (if there is one) $\mu \sim m$.*

EXACTNESS.

There are many ways of proving ergodicity. Sometimes, it's easier to prove a stronger property.

A non-singular transformation T of the measure space $(X, \mathcal{B}, m)$ is called *exact* if

$$\bigcap_{n=1}^{\infty} T^{-n}\mathcal{B} = \{\emptyset, X\} \quad \text{mod } m.$$

As above, the σ-algebra $\mathfrak{T}(T) := \bigcap_{n=1}^{\infty} T^{-n}\mathcal{B}$ is called the *tail* of T.

Evidently T-invariant measurable sets are in the tail and so exactness implies ergodicity. The converse is false due to the existence of invertible, ergodic transformations.

We show first that expanding maps of the unit interval are exact. Let T be a piecewise onto, C^2 interval map satisfying Adler's condition and with basic partition α.

It follows from (1.5) that for $n \geq 1$, $a \in \alpha_0^{n-1}$ and $B \in \mathcal{B}$,

$$m(a \cap T^{-n}B) = \int_B \widehat{T}^n 1_a dm = e^{\pm K} m(B)m(a).$$

Now suppose that $B \in \mathfrak{T}$, and $a \in \alpha_0^{n-1}$; then $\exists\ B_n \in \mathcal{B}$ such that $B = T^{-n}B_n$. Thus

$$m(a \cap B) = m(a \cap T^{-n}B_n) = e^{\pm K} m(B_n)m(a) = e^{\pm 2K} m(B)m(a).$$

This remains true for $a \in \mathcal{B}$, whence $0 = m(B^c \cap B) = e^{\pm 2K} m(B)m(B^c)$ and $m(B) = 0, 1$.

Non-expanding interval maps II

As a consequence of this, we obtain from proposition 3.2 that the non-expanding maps of the unit interval are also ergodic. It now follows proposition 3.3 that (having an absolutely continuous invariant infinite measure) they admit no absolutely continuous invariant probability. In fact ([45]) the non-expanding maps of the unit interval considered here are themselves exact.

The following is a useful criterion for exactness.

3.4 Theorem [37]. *Let T be a non-singular transformation of $(X, \mathcal{B}, m)$. Then*

$$T \text{ is exact } \Leftrightarrow \ \|\widehat{T}^n f\|_1 \to_{n\to\infty} 0 \ \forall f \in L^1, \int_X f dm = 0.$$

Boole's transformations III

1) Suppose that T is a non-Möbius Boole's transformation and suppose that $\omega \in \mathbb{R}^{2+}$ and $T(\omega) = \omega$. As shown above, the measure P_ω is T-invariant. We'll show using theorem 3.4 that T is exact.

Since T is not Möbius, $|T'(\omega)| < 1$ and $T^n(z) \to \omega\ \forall\ z \in \mathbb{R}^{2+}$.

We'll show exactness via theorem 3.4 using $f * \varphi_{ib} \to f$ in L^1 as $b \to 0$, $\forall\ f \in L^1$ where $f * g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy$.

Fix $f \in L^1$. For $b > 0$, $f * \varphi_{ib} = \int_{\mathbb{R}} f(y)\varphi_{y+ib}dy$ whence $\widehat{T}^n(f * \varphi_{ib}) = \int_{\mathbb{R}} f(y)\varphi_{T^n(y+ib)}dy$ and

$$\|\widehat{T}^n(f * \varphi_{ib}) - \int_{\mathbb{R}} f dm_{\mathbb{R}} \varphi_\omega\|_1 \le \int_{\mathbb{R}} |f(y)| \|\varphi_{T^n(y+ib)} - \varphi_\omega\|_1 \to 0,$$

$$\begin{aligned}\limsup_{n\to\infty} \|\widehat{T}^n f - \int_{\mathbb{R}} f dm_{\mathbb{R}} \varphi_\omega\|_1 &\le \limsup_{n\to\infty} \|\widehat{T}^n f - \widehat{T}^n(f * \varphi_{ib})\|_1 \\ &\le \|f - f * \varphi_{ib}\|_1 \to 0\end{aligned}$$

as $b \to 0$. Exactness follows from theorem 3.4.

2) Now consider

$$Tx = x + \sum_{k=1}^{n} \frac{p_k}{t_k - x}$$

where $n \ge 1$, $p_1, \dots, p_n \ge 0$ (not all zero) and $t_1, ..., t_n \in \mathbb{R}$.

Recall from above that T is a conservative, measure preserving transformation of $\mathbb{R}$ equipped with Lebesgue measure.

We claim that T is exact. This can be shown using theorem 3.4 in the following steps ([4]):

- By (2.3), $\|\varphi_{T^n(\omega)} - \varphi_{T^n(\omega')})\|_1 \to 0$ as $n \to \infty\ \forall\ \omega,\ \omega' \in \mathbb{R}^{2+}$.
- For $f \in L^1, \int_X f dm = 0$ fixed, if $b > 0$ then $f * \varphi_{ib} = \int_{\mathbb{R}} f(y)\varphi_{y+ib}dy$ whence

$$|\widehat{T}^n(f * \varphi_{ib})| = |\int_{\mathbb{R}} f(y)\varphi_{T^n(y+ib)}dy| \le \int_{\mathbb{R}} |f(y)||\varphi_{T^n(y+ib)} - \varphi_{T^n(ib)}|dy,$$

and

$$\|\widehat{T}^n(f * \varphi_{ib})\|_1 \le \int_{\mathbb{R}} |f(y)| \|\varphi_{T^n(y+ib)} - \varphi_{T^n(ib)}\|_1 dy \to 0 \text{ as } n \to \infty.$$

- As above,

$$\begin{aligned}&\limsup_{n\to\infty} \|\widehat{T}^n f\|_1 \\ &\le \limsup_{n\to\infty} \|\widehat{T}^n(f * \varphi_{ib})\|_1 + \limsup_{n\to\infty} \|\widehat{T}^n(f * \varphi_{ib} - f)\|_1 \\ &\le \|f * \varphi_{ib} - f\|_1 \to 0 \text{ as } b \to 0.\end{aligned}$$

Ergodic Theorems

The original "proof of the ergodic theorem" was

4.1 Birkhoff's ergodic theorem [14]. *Suppose that T is a probability preserving transformation of $(X, \mathcal{B}, m)$. Then*

$$\frac{1}{n}\sum_{k=1}^{n} f(T^k x) \to E(f|\mathfrak{I})(x) \text{ as } n \to \infty \text{ for a.e. } x \in X,\ \forall f \in L^1(m),$$

where $\mathfrak{I}$ is the σ-algebra of T-invariant sets in $\mathcal{B}$.

Generalizations of Birkhoff's theorem were given by E.Hopf and Stepanov ([30], [44]) for infinite measure preserving transformations, W.Hurewicz [31] (for non-singular transformations), and R.Chacon, D.Ornstein [18] (for Markov operators).

4.2 Hopf-Stepanov ergodic Theorem [30], [44]. *Suppose that T is a conservative, measure preserving transformation of $(X,\mathcal{B},m)$. Then*

$$\frac{\sum_{k=1}^n f\circ T^k(x)}{\sum_{k=1}^n p\circ T^k(x)} \to E_{m_p}(\frac{f}{p}|\mathfrak{I})(x),$$

as $n\to\infty$ for a.e. $x\in X$, $\forall f,p\in L^1(m)$, $p>0$, where $dm_p = pdm$.

4.3 Hurewicz's Ergodic Theorem [31]. *Suppose that T is a conservative non-singular transformation of $(X,\mathcal{B},m)$. Then*

$$\frac{\sum_{k=1}^n \widehat{T}^k f(x)}{\sum_{k=1}^n \widehat{T}^k p(x)} \to E_{m_p}(\frac{f}{p}|\mathfrak{I})(x)$$

as $n\to\infty$ for a.e. $x\in X$, $\forall f,p\in L^1(m)$, $p>0$.

Note that, when T is ergodic, $E_{m_p}(\frac{f}{p}|\mathfrak{I})(x) = \frac{\int_X fdm}{\int_X pdm}$.

Absolutely Normalized Convergence

Given a conservative, ergodic measure preserving transformation T, it is natural to ask the rate at which $\sum_{k=1}^n p\circ T^k \to\infty$ for $p\in L^1$, $p\ge 0$. For probability preserving T, $\sum_{k=1}^n p\circ T^k \sim n\int_X pdm$ a.e., however for a conservative, ergodic infinite measure preserving transformation T one only obtains (from the Hopf-Stepanov ergodic theorem) that $\sum_{k=1}^n p\circ T^k = o(n)$ a.e. as $n\to\infty$ for $p\in L^1$, $p\ge 0$.

We are led to ask for "absolutely normalized ergodic convergence" i.e. for constants $a_n>0$ such that $\frac{1}{a_n}\sum_{k=1}^n p\circ T^k \to \int_X pdm$ in some sense.

The Hopf-Stepanov ergodic theorem does not provide this absolutely normalized convergence for ergodic, infinite measure preserving transformations, and the next result shows that absolutely normalized pointwise convergence is impossible for ergodic, infinite measure preserving transformations.

5.1 Theorem [1]. *Suppose that T is a conservative, ergodic measure preserving transformation of the σ-finite, infinite measure space $(X,\mathcal{B},m)$, and let $a_n > 0$ $(n\ge 1)$. Then*

$$\text{(1)}\qquad \textit{either } \liminf_{n\to\infty}\frac{S_n(f)}{a_n} = 0 \textit{ a.e.}, \forall f\in L^1(m)_+,$$

$$\text{(2)}\qquad \textit{or } \exists n_k\uparrow\infty \textit{ such that } \frac{S_{n_k}(f)}{a_{n_k}} \to_{n\to\infty} \infty \textit{ a.e.}, \forall f\in L^1(m)_+.$$

POINTWISE DUAL ERGODICITY.

The situation is different for the duals of some non-invertible transformations.

Boole's transformations IV.

Suppose that $T:\mathbb{R}\to\mathbb{R}$ is defined by $Tx = x+\sum_{k=1}^n \frac{p_k}{t_k - x}$ where $n\ge 1$, $p_1,\dots,p_n\ge 0$ and $t_1,...,t_n\in\mathbb{R}$. Then

$$\frac{1}{a_n}\sum_{k=1}^n \widehat{T}^k f \;\to\; \int_{\mathbb{R}} fdm_{\mathbb{R}} \textit{ a.e. as } n\to\infty,\quad \forall f\in L^1(m_{\mathbb{R}}),$$

where $a_n := \sqrt{\frac{2n}{\pi^2\nu}}$ *and* $\nu := \sum_{k=1}^n p_k$.

Proof ([4]).
For $\omega \in \mathbb{R}^{2+}$, $T^n(\omega) = u_n + iv_n$ where by (2.3) $\sup_{n\ge 1}|u_n| < \infty$ and $v_n \sim \sqrt{2\nu n}$ as $n \to \infty$. It follows that

$$\pi \widehat{T}^n \varphi_\omega(x) = \text{Im}\, \frac{1}{x - T^n(\omega)} = \frac{v_n}{(x-u_n)^2 + v_n^2} \sim \frac{1}{\sqrt{2\nu n}}$$

whence

$$\sum_{k=1}^n \widehat{T}^k \varphi_\omega \sim \sum_{k=1}^n \frac{1}{\pi\sqrt{2\nu n}} = a_n$$

and for $f \in L^1(m_{\mathbb{R}})$, using Hurewicz's theorem,

$$\frac{\sum_{k=1}^n \widehat{T}^k f(x)}{a_n} \approx \frac{\sum_{k=1}^n \widehat{T}^k f(x)}{\sum_{k=1}^n \widehat{T}^k \varphi_\omega(x)} \to \int_X f dm \text{ a.e. as } n \to \infty.$$

□

A conservative, ergodic, measure preserving transformation is called *pointwise dual ergodic* if there are constants a_n such that

$$\frac{1}{a_n}\sum_{k=0}^{n-1} \widehat{T}^k f \to \int_X f dm_T \text{ a.e. as } n \to \infty\ \forall\ f \in L^1(X_T).$$

The sequence a_n is called a *return sequence* of T and denoted $a_n(T)$. Its asymptotic proportionality class $\mathcal{A}(T) := \{(a'_n)_{n\in\mathbb{N}} :\ \exists \lim_{n\to\infty} \frac{a'_n}{a_n(T)} \in \mathbb{R}_+\}$ is called the *asymptotic type* of T.

Non-expanding interval maps III
Let T be a piecewise onto, C^2 interval map with basic partition $\alpha = \{(0,u)\} \cup \alpha_0$ satisfying Adler's condition. Suppose that

$$Tx = x + cx^{1+p} + o(x^{1+p}) \quad \text{as } x \to 0$$

where $c > 0,\ p \ge 1,\ T(u) = 1,\ T'' \ge 0$ on $[0,u]$ and $\exists\ \kappa > 1$ such that

$$T'x \ge \kappa \quad \forall\ x \in\ a \in \alpha_0,$$

then ([7], [47]) T is pointwise dual ergodic and

$$a_n(T) \propto \begin{cases} n^{\frac{1}{p}} & p > 1, \\ \frac{n}{\log n} & p = 1. \end{cases}$$

Examples of piecewise onto, C^2 interval maps with more general return sequences can be found in [1], §4.8.

Markov shifts

Let S be a countable set, $T : S^{\mathbb{N}} \to S^{\mathbb{N}}$ be the *shift* map, defined by $T(x_1, x_2, ...) = (x_2, x_3, ...)$.

Given a *stochastic matrix* P on S ($p : S \times S \to \mathbb{R},\ p_{s,t} \ge 0,\ \sum_{t\in S} p_{s,t} = 1\ \forall s \in S$) and a probability π on S such that $\pi_s > 0\ \forall s \in S$, we specify the probability $m = m_\pi$ on $S^{\mathbb{N}}$ such that

$$m([s_1, \dots, s_n]) = \pi_{s_1} p_{s_1,s_2} \cdots p_{s_{n-1},s_n}\ \forall s_1, \dots, s_n \in S^n,\ n \in \mathbb{N}$$

where $[s_1, \dots, s_n] := \{x = (x_1, x_2, ...) \in S^{\mathbb{N}} : x_1 = s_1, ..., x_n = s_n\}$.

The *Markov shift* of P (with initial distribution π) is the quadruple

$$(S^{\mathbb{N}}, \mathcal{B}, m, T),$$

where $\mathcal{B} = \sigma(\{[s_1, \dots, s_n] :\ s_1, \dots, s_n \in S^n,\ n \in \mathbb{N}\}$.

Since $T[s] = \bigcup_{t\in S, p_{s,t}>0} [t] \mod m$, we see T is a non-singular transformation of $(S^{\mathbb{N}}, \mathcal{B}, m)$ iff

$$\forall t \in S,\ \exists s \in S,\ p_{s,t} > 0$$

and in this situation,

$$\widehat{T}^k 1_{[t]}(x) = \frac{\pi_t}{\pi_{x_1}} p^{(k)}_{t,x_1}\ \forall\ k \in \mathbb{N},\ t \in S.$$

A state $s \in S$ is called *recurrent* if $\sum_{n=0}^{\infty} p^{(n)}_{s,s} = \infty$.

We denote the collection of recurrent states by S_r and call P *recurrent* if $S_r = S$.

Evidently, if P is recurrent, then T is non-singular.

As shown in [28], if T is non-singular, then $\mathfrak{C}(T) = \bigcup_{s\in S_r} [s]$ (this also follows from proposition 2.1). In particular, if P is recurrent, then T is non-singular and conservative.

A stochastic matrix is called *irreducible* if

$$\forall s, t \in S, \quad \exists n \in \mathbb{N}\ \ni p^{(n)}_{s,t} > 0.$$

Evidently again, if a stochastic matrix is irreducible, then its shift is non-singular.

We'll sketch how to show that the shift of an irreducible, recurrent stochastic matrix is ergodic ([28]), and has a σ-finite invariant measure which comes from an invariant distribution on states ([19]).

First, fix $s \in S_r$, and let m_s be that constant multiple of $m|_{[s]}$ with $m_s([s]) = 1$, then $T_{[s]}$ is an exact measure preserving transformation of $([s], \mathcal{B} \cap [s], m_s)$. This is because if $\alpha = \{[s, t_1, \dots, t_{n-1}, s] :\ n \ge 1,\ t_1, \dots, t_{n-1} \ne s\}$ then $\{T_{[s]}^{-n}\alpha\}$ are statistically independent and $\sigma(\{T_{[s]}^{-n}\alpha\}) = \mathcal{B} \cap [s]$ whence the tail is trivial by Kolmogorov's zero-one law ([34]).

Next, suppose that $T^{-1}A = A \in \mathcal{B}_+$ then $T_{[s]}^{-1}(A \cap [s]) = A \cap [s]\ \forall\ s \in S$, whence $m(A \cap [s]) > 0\ \Rightarrow\ A \supset [s] \mod m$. Since $m(A) > 0,\ \exists\ t \in S$ such that $A \supset [t] \mod m$.

Let $s \in S$, then, by irreducibility, there exists $n \ge 1$ such that $p^{(n)}_{s,t} > 0$, whence

$$m([s] \cap A) = m([s] \cap T^{-n}A) \ge m([s] \cap T^{-n}[t]) > 0\ \Rightarrow\ A \supset [s] \mod m.$$

This shows that T is ergodic.

For $s \in S$, let

$$\mu_s(B) = \int_{[s]} \left(\sum_{k=0}^{\varphi_{[s]}-1} 1_B \circ T^k \right) dm_s,$$

then, by proposition 2.5, $\mu_s \circ T^{-1} = \mu_s << m$.

By unicity of invariant measure (theorem 3.3), $\mu_t \equiv \mu_s([t])\mu_s \sim m \;\; \forall\, t \in S$, and, necessarily, $0 < \mu_s([t]) < \infty$ for $s, t \in S$.

Since $\mu_s|_{[s]} = m_s|_{[s]} \;\forall\, s \in S$, it follows that

$$\mu_s([t_1, ..., t_n]) = \mu_s([t_1]) \prod_{k=1}^{n-1} p_{t_k, t_{k+1}},$$

whence

$$\sum_{t \in S} \mu_s([t]) p_{t,u} = \mu_s(T^{-1}[u]) = \mu_s([u]).$$

A calculation shows ([19]) that

$$\mu_s([t]) = \sum_{n=1}^{\infty} {}_s p_{s,t}^{(n)}$$

where

$${}_s p_{s,t}^{(1)} = p_{s,t}, \quad {}_s p_{s,t}^{(n+1)} = \sum_{u \neq s} {}_s p_{s,u}^{(n)} p_{u,t} \quad (n \geq 1).$$

Consider now T as a measure preserving transformation of $(X, \mathcal{B}, \mu)$ where $\mu = \mu_s$. Write $\mu_s([t]) = c_t$. It follows that

$$\widehat{T}^k 1_{[t]}(x) = \frac{c_t}{c_{x_1}} p_{t,x_1}^{(k)} \;\forall\, k \in \mathbb{N},\; t \in S,$$

whence T is pointwise dual ergodic with $a_n(T) \sim \sum_{k=0}^{n-1} p_{s,s}^{(k)}$.

An irreducible stochastic matrix is said to be *aperiodic* if for some (and hence all) $s \in S$, g.c.d. $\{n \geq 1 : p_{s,s}^{(n)} > 0\} = 1$. It is shown in [16] that the Markov shift of an aperiodic, recurrent stochastic matrix is is exact (see also theorem 3.2 of [10]).

RATIONAL ERGODICITY.

No invertible, conservative, ergodic, measure preserving transformation of an infinite measure space can be pointwise dual ergodic because $\widehat{T}f = f \circ T^{-1}$ and pointwise dual ergodicity would violate theorem 5.1. There are invertible rationally ergodic transformations.

A conservative, ergodic, measure preserving transformation T of $(X, \mathcal{B}, m)$ is called *rationally ergodic* if there is a set $A \in \mathcal{B},\; 0 < m(A) < \infty$ satisfying a *Renyi inequality*: $\exists\, M > 0$ such that

$$\int_A (S_n(1_A))^2\, dm \leq M \left(\int_A S_n(1_A) dm \right)^2 \quad \forall\, n \geq 1.$$

5.2 Theorem [2], [3]. *Suppose that T is rationally ergodic. Then there is a sequence of constants $a_n \uparrow \infty$, unique up to asymptotic equality, such that whenever $A \in \mathcal{B}$ satisfies a Renyi inequality,*

$$\frac{1}{a_n}\sum_{k=0}^{n-1} m(B \cap T^{-k}C) \to m(B)m(C) \text{ as } n \to \infty \ \forall\ B, C \in \mathcal{B} \cap A$$

$$\underline{\lim}_{n\to\infty} \frac{1}{a_n}\sum_{k=0}^{n-1} m(B \cap T^{-k}C) \ge m(B)m(C) \ \forall\ B, C \in \mathcal{B},$$

and

$$\forall\ m_\ell \uparrow \infty\ \exists\ n_k = m_{\ell_k} \uparrow \infty$$

such that

$$\frac{1}{N}\sum_{k=1}^{N}\frac{1}{a_{n_k}}\sum_{j=0}^{n_k-1} f\circ T^j \to \int_X f dm \text{ a.e. as } n \to \infty \ \forall\ f \in L^1(m). \tag{3}$$

A conservative, ergodic, measure preserving transformation satisfying (3) with respect to some sequence of constants $\{a_n\}$ is called *weakly homogeneous.*

5.3 Proposition [4]. *Suppose that T is pointwise dual ergodic. Then T is rationally ergodic and satisfies (3) with respect to its return sequence $a_n(T)$.*

The sequence of constants $\{a_n\}$ appearing in (3) may therefore be also called the *return sequence* of T without ambiguity in this choice of name.

ASYMPTOTIC DISTRIBUTIONAL BEHAVIOUR.

5.4 Proposition [6]. *If T is a conservative, ergodic measure preserving transformation of $(X, \mathcal{B}, m)$, and $n_k \to \infty$, and $d_k > 0$, then $\exists\ m_\ell := n_{k_\ell} \to \infty$, and a random variable Y on $[0, \infty]$ such that*

$$g\left(\frac{1}{d_{k_\ell}}\sum_{j=0}^{m_\ell-1} f\circ T^j\right) \to E(g(\mu(f)Y)), \tag{$*$}$$

weak $*$ in L^∞, $\forall\ f \in L^1(m),\ f \ge 0,\ g \in C([0,\infty])$, where $\mu(f) := \int_X f dm$.

In the situation of $(*)$, we'll write

$$\frac{S^T_{m_\ell}}{d_{k_\ell}} \xrightarrow{\mathfrak{d}} Y,$$

and call Y a *distributional limit of T along $\{m_\ell\}$.*

Clearly $Y = 0$ and $Y = \infty$ are distributional limits of any conservative, ergodic, measure preserving transformation. We'll consider a random variable Y on $[0, \infty]$ *trivial* if it is supported on $\{0, \infty\}$.

Mittag-Leffler distribution. Let $\alpha \in [0, 1]$. The random variable Y_α on $\mathbb{R}_+$ has the *normalized Mittag-Leffler distribution of order α* if

$$E(e^{zY_\alpha}) = \sum_{p=0}^{\infty}\frac{\Gamma(1+\alpha)^p z^p}{\Gamma(1+p\alpha)}$$

(the normalization being $E(Y_\alpha) = 1$).

As can be checked, Y_0 is exponentially distributed, $Y_{\frac{1}{2}}$ is distributed as the absolute value of a centered Gaussian random variable, and $Y_1 = 1$.

5.5 Darling- Kac Theorem ([6], c.f. [20]). *Suppose that T is pointwise dual ergodic, and that $a_n(T)$ is regularly varying with index $\alpha \in [0,1]$ as $n \to \infty$. Then*

$$\frac{S_n^T}{a_n(T)} \xrightarrow{\mathfrak{d}} Y_\alpha,$$

where Y_α has the normalized Mittag-Leffler distribution of order α.

Examples

Boole's transformation $T : \mathbb{R} \to \mathbb{R}$ is defined by $Tx = x + \sum_{k=1}^n \frac{p_k}{t_k - x}$, where $n \geq 1$, $p_1, \ldots, p_n \geq 0$ and $t_1, ..., t_n \in \mathbb{R}$, satisfy $\frac{S_n^T}{a_n(T)} \xrightarrow{\mathfrak{d}} Y_{\frac{1}{2}}$, where $a_n := \sqrt{\frac{2n}{\pi^2 \nu}}$ and $\nu := \sum_{k=1}^n p_k$.

If $T : [0,1] \to [0,1]$ is defined by $Tx = \{\frac{1}{1-x}\}$, then $\frac{S_n^T}{a_n(T)} \xrightarrow{\mathfrak{d}} Y_1$ where $a_n := \frac{n}{\log n}$; equivalently $\frac{1}{a_n(T)} \sum_{k=0}^{n-1} f \circ T^k \to \int_I f(x) \frac{dx}{x}$ in measure $\forall\ f \in L^1(\frac{dx}{x})$.

The Mittag-Leffler distributions are related to the positive stable distributions, indeed $Y_\alpha^{-\alpha}$ has positive stable distribution of order α:

$$E(e^{-tY_\alpha^{-\frac{1}{\alpha}}}) = e^{-c_\alpha t^\alpha}. \tag{4}$$

See [17] and [6].

For distributional convergence to stable laws as a consequence of the Darling-Kac theorem, see [6].

Structure

For $c \in (0, \infty]$, a *c-factor map* from a measure preserving transformation $(X, \mathcal{B}, \mu, S)$ onto a measure preserving transformation $(Y, \mathcal{C}, \nu, T)$ is a measurable map $\pi : X' \to Y'$ (where $X' = S^{-1}X' \in \mathcal{B}$ and $Y' = T^{-1}Y' \in \mathcal{B}\mathcal{C}$ are sets of full measure) satisfying

$$\mu \circ \pi^{-1}(A) = c\nu(A)\ \forall\ A \in \mathcal{B}_T, \text{ and } \pi \circ S = T \circ \pi.$$

We shall denote this situation by $\pi : S \xrightarrow{c} T$.

A c-isomorphism is an invertible c-factor map π (denoted $\pi : S \xleftrightarrow{c} T$).

It is necessary to consider c-factor maps with $c \neq 1$ as our measure spaces are not normalized (being infinite).

If, for some $c \in \mathbb{R}_+$ there exists $\pi : S \xrightarrow{c} T$, we shall call T a *factor* of S, and S an *extension* of T, denoting this by $S \to T$.

Clearly if $\pi : S \xrightarrow{c} T$, then $\pi^{-1}\mathcal{B}_T$ is a σ-finite, sub-σ-algebra of $\mathcal{B}_S$ which is T-*invariant* in the sense that $S^{-1}(\pi^{-1}\mathcal{B}_T) \subset \pi^{-1}\mathcal{B}_T$. In case T is invertible, $\pi^{-1}\mathcal{B}_T$ is *strictly T-invariant* : $S^{-1}(\pi^{-1}\mathcal{B}_T) = \pi^{-1}\mathcal{B}_T$.

6.1 Factor Proposition. *Suppose that T is a measure preserving transformation of the σ-finite, standard measure space $(X, \mathcal{B}, m)$.*

For every T-invariant, sub-σ-algebra of $\mathcal{F} \subset \mathcal{B}$, there is a factor U, and a factor map $\pi : T \xrightarrow{1} U$ with $\mathcal{F} = \pi^{-1}\mathcal{B}_U$.

The factor is invertible iff the sub-σ-algebra is strictly T-invariant.

Two measure preserving transformations are called *strongly disjoint* if they have no common extension, and *similar* otherwise.

No two probability preserving transformations are strongly disjoint. Indeed, if S, T are probability preserving transformations, the Cartesian product transformation $R = S \times T$ is a probability preserving transformation, and

$$S \leftarrow R \rightarrow T.$$

Recall from [23] that the probability transformations S, T are called *disjoint* if any common extension R has the Cartesian product as factor (i.e. $R \to S \times T$).

When S, and T are infinite measure preserving transformations, the Cartesian product is not an extension of either transformation and moreover, strong disjointness is not uncommon among infinite measure preserving transformations.

Notwithstanding, similarity is an equivalence relation.

6.2 Proposition. *Suppose that S and T are similar measure preserving transformations. Then S is conservative if and only if T is conservative.*

This follows from proposition 2.1.

6.3 Proposition [3]. *Suppose that S and T are similar measure preserving transformations, both conservative and ergodic.*

If S is weakly homogeneous, then so is T and

$$\exists \lim_{n\to\infty} \frac{a_n(S)}{a_n(T)} \in \mathbb{R}_+.$$

Thus, Boole's transformations $Tx = x + \sum_{k=1}^n \frac{p_k}{t_k - x}$ (with return sequence $a_n(T) \propto \sqrt{n}$) are all strongly disjoint from $Sx = \{\frac{1}{1-x}\}$ (return sequence: $a_n(S) \propto \frac{n}{\log n}$).

A one parameter family of pairwise strongly disjoint pointwise dual ergodic transformations is given by the non-expanding interval maps considered above. For $p \geq 1$ let T_p be a piecewise onto, C^2 interval map with basic partition $\alpha = \{(0, \frac{1}{2}), (\frac{1}{2}, 1)\}$ with

$$T_p x = \begin{cases} x + c_p x^{1+p} + o(x^{1+p}) & x \in (0, \frac{1}{2}), \\ 2x - 1 \quad x \in (\frac{1}{2}, 1) \end{cases}$$

where $c_p > 0$ is fixed so that $T_p(x) \to 1$ as $t \to \frac{1}{2}-$.

As before, each T_p is pointwise dual ergodic and

$$a_n(T_p) \propto \begin{cases} n^{\frac{1}{p}} & p > 1, \\ \frac{n}{\log n} & p = 1, \end{cases}$$

whence T_p and $T_{p'}$ are strongly disjoint for $p \neq p'$.

NATURAL EXTENSION.

Let $(X, \mathcal{B}, m, T)$ be a measure preserving transformation of a σ-finite, standard measure space. As in [42], a *natural extension* of T is an invertible extension $(X', \mathcal{B}', m', T')$ of T which is minimal in the sense that and $T'^n \pi^{-1}\mathcal{B} \uparrow \mathcal{B}' \mod m'$ where $\pi : T' \xrightarrow{1} T$ is the extension map.

6.4 Theorem: Existence and uniqueness of natural extensions ([42]).
Any measure preserving transformation of a standard, σ-finite measure space has a natural extension (also on a standard space).

All natural extensions of the same measure preserving transformation are isomorphic.

6.5 Theorem [39]. *The natural extension of a conservative, ergodic measure preserving transformation is conservative, and ergodic.*

Intrinsic Normalizing Constants and Laws of Large Numbers

Although infinite measure spaces are not canonically normalized, it may be that a measure preserving transformation T is intrinsically normalized, for example, in the sense that $T \overset{c}{\not\leftrightarrow} T$ for each $c \neq 1$. To this end, we consider (for T a measure preserving transformation)

$$\Delta_0(T) := \{c \in (0, \infty) : \ T \overset{c}{\leftrightarrow} T\}$$

first introduced in [24]. Clearly, this is a multiplicative subgroup of $\mathbb{R}_+$, and if S and T are isomorphic, then $\Delta_0(S) = \Delta_0(T)$.

7.1 Proposition. *If T is a conservative, ergodic measure preserving transformation of $(X, \mathcal{B}, m)$, then $\Delta_0(T)$ is a Borel subset of $\mathbb{R}$.*

For example, let T be a totally dissipative measure preserving transformation of the standard σ-finite, non-atomic measure space $(X, \mathcal{B}, m)$. There is a wandering set $W \in \mathcal{B}$ such that $X = \bigcup_{n \in \mathbb{Z}} T^n W$, and it is not hard to see that

$$\Delta_0(T) = \begin{cases} \{1\} & m(W) < \infty, \\ \mathbb{R}_+ & m(W) = \infty. \end{cases}$$

The first conservative, ergodic example with $\Delta_0(T) = \{1\}$ was given in [25].

Unfortunately, it is not the case that $\Delta_0(S) = \Delta_0(T)$ for similar conservative, ergodic measure preserving transformations (see [8]).

Accordingly, we consider the *intrinsic normalizing constants* of a conservative, ergodic, measure preserving transformation T, namely the collection

$$\Delta_\infty(T) := \{c \in (0, \infty] : \ \exists \text{ a conservative, ergodic m.p.t. } \ R \overset{1}{\to} T, \ \ R \overset{c}{\to} T\}.$$

It is shown in [8] that $\Delta_\infty(T) \cap \mathbb{R}_+$ is an analytic subset of $\mathbb{R}_+$.

7.2 Proposition [8]. *If S and T are similar conservative ergodic measure preserving transformations, then*

$$\Delta_\infty(S) = \Delta_\infty(T).$$

7.3 Proposition [8]. *If T is a conservative, ergodic, measure preserving transformation, then $\Delta(T) := \Delta_\infty(T) \cap \mathbb{R}_+$ is a multiplicative subgroup of $\mathbb{R}_+$.*

7.4 Proposition [6]. *If T is weakly homogeneous, then $\Delta_\infty(T) = \{1\}$.*

This will follow from proposition 7.8 below.

Examples where $\Delta_\infty(T) \neq \{1\}$

Let $\Omega = \{x = (x_1, x_2, \dots) : \ x_n = 0, 1\}$ be the group of dyadic integers, and let $(\tau x)_n := x_n + \epsilon_n \mod 2$ where $\epsilon_1 = 1$ and $\epsilon_{n+1} = 1$ if $x_n + \epsilon_n \geq 2$ and $\epsilon_{n+1} = 0$ otherwise.

For $p \in (0, 1)$, define a probability μ_p on Ω by

$$\mu_p([\epsilon_1, ..., \epsilon_n]) = \prod_{k=1}^{n} p(\epsilon_k)$$

where $p(0) = 1 - p$ and $p(1) = p$.

Recall that $\mu_{\frac{1}{2}} = m$ is Haar measure on Ω, whence $\mu_{\frac{1}{2}} \circ \tau = \mu_{\frac{1}{2}}$. It is no longer true that τ preserves μ_p if $p \neq 1/2$, however $\mu_p \circ \tau \sim \mu_p$ and

$$\frac{d\mu_p \circ \tau}{d\,\mu_p} = \left(\frac{1-p}{p}\right)^{\phi}$$

where

$$\phi(x) = \min\{n \in \mathbb{N} : x_n = 0\} - 2.$$

7.5 Proposition. *τ is an invertible, conservative, ergodic non-singular transformation of*
$(\Omega, \mathcal{B}, \mu_p)$.

Proof sketch.

A calculation shows that τ is non-singular (and indeed measure preserving when $p = \frac{1}{2}$).

All τ-invariant sets are in the *tail* σ-algebra $\mathfrak{T} := \bigcap_{n=1}^{\infty} \sigma(x_n, x_{n+1}, \dots)$, which is trivial by Kolmogorov's zero-one law ([34]). $\square$

In fact, as shown in [13], there is no τ-invariant σ-finite, μ_p-absolutely continuous measure on Ω when $p \neq 1/2$. The next result is a strengthening of this.

For $0 < p < 1$, we consider the measure preserving transformations $T_p := (X, \mathcal{B}, m_p, T)$ where

$$X = \Omega \times \mathbb{Z}, \ \mathcal{B} := \mathcal{B}(\Omega) \otimes 2^{\mathbb{Z}},$$

$$m_p(A \times \{n\}) = \mu_p(A)\left(\frac{1-p}{p}\right)^n, \text{ and } T(x, n) = (\tau x, n - \phi(x)).$$

7.6 Theorem [24]. *$(X, \mathcal{B}, m_p, T)$ is a conservative, ergodic measure preserving transformation.*

The result of [13] follows from this since the existence of a τ-invariant σ-finite, μ_p-absolutely continuous measure on Ω when $p \neq 1/2$ entails $\phi = f - f \circ \tau$ for some $f : \Omega \to \mathbb{Z}$ measurable contradicting ergodicity of $(X, \mathcal{B}, m_p, T)$.

7.7 Proposition [8]. *For* $p \neq \frac{1}{2}$,

$$\Delta_0(T_p) = \Delta_\infty(T_p) = \left\{ \left(\frac{1-p}{p} \right)^n \ : \ n \in \mathbb{Z} \right\}.$$

When $p = \frac{1}{2}$ the situation is different: $T_{\frac{1}{2}}$ is rationally ergodic with $a_n(T_{\frac{1}{2}}) \asymp \frac{n}{\sqrt{\log n}}$ (whence by proposition 7.4, $\Delta_\infty(T_{\frac{1}{2}}) = \{1\}$). For this, and further results on the transformations T_p, see [11].

In [8], examples of conservative, ergodic, measure preserving transformations T with $\Delta(T) = \Delta_0(T)$ of arbitrary Hausdorff dimension were given.

For a conservative, ergodic, measure preserving transformation T with $\Delta_\infty(T) = \{1, \infty\}$, see proposition 2.1 of [9].

Definition: Law of large numbers. A *law of large numbers* for a conservative, ergodic, measure preserving transformation T of $(X, \mathcal{B}, m)$ is a function $L : \{0,1\}^{\mathbb{N}} \to [0, \infty]$ such that $\forall \ A \in \mathcal{B}$, for a.e. $x \in X$,

$$L(1_A(x), 1_A(Tx), \dots) = m(A).$$

For example, if $(X, \mathcal{B}, m, T)$ is weakly homogeneous, then $\exists \ n_k \to \infty$ such that

$$\frac{1}{N} \sum_{k=1}^{N} \frac{1}{a_{n_k}(T)} \sum_{j=0}^{n_k - 1} f \circ T^j \to \int_X f dm \text{ a.e. as } n \to \infty \ \forall \ f \in L^1(m)$$

and a law of large numbers for T is given by

$$L(\epsilon_1, \epsilon_2, \dots) := \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{1}{a_{n_k}(T)} \sum_{j=0}^{n_k - 1} \epsilon_j.$$

7.8 Proposition [1]. *If* $(X, \mathcal{B}, m, T)$ *is conservative and ergodic, and every conservative, ergodic measure preserving transformation similar to* T *has a law of large numbers, then* $\Delta_\infty(T) = \{1\}$.

Proof.

Let $c \in \Delta_\infty(T)$. Suppose that $(Y \mathcal{C}, \mu, U)$ is conservative and ergodic, and let $\varphi : U \xrightarrow{1} T$, and $\psi : U \xrightarrow{c} T$ be 1-, and c-factor maps (respectively). Let L be a law of large numbers for U, and fix $A \in \mathcal{B}$, $m(A) = 1$ then

$$L(1_A(\varphi x), 1_A(T\varphi x), \dots) = L(1_{\varphi^{-1}A}(x), 1_{\varphi^{-1}A}(Ux), \dots) = \nu(\varphi^{-1}A) = 1,$$

ν-a.e., whence $L(1_A(x), 1_A(Tx), \dots) = 1$ m-a.e., but also

$$L(1_A(\psi x), 1_A(T\psi x), \dots) = L(1_{\psi^{-1}A}(x), 1_{\psi^{-1}A}(Ux), \dots) = \nu(\psi^{-1}A) = c$$

ν-a.e., whence $L(1_A(x), 1_A(Tx), \dots) = c$ m-a.e. with the conclusion that $c = 1$. □

It turns out that the existence of laws of large numbers is close to characterising absence of intrinsic normalising constants.

7.9 Theorem [1]. *If $\Delta_\infty(T) = \{1\}$, then T has a law of large numbers.*

This result can be used to prove the converse of proposition 7.8.

As another application, let $(X, \mathcal{B}, m)$ be a compact, metric group equipped with normalised Haar measure and let T be an ergodic translation of X. Suppose that $d \in \mathbb{N}$ and that $\phi : X \to \mathbb{Z}^d$ is measurable. Define the skew product transformation $T_\phi : X \times \mathbb{Z}^d \to X \times \mathbb{Z}^d$ by $T_\phi(x, y) = (Tx, y + \phi(x))$. This preserves the product measure $m \times$ counting measure. It is shown in [1] (by considering self joinings of T_ϕ) that if T_ϕ is ergodic, then $\Delta_\infty(T_\phi) = \{1\}$ and thus, by theorem 7.9, T_ϕ has a law of large numbers.

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