CALCULUS 1A
LECTURE NOTES SPRING 2017.

JON AARONSON'S `LECTURE NOTES`

Lecture # 1 [1]

SETS AND NUMBERS

Some set theory notations.

A *set* (קבוצה) is a collection of *elements* (איברים).

• If an object a is an element of - or *belongs to* - (שייך ל־) the set A, we denote this $a \in A$. Otherwise the object a is not an element of the set A (denoted $a \notin A$).

• Let A, B be sets. We say that A is a *subset* (תת־קבוצה) of B (written $A \subset B$) if every element of A belongs to B (i.e. $x \in A \Rightarrow x \in B$).

Let X be a set and let $A,\ B \subset X$ be subsets.

• The *union* (איחוד) of A and B is the set

$$A \cup B := \{x \in X : \texttt{ either } x \in A \texttt{ or } x \in B\}.$$

• The *intersection* (חיתוך) of A and B is the set

$$A \cap B := \{x \in X : \ \ x \in A \texttt{ and } x \in B\}.$$

• The *complement* (משלים) of A (in X) is

$$A^c := \{x \in X : \ x \notin A\}.$$

• The sets $A,\ B$ are *disjoint* (זרות) if the set $A \cap B$ is empty (written $A \cap B = \varnothing$). Note that $A \cap B = \varnothing$ if and only if $A \subset B^c$.

• The *set difference* (הפרש קבוצתי) between B and A is

$$B \smallsetminus A := \{x \in B : \ x \notin A\} \overset{(!)}{=} B \cap A^c.$$

`Lectures were ~ 50 minutes.`

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Functions.

Let A, B be sets. A *function* (פונקציה) $f : A \to B$ is an "`assignment`" ("קביעה") to each element $a \in A$ of an element $f(a) \in B$. This is sometimes denoted $x \mapsto f(x)$, or $x \overset{f}{\mapsto} f(x)$.

• A is called the *domain* (תחום הגדרה) of f, $f(A) := \{f(a) : a \in A\}$ is called the *range* (טווח) of f.

e.g. (i) $A := \{\texttt{TAU students}\}$, $B := \{0, 1, 2, 3, \dots\}$ and $t(a) :=$ number of teeth a possesses.

e.g. (ii) A X-valued *sequence* (סדרה) (X some set) is a function $a : \mathbb{N} \to X$ where $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of "`natural`" numbers defined below. The sequence $a : \mathbb{N} \to X$ is sometimes denoted $(a_1, a_2, \dots)$.

• The function $f : A \to B$ is called 1-1 (חד־חד ערכית) (*injective* or *an injection*) if $f(x) = f(y) \;\Rightarrow\; x = y$.

• The above "`tooth function`" $t : A \to B$ is possibly not 1-1, but the function $\texttt{ID} : A \to B$ defined by $\texttt{ID}(a) :=$ student number of a should be 1-1 so that students are identified by their student numbers.

• The function $f : A \to B$ is called *onto* (על) B (*surjective* or *a surjection*) if $f(A) = B$. The above "`tooth function`" $t : A \to B$ is not onto, since `(anatomically)` $t(A) \subseteq \{0, 1, \dots, 32\} \subsetneq B$. Note that any function $f : A \to B$ is onto its range $f(A)$.

• The function $f : A \to B$ is called *bijective*, *a bijection* or a *set correspondence* (התאמה בין קבוצות) if it is 1-1 and onto.

• The sets A, B are *in correspondence* (בהתאמה) if $\exists$ a bijection $f : A \to B$. For example in a dog trainers school (where each kid attending learns to train its dog) there is a correspondence between the kids and the dogs `(by the leashes)`.

• If $f : A \to B$ is a bijection, then for each $b \in B$ there is a unique $a =: f^{-1}(b) \in A$ so that $f(a) = b$. This defines a function $f^{-1} : B \to A$ called the *inverse function*. It is also a bijection(!).

Product sets. Let A, B be sets. The `Cartesian` *product set* (מכפלה קרטזית) is the set of `ordered pairs` (זוגות סדורים)

$$A \times B := \{(a, b) : a \in A,\ b \in B\}.$$

The *graph* (גרף) of the function $f : A \to B$ is the subset

$$G_f := \{(a, f(a)) \in A \times B : a \in A\}.$$

• Note that $\forall\ a \in A,\ \exists\ b \in B,\ (a, b) \in G$ and if $(a, b),\ (a, b') \in G_f$ then $b = b'$. Conversely, any subset of $A \times B$ with these properties is the graph of some function $f : A \to B$. This is considered a better way to define "`function`" (as it doesn't use the word "`assignment`").

Collections of functions. The collection of functions defined on the set A and taking values in the set B is denoted $B^A := \{f : A \to B\}$. For example if B is a set, then the collection of B-valued sequences is $B^{\mathbb{N}}$.

Proposition *$B^{\{1,2\}}$ is in correspondence with $B \times B$.*

Proof To define a bijection $f : B^{\{1,2\}} \to B \times B$, let $H \in B^{\{1,2\}}$, then $H : \{1,2\} \to B$. Define $f(H) := (H(1), H(2))$, then $f : B^{\{1,2\}} \to B \times B$. It is easy to see (!) that $f : B^{\{1,2\}} \to B \times B$ is 1-1 and onto. ☑

Collections of numbers.

$\mathbb{N} := \{1, 2, 3, 4, \dots\}$ the *natural numbers,* המספרים הטבעיים;

$\mathbb{Z} := \{0, n, \ -n : \ n \in \mathbb{N}\}$ the *integers*, המספרים השלמים;

$\mathbb{Q} := \{\frac{m}{q} : \ m \in \mathbb{Z}, \ q \in \mathbb{N}\}$ the *rational numbers*, המספרים הרציונאלים.

$\mathbb{R} := \{$`all the points on a line`$\}$ the *real line*, המספרים הממשיים.

$\mathbb{C} := \{x + \sqrt{-1}y : \ x, \ t \in \mathbb{R}\}$ the *complex numbers*, המספרים המרוכבים.

All of these need rigorous definition.

Definition of $\mathbb{N}$ and Induction (אנדוקציה)

`"I've told you` n `times, I've told you` $n+1$ `times..."`

`Peano's axiom` is that $\exists$ a set $\mathbb{N}$ with the following properties:

N1 $1 \in \mathbb{N}$: the *initial element*;

N2 (`Follower map`) $\exists \ f : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ 1-1.

N3 (`Induction` axiom) If $K \subset \mathbb{N}$ satisfies $1 \in K$ and $x \in K \Rightarrow f(x) \in K$, then $K = \mathbb{N}$.

☺ `If you've "counted" a set of numbers (i.e. each number counted has its follower counted too) AND you started at one, then you counted all the numbers.`

☹ Domino theory: `If the first domino falls and each falling domino topples the next one, then all the dominoes fall.`

- Peano showed that any two such systems are `isomorphic` (i.e. $\exists$ bijection between them preserving the follower maps).

! Given $\mathbb{N}$ as above, find $F : \mathbb{N} \to \mathbb{N}$ satisfying **N2** but not **N3**.

- **Addition** (חיבור) so that $n+1$ is the follower of n and **multiplication** (כפל) can be defined (uniquely) on $\mathbb{N}$ satisfying
associativity (צירוף): $(a+b)+c=a+(b+c),\ (ab)c=a(bc)$,
commutativity (חילוף): $a+b=b+a,\ ab=ba$ and
distributivity (פילוג): $a(b+c)=ab+ac$ and then
- **Order** (סדר) is defined on $\mathbb{N}$ by $a<b$ if $\exists\ c\in\mathbb{N},\ b=a+c$. This order is `linear` satisfying:
- $\forall\ a,b\in\mathbb{N}$, either $a<b$, or $b<a$ or $a=b$;
- $a<b,\ b<c\ \Rightarrow\ a<c$;
- $a<b\ \Rightarrow\ ac<bc\ \forall\ c\in\mathbb{N}$.

We write $u>v$ in case $v<u$ and $a\le b$ (or $b\ge a$) in case either $a<b$ or $a=b$.

Equivalence relations & the rationals

When you define the rationals, you consider identities such as

$$\frac{4}{8}=\frac{3}{6}=\frac{1}{2}.$$

This is formalized via `equivalence relations`.

Binary relation on a set. Let S be a set. A *binary relation on S* is a set $R\subset S\times S$. You think of R as defining a "`relationship`" between elements of S and write

$$x\ \sim\ y\ \iff\ (x,y)\in R.$$

Equivalence relation on a set. This is the simplest kind of binary relation. Let S be a set. An *equivalence relation on S* is a binary relation

$$R=\{(x,y)\in S\times S:\ x\sim y\}\subset S\times S$$

which is:

`reflexive`: $x\sim x\ \forall\ x\in S$, equivalently

$$R\supseteq \mathtt{diag}\,(S\times S):=\{(x,x):\ x\in S\};$$

`symmetric`: $x\sim y\ \Rightarrow\ y\sim x$;

`transitive`: $x\sim y\ \&\ y\sim z\ \Rightarrow\ x\sim z$.

The simplest example of an equivalence relation on S is `equality`. Here

$$x\sim y\ \iff\ x=y\ \ \&\ \ R=\mathtt{diag}\,(S\times S).$$

The next simplest example is `equality of an image`. Here, you're given another set T and a map $f : S \to T$. The relation is

$$x \sim y \iff f(x) = f(y)$$
$$\texttt{i.e.}\quad R = \{(x,y) \in S \times S :\ (f(x), f(y)) \in \mathtt{diag}\,(T \times T)\}.$$

This means that the *equivalence classes* of $\sim$:

$$[x]_\sim := \{y \in S :\ x \sim y\} = \{y \in S :\ f(y) = f(x)\} \quad x \in S$$

form a *partition* of S i.e.

$$S = \dot{\bigcup_{t \in f(S)}} \{x \in S :\ f(x) = t\}$$

and for any $x,\ y \in S$,
either $f(x) = f(y)$ & $[x]_\sim = [y]_\sim$,
or $f(x) \neq f(y)$ & $[x]_\sim \cap [y]_\sim = \varnothing$.

It is easy to see that this last is true of the equivalence classes of any equivalence relation and so we have the

Theorem *All equivalence relations are of the form* `equality of an image`.

You can take the "image space" T to be the partition of S into disjoint equivalence classes which is denoted $S/\sim$ and the map $f : S \to T$ defined by $f(x) := [x]_\sim$.

`RATIONALS.` It's assumed you know about addition and multiplication on $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}) \cup \{0\}$ (eg from the exercise class).

Define

$$\widehat{\mathbb{Q}} := \mathbb{Z} \times \mathbb{N}$$

and addition and multiplication on $\widehat{\mathbb{Q}}$ by

$$(k,p) + (\ell,q) := (qk + p\ell, pq), \quad (k,p) \cdot (\ell,q) := (k \cdot \ell, pq).$$

We'll define the *rationals* $\mathbb{Q} := \widehat{\mathbb{Q}}/\sim$ where $\sim$ is an appropriate equivalence relation on $\widehat{\mathbb{Q}}$.

To this end, define the `relation` $\sim$ on $\widehat{\mathbb{Q}}$ by

- $(k,p) \sim (\ell,q)$ if $kq = \ell p$, then $\sim$ is an `equivalence relation`.

Motivation: $$\frac{k}{p} = \frac{\ell}{q}.$$

If $x, x', y, y' \in \widehat{\mathbb{Q}}$ and $x \sim x',\ y \sim y'$, then $x+y \sim x'+y'$ and $x \cdot y \sim x' \cdot y'$, so one can define addition and multiplication on

$$\mathbb{Q} := \widehat{\mathbb{Q}}/\sim$$

by

$$(+) \qquad [(k,p)] + [(\ell,q)] := [(k,p) + (\ell,q)],$$

$$(\times) \qquad [(k,p)] \cdot [(\ell,q)] := [(k,p) \cdot (\ell,q)].$$

It's not hard to see that these satisfy associativity, commutativity and distributivity.

By writing $[(k,p)] = \frac{k}{p}$ you get back the "traditional idea" of a "reduced fraction".

`Order` on $\mathbb{Q}$ defined by $a < b$ if $\exists\ c \in \mathbb{N} \times \mathbb{N},\ b = a + [c]$ is linear.

Recall that the integers $\mathbb{Z}$ can be defined using an equivalence relation over $\widehat{\mathbb{Z}} := \mathbb{N} \times \mathbb{N}$ to express identities such as

$$m - n = (m + J) - (n + J).$$

Countability

- **Finite set**

  A set A is called *finite* (סופי) if it is in correspondence with $\{k \in \mathbb{N} :\ k \le n\}$ for some $n \in \mathbb{N}$.
- **Infinite set**

  A set A is called *infinite* (אינסופי) if it is not finite.
- **Countable set**

  A set A is called *countable* (aka *denumerable*, בת מנייה) if it is in correspondence with $\mathbb{N}$.
- **At most countable set**

  A set A is called *at most countable* (לכל היותר בת מנייה) if it is finite or countable.

Proposition $\mathbb{Z}$ *is countable.*

Proof Define $f : \mathbb{N} \to \mathbb{Z}$ by $f = (0, 1, -1, 2, -2, \dots)$ equivalently

$$f(j) = \begin{cases} 0 & j = 1, \\ n & j = 2n \quad (n \in \mathbb{N}), \\ -n & j = 2n+1 \quad (n \in \mathbb{N}). \end{cases}$$

Evidently $f : \mathbb{N} \to \mathbb{Z}$ is a bijection. ☑

Next topics WELL ORDERING PRINCIPLE, COUNTABLE UNIONS OF COUNTABLE SETS, $\mathbb{Q}$ IS COUNTABLE, $\sqrt{2} \notin \mathbb{Q}$, FIELDS, $\sum$ NOTATION, ORDERED SETS

Lecture # 2 [2]

Well Ordering Principle (עקרון הסדר הטוב)

Every nonempty subset A of $\mathbb{N}$ has a minimal element (איבר מינימאלי) *$a \in A$ (i.e. such that $a \le b \ \forall \ b \in A$).*

Proof Suppose that $A \subset \mathbb{N}$ has no minimal element and let $B := \mathbb{N} \setminus A$.

We'll show (using the `induction` property of $\mathbb{N}$) that $B = \mathbb{N}$ (whence $A = \varnothing$).

To this end we claim :

¶1 $1 \in B$ (else 1 would be a minimal element for A);

¶2 if $k \in B \ \forall \ 1 \le k \le n$, then $n+1 \in B$ (else $n+1$ would be a minimal element for A).

¶3 Let $C := \{n \in \mathbb{N} : \ k \in B \ \forall \ 1 \le k \le n\}$, then $C \subset B$ and:

- by ¶1, $1 \in C$; and
- by ¶2 $n \in C \ \Rightarrow \ n+1 \in C$.

By induction $C = B = \mathbb{N}$ and $A = \varnothing$. ⊠

Countable unions of countable sets & $\mathbb{Q}$.

Theorem

If A_n is non-empty and at most countable for $n \in \mathbb{N}$, then $U = \bigcup_{n \in \mathbb{N}} A_n := \{x : \ x \in A_n \texttt{ for some } n \in \mathbb{N}\}$ is at most countable.

The proof of this theorem is in a series of lemmas.

Lemma 1

Under the conditions of the theorem, there is a surjection $f : \mathbb{N} \to U := \bigcup_{n \in \mathbb{N}} A_n$.

Proof For each $i \in \mathbb{N}$ there is a surjection $f_i : \mathbb{N} \to A_i$. Define $f : \mathbb{N} \to U$ by $(f(1), f(2), \dots) = (B^1, B^2, \dots)$ where

$$\begin{aligned} B^n &= (B_1^{(n)}, B_2^{(n)}, \dots, B_{n^2}^{(n)}) \\ &=: (f_1(1), \dots, f_1(n), f_2(1), \dots, f_2(n), \dots, f_n(1), \dots, f_n(n)). \end{aligned}$$

Now, let $n, \ j \in \mathbb{N}$ and let $N \ge n, j$. It is not hard to see that

$$B_{(n-1)N+j}^{(N)} = f_n(j).$$

Thus

$$f(S_N + (n-1)N + j) = f_n(j)$$

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where $S_N = 1 + 2^2 + \cdots + (N-1)^2$, and

$$f(\mathbb{N}) = \{f_n(j) :\ n,\ j \in \mathbb{N}\} = \bigcup_{n=1}^{\infty} f_n(\mathbb{N}) = \bigcup_{n=1}^{\infty} A_n = U. \quad \boxtimes$$

Lemma 2 *An infinite subset $K \subset \mathbb{N}$ is countable.*

Proof Enumerate K as follows:

$$n_1 := \min K,\ n_2 := \min K \setminus \{n_1\},\ n_3 := \min K \setminus \{n_1, n_2\},\ \cdots$$

$$n_k := \min K \setminus \{n_1, n_2, \ldots, n_{k-1}\}.$$

The process does not stop as K is infinite and $n_{k+1} > n_k$ for each $k \geq 1$.

The map $k \mapsto n_k$ defines an injection of $\mathbb{N}$ into K. To see that it is onto, fix $\nu \in K$ and $n_k > \nu$. We claim that $\nu = n_J$ for some $J < k$ since, if not $n_k = \min K \setminus \{n_1, n_2, \ldots, n_{k-1}\} = \nu$ contradicting $n_k > \nu$. $\boxtimes$

Lemma 3 *If A is a set and there exists a surjection $f : \mathbb{N} \to A$, then A is at most countable.*

Proof For each $a \in A$, let $k_a := \min\{k \in \mathbb{N} :\ f(k) = a\}$ and let $K := \{k_a :\ a \in A\}$, then $K \subset \mathbb{N}$ and $f : K \to A$ is a bijection. If A is infinite, so is K which by lemma 2 is countable. $\boxtimes$

The theorem follows from lemmas 1 & 3.

Countability of $\mathbb{Q}$.

As shown above,

$$\mathbb{Q} \cong \{\frac{p}{q} :\ p \in \mathbb{Z},\ q \in \mathbb{N},\ \mathtt{gcd}\{p, q\} = 1\}$$

where $\cong$ denotes set correspondence and for $K \subset \mathbb{Z} \setminus \{0\}$, $\mathtt{gcd}\, K$ is the `greatest common divisor`
(מחלק משותף מרבי) of K

$$\mathtt{gcd}\, K := \max\{n \in \mathbb{N} :\ n | k\ \forall\ k \in K\}$$

where for $n \in \mathbb{N},\ k \in \mathbb{Z} \setminus \{0\},\ n | k$ means $k = an$ for some $a \in \mathbb{Z}$.

For each $q \in \mathbb{N}$, $\{\frac{p}{q} :\ p \in \mathbb{Z},\ \mathtt{gcd}\{p, q\} = 1\}$ is at most countable, and so by the theorem,

$$\mathbb{Q} = \bigcup_{q \in \mathbb{N}} \{\frac{p}{q} :\ p \in \mathbb{Z},\ \mathtt{gcd}\{p, q\} = 1\}$$

is at most countable, and being infinite, countable.

Uncountable sets.

A set is called *uncountable* if it is infinite, but not countable. Cantor first proved existence of uncountable sets.

Cantor's Theorem

$2^{\mathbb{N}} := \{\texttt{subsets of } \mathbb{N}\}$ *is uncountable.*

Proof Since $2^{\mathbb{N}} \cong \Omega := \{0,1\}^{\mathbb{N}}$ by $(a_1, a_2, \dots) \simeq \{n \in \mathbb{N} : a_n = 1\})$, it suffices to show that Ω is uncountable.

Suppose otherwise, then $\Omega = \{x_n : n \in \mathbb{N}\}$ and write $x_n = (a_1^{(n)}, a_2^{(n)}, \cdots)$. For each $n \geq 1$, let $b_n := 1 - a_n^{(n)}$ and let $Z := (b_1, b_2, \cdots)$, then $Z \in \Omega$ (since $b_i = 0, 1 \ \forall \ i \geq 1$). However $Z \neq x_n \ \forall \ n \geq 1$ because $b_n \neq a_n^{(n)}$.⊠

FIELDS

A *field* (שדה)is a nonempty set F (say) on which are defined two binary operations :*addition* $(a,b) \mapsto a+b$ and *multiplication* $(a,b) \mapsto ab$ satisfying the

Addition axioms

A1) (commutative law) (חוק החילוף) $a + b = b + a \ \forall \ a, b \in F$
A2) (associative law) (חוק הקיבוץ) $(a + b) + c = a + (b + c) \ \forall \ a, b, c \in F$
A3) (neutral element) (איבר נטראלי) $\exists \ 0 \in F$ such that $a + 0 = a \ \forall \ a \in F$
A4) (inverse element) (איבר נגדי) $\forall \ a \in F$, $\exists \ (-a) \in F$ such that $a+(-a) = 0$

Multiplication axioms

M1) (commutative law) $ab = ba \ \forall \ a, b \in F$
M2) (associative law) $(ab)c = a(bc) \ \forall \ a, b, c \in F$
M3) (neutral element) $\exists \ 1 \in F \setminus \{0\}$ such that $a1 = a \ \forall \ a \in F$;
M4) (multiplicative inverse element) (איבר נגדי כפלי) $\forall \ a \in F$, $a \neq 0$, $\exists \ \frac{1}{a} \in F$ such that $a\frac{1}{a} = 1$.

Distributive law (חוק הפילוג):

$$\text{(D)} \qquad a(b + c) = ab + ac.$$

- $\mathbb{Q}$ is a field.
- $\mathbb{Q}_{10} := \{\frac{p}{10^n} : p \in \mathbb{Z}, \ n \in \mathbb{N}\}$ is not a field because M4 fails (e.g. $r := \frac{3}{10} \in \mathbb{Q}_{10}$ but $\frac{1}{r} = \frac{10}{3} \notin \mathbb{Q}_{10}$).

Square root rational field extension.

For $a \in \mathbb{Q}$, define the *extension field extension* using the "external symbol" $\sqrt{a}$:

$$\mathbb{Q}(\sqrt{a}) := \{x + \sqrt{a}y : x, y \in \mathbb{Q}\}$$

with arithmetic defined by

$$(x + \sqrt{a}y) + (x' + \sqrt{a}y') := (x + x') + \sqrt{a}(y + y'),$$

$$(x+\sqrt{a}y)\cdot(x'+\sqrt{a}y') := (xx'+ayy')+\sqrt{a}(xy'+x'y).$$

Exercise

Show that $(\mathbb{Q}(\sqrt{a}),+,\cdot)$ is a field and that $\mathbb{Q}(\sqrt{a})=\mathbb{Q}$ iff $\exists\ b\in\mathbb{Q},\ b^2=a$.

$\sum$ NOTATION

Finite sums and products.

Let $(F,+,\cdot)$ be a field.

Any finite sequence $(a_1,a_2,\dots,a_n)$ of elements of F has a sum

$$a_1+a_2+\cdots+a_n = (\dots(a_1+a_2)+a_3)\dots)+a_{n-1})+a_n\in F$$

and a product

$$a_1\cdot a_2\cdot\dots\cdot a_n = (\dots(a_1\cdot a_2)\cdot a_3)\dots)\cdot a_{n-1})\cdot a_n\in F.$$

• By associativity, the sum obtained does not depend on the order in which the operations are performed and so the brackets (`showing order of operation`) are removed. The proof of this (not given) is by induction.

Higher order commutativity. Let $\sigma:\{1,2,\dots,n\}\to\{1,2,\dots,n\}$ be a `permutation` (תמורה) (aka *bijection*).

A proof by induction (not given) shows that

$$a_{\sigma(1)}+a_{\sigma(2)}+\cdots+a_{\sigma(n)} = a_1+a_2+\cdots+a_n$$

and

$$a_{\sigma(1)}\cdot a_{\sigma(2)}\cdot\dots\cdot a_{\sigma(n)} = a_1\cdot a_2\cdot\dots\cdot a_n.$$

Sum over a finite set.

Let A be a finite set and let $f:A\to F$. The set A is in correspondence with $\{1,2,\dots,n\}$ (where $n=\#A$) and we can define the sum

$$\sum_{x\in A} f(x) := f(\pi(1))+f(\pi(2))+\cdots+f(\pi(n));$$

and the product

$$\prod_{x\in A} f(x) := f(\pi(1))\cdot f(\pi(2))\cdot\dots\cdot f(\pi(n))$$

where is any bijection $\pi:\{1,2,\dots,n\}\to A$ because the expressions do not depend on π by higher order commutativity.

Higher order distributivity.

$$\sum_{x\in A}(af(x)+bg(x)) = a\sum_{x\in A} f(x) + b\sum_{x\in A} g(x).$$

This follows from the distributive law. The proof (not given) uses induction.

Change of variables in finite sums and products

Suppose that A, B are finite sets, $f : A \to F$ and $g : B \to A$ is a bijection (the `change of variables`*), then*

$$\sum_{y\in B} f(g(y)) = \sum_{x\in A} f(x), \quad \prod_{y\in B} f(g(y)) = \prod_{x\in A} f(x).$$

Proof If $\pi : \{1,2,\dots,n\} \to B$ is a bijection, then so is $g\circ\pi : \{1,2,\dots,n\} \to A$ (defined by $g\circ\pi(k) := g(\pi(k))$). The result follows by higher order commutativity. ☑

Sum notation. For $J,\ K \in \mathbb{Z},\ J \le K$, write

$$\sum_{k=J}^{K} f(k) := \sum_{t\in[J,K]\cap\mathbb{Z}} f(t) = f(J) + f(J+1) + \cdots + f(K).$$

Here $[J,K]\cap\mathbb{Z} := \{j \in \mathbb{Z} :\ J \le j \le K\}$.

Proposition (Gauss)

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}.$$

Proof By the change of variables formula

$$\sum_{k=1}^{N} k = \sum_{k=1}^{N}(N+1-k)$$

(the change of variables here is $k \mapsto N+1-k$).

Thus

$$\begin{aligned} 2\sum_{k=1}^{N} k &= \sum_{k=1}^{N} k + \sum_{k=1}^{N} k = \sum_{k=1}^{N} k + \sum_{k=1}^{N}(N+1-k) \\ &= \sum_{k=1}^{N}[k + (N+1-k)] \quad \texttt{by higher order distributivity} \\ &= N(N+1). \ ☑ \end{aligned}$$

`Next topics` ORDERED SETS, COMPLETENESS, ORDERED FIELDS, $\mathbb{R}$, LIMITS.

Lecture # 3

3[3]

Linearly ordered sets

- A "`line`" involves a notion of `linear order` which we now define.

A *(linearly) ordered set* (קבוצה סדורה) is a pair $(F,<)$ where F is a set and $<$ is a *relation* (יחס) (i.e. a subset $\mathcal{R}\subset F\times F$ defining $x<y \iff (x,y)\in\mathcal{R}$) satisfying

(i) $\forall\ x,\ y\in F$, precisely one of the following holds:

- $x<y$ or $y<x$ or $x=y$;

(ii) $x<y,\ y<z\ \Rightarrow\ x<z$.

We write $u>v$ in case $v<u$ and $a\le b$ (or $b\ge a$) in case either $a<b$ or $a=b$.

- Evidently if $(F,<)$ is an ordered set and $F'\subset F$, then $(F',<)$ is also an ordered set.
- $\mathbb{Q}$ is an ordered set under the appropriate definition of order:

If $x,y\in\mathbb{Q}$, then $x<y$ if $x+\frac{p}{q}=y$ for some $p,\ q\in\mathbb{N}$.

Intervals.

Let $(F,<)$ be an ordered set. An *interval* (קטע) in F is a set of one of the following forms

- $[a,b]:=\{x\in F:\ a\le x\le b\}$;
- $(a,b):=\{x\in F:\ a<x<b\}$;
- $[a,b):=\{x\in F:\ a\le x<b\}$;
- $(a,b]:=\{x\in F:\ a<x\le b\}$.

Here, $a,\ b\in F$ are aka the *endpoints* (קצוות).

Ordered fields

An *ordered field* (שדה סדור) is a quadruple $(F,<,+,\cdot)$ where $(F,<)$ is an ordered set and $(F,+,\cdot)$ is a field so that

$$a+b,\ a\cdot b>0\ \forall\ a,b>0.$$

The collection of *positive elements* of the ordered field F is

$$F_+:=\{x\in F:\ x>0\}.$$

- $\mathbb{Q}$ is an ordered field with $\mathbb{Q}_+:=\{\frac{p}{q}:\ p,\ q\in\mathbb{N}\}$.

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Proposition *Let $(F,<,+,\cdot)$ be an ordered field, then*

(1) $a,b\in F,\ a>b \iff a-b\in F_+$;

(2) $\forall\ a\in F$, `precisely one of the following holds:`
$a\in F_+,\ (-a)\in F_+,\ a=0$;

(3) $a,b,c\in F,\ a<b \Rightarrow a+c<b+c\ \ \&\ \begin{cases} ac<bc & c>0; \\ ac>bc & c<0; \\ ac=bc=0 & c=0. \end{cases}$

Proof Exercise.

Absolute value. Let $(F,<,+,\cdot)$ be an ordered field. Define the *absolute value* of $x\in F$ by

$$|x| = \begin{cases} x & x\ge 0, \\ -x & x<0. \end{cases}$$

Proposition (`triangle inequality`)

$$|a+b|\le |a|+|b|\ \forall\ a,b\in F$$

with equality iff either $a,b\ge 0$ or $a,b\le 0$.

Proof Exercise.

Orderable field.

A field $(F,+,\cdot)$ is called *orderable* (שדה ניתן לסדור) if there is a relation $<$ on F so that $(F,<,+,\cdot)$ is an ordered field.

Exercise

Let $(F,+,\cdot)$ be a field. Let $P\subset F$ satisfy

(0) $0\notin P,\quad 1\in P$

(1) $a+b,\ a\cdot b\in P\ \forall\ a,b\in P$

(2) $\forall\ a\in F$, `precisely one of the following holds:` $a\in P,\ (-a)\in P,\ a=0$.

Define the relation $<$ on F by $x>y \iff x-y\in P$.

(a) Show that $(F,<,+,\cdot)$ is an ordered field and that $F_+=P$.
A set $P\subset F$ as in (a) is called a *pre-ordering* for F.

(b) Show that the field F is orderable iff it has a pre-ordering.

Exercise

Show that

(i) the only pre-ordering for $\mathbb{Q}$ is $\{\frac{p}{q}:\ p,\ q\in\mathbb{N}\}$.

(ii) $\mathbb{Q}(\sqrt{-1})$ is not orderable.

(iii)☆ Show that $\mathbb{Q}(\sqrt{2})$ is orderable.

"Holes" in $\mathbb{Q}$

This section is illustrative, leading up to precise formulations in the sequel.

We'll see that there is a "hole in $\mathbb{Q}$ at $\sqrt{2}$" and how this hole can be "filled".

Pythagoras' irrationality theorem

$\nexists\ x \in \mathbb{Q}$ *with* $x^2 = 2$.

Proof

Suppose that $x \in \mathbb{Q},\ x^2 = 2$, then without loss of generality $x > 0$ (as $(-x)^2 = 2$ also) and

$$A := \{q \in \mathbb{N} :\ qx \in \mathbb{N}\} \neq \varnothing.$$

We show that this is impossible contradicting the assumption that $x \in \mathbb{Q},\ x^2 = 2$ and proving the theorem.

- By well ordering, the set A has a minimal element Q. Write $Qx =: N \in \mathbb{N}$, then $2Q^2 = Q^2x^2 = N^2$.
- Thus N^2 is even, and (!) so is N.
- Let $M := \frac{N}{2} \in \mathbb{N}$, then $Q^2 = 2M^2,\ Mx = Q \in \mathbb{N}$ and so $M \in A$ but $M < Q$ contradicting minimality of Q in A. ☐

Note also that $\nexists\ x \in \mathbb{Q}$ with $x^2 = -1$ but nobody thinks there's a "hole in $\mathbb{Q}$ at $\sqrt{-1}$".

To illustrate that $\mathbb{Q}$ has a "hole at $\sqrt{2}$" let

$$A_< := \{x \in \mathbb{Q}_+ :\ x^2 < 2\}\ \text{``} = (0, \sqrt{2})''$$

and

$$A_> := \{x \in \mathbb{Q}_+ :\ x^2 < 2\}\ \text{``} = (\sqrt{2}, \infty)'',$$

then

- $x \in A_<,\ y \in A_> \implies x < y$

  and
- $\mathbb{Q}_+ = A_< \cup A_>$ by Pythagoras' irrationality theorem.

Dedkind's proposition (filling the hole)

$$A_< \cdot A_< := \{xy :\ x,\ y \in A_<\} = (0, 2)\ \&\ A_> \cdot A_> = (2, \infty).$$

For the proof of this, you need to know

- `Archimedean property of` $\mathbb{Q}$: For $x,\ y \in \mathbb{Q}_+,\ \exists\ N \in \mathbb{N},\ Nx > y$.

• `Bernoulli's inequality`: For $x \in \mathbb{Q}_+,\ (1+x)^n > 1+nx\ \forall\ n \geq 1$. This is proved by induction.

Proof that $A_< \cdot A_< = (0,2)$

If $x, y \in A_<,\ x \leq y$, then $xy \leq y^2 < 2$, thus $A_< \cdot A_< \subseteq (0,2)$.

The reverse inequality relies on

(a) If $y \in A_< \cdot A_<$, then $(0,y) \subset A_< \cdot A_<$; and

(b) For each $z \in \mathbb{Q}_+,\ z < 2,\ \exists\ x \in A_<,\ x^2 > z$.

`Proof of (b)` Suppose that (b) is wrong, then $\exists\ \Delta \in \mathbb{Q}_+ \cap (0, \frac{1}{2})$ so that

$$(\text{🕮}) \qquad x \in A_< \implies x^2 < 2 - 4\Delta.$$

To obtain a contradiction from this fix $z_0 \in A_<$, then by (🕮), $z_0^2 < 2 - 4\Delta$, whence

$$z_0^2 < 2(1-2\Delta) < 2(1-\Delta)^2$$

and $z_1^2 < 2$ where $z_1 := \frac{z_0}{1-\Delta} \in A_<$.

Thus, again, by (🕮), $z_1^2 < 2 - 4\Delta$ and by the same argument, $z_2 := \frac{z_1}{1-\Delta} = \frac{z_0}{(1-\Delta)^2} \in A_<$.

Induction shows that $z_n := \frac{z_0}{(1-\Delta)^n} \in A_<\ \forall\ n \geq 1$.

This is impossible since

$$\begin{aligned} z_n &= \tfrac{z_0}{(1-\Delta)^n} \\ &= z_0(1 + \tfrac{\Delta}{1-\Delta})^n \\ &> z_0(1 + \tfrac{n\Delta}{1-\Delta}) \\ &> 2\ \forall\ n \text{ large enough, specifically: } n > \tfrac{(2-z_0)(1-\Delta)}{z_0\Delta}. \quad ☑ \end{aligned}$$

Exercises: `Generalizations of Pythagoras' irrationality theorem`

(i) For $p \geq 2$ prime, $\nexists\ x \in \mathbb{Q}$ with $x^2 = p$.

(ii)If $n = pq$ with $p, q \geq 2$ prime, then $\nexists\ x \in \mathbb{Q}$ with $x^2 = n$.

(iii)* If $x \in \mathbb{Q},\ x > 0$ and $x^2 \in \mathbb{N}$, then $x \in \mathbb{N}$.

Bounds for sets in $(F, <)$

Upper bounds. Let $(F, <)$ be an ordered set. A set $A \subset F$ is called *bounded from above* (חסום מלעיל) if $\exists M \in F$ such that $x \leq M\ \forall\ x \in A$. In this case, M is called an *upper bound* (חסם מלעיל) for A.

A set $A \subset F$ is called *bounded from below* (חסום מלרע) if $\exists M \in F$ such that $x \geq M\ \forall\ x \in A$. In this case, M is called a *lower bound* (חסם מלרע) for A.

A set $A \subset F$ is called *bounded* if it is bounded from above and below.

- Let $A \subset F$ be bounded above.

We seek the `least` upper bound (if it exists), (which is "`best`" since if M is an upper bound for A, so is every $M' \geq M$).

- Let $A \subset F$ be bounded above. Call $s \in F$ a *least upper bound* (LUB) (חסם מלעיל מזערי = חסם עליון) for A if

(i) s is an upper bound for A,

and

(ii) $\forall\ t < s\ \exists\ x \in A$ with $x > t$; `(i.e. there is no smaller upper bound for` A`).`

- We denote the LUB of A (if it exists) by $\texttt{LUB}\,A$

`(aka` $\sup A$ `where sup is short for` *supremum* `which is a word in Latin).`

- For example if the set A has a *maximal element* (איבר הכי גדול), (i.e. $\exists\ \max A \in A$ such that $x \leq \max A\ \forall\ x \in A$) then this is a least upper bound for A.

- There are also bounded sets $A \subset F$ with $\texttt{LUB}\,A \in F \smallsetminus A$;

e.g. (!) in $\mathbb{Q}$, $\texttt{LUB}\ (0,1) = 1 \notin (0,1)$.

- There can exist at most one LUB of a set A which is bounded from above.

Proof Suppose that A is bounded from above and that a, b are both LUB's of A. If $a > b$, then property (ii) applied to a says that b is not an upper bound for A and therefore not a LUB.☑

Lower bounds. Let $A \subset F$ be bounded below. An element $s \in F$ is called a *greatest lower bound* (GLB) (חסם תחתון או חסום מלרע מרבי) A if

(i) s is a lower bound for A,

and

(ii) $\forall\ t > s\ \exists\ x \in A$ with $x < t$; i.e. there is no larger lower bound for A.

As above, there can exist at most one GLB of A and we denote it (if it exists) by $\texttt{GLB}\,A$ `(or` $\inf A$ `where inf is for infimum, another latin word).`

Complete ordered sets and fields

An ordered set $(F, <)$ is called *complete*, (שלם) if $\forall\ A \subset F$ bounded above, $\exists\ \texttt{LUB}\ \ A \in F$.

Proposition

The ordered set $(F,<)$: *is complete iff* $\forall\ A \subset F$ *bounded below,* $\exists$ `GLB` $A \in F$.

Proof of $\Rightarrow$

Let $A \subset F$ be bounded below, and let $B := \{$`lower bounds for` $A\}$, then $a \geq b\ \forall\ a \in A,\ b \in B$ and in particular every $a \in A$ is an upper bound for B (which is thus bounded above).

By completeness, $\exists$ `LUB` $B =: Q \in F$.

We claim that $Q \in B$.

Proof If not then Q is not a lower bound for A and $\exists\ a \in A,\ a < Q$. This a is a smaller upper bound for B contradicting $Q =$ `LUB` B.

It follows from $Q \in B$ that Q is a maximal element in B, i.e. $Q =$ `GLB` A. ☑

Proof of $\Leftarrow$

Let $A \subset F$ be bounded above, and let $B := \{$`upper bounds for` $A\}$, then $a \leq b\ \forall\ a \in A,\ b \in B$ and B is bounded below. By assumption $\exists$ `GLB` $B \in F$ and, as above `GLB` $B =$ `LUB` A. ☑

Evidently, $\mathbb{N}$ is complete as bounded sets are finite.

Definition: Complete ordered field.

The ordered field $(F,<,+,\cdot)$ is *complete* if the ordered set $(F,<)$ is complete.

Some facts:

Uniqueness Any two complete, ordered fields are `isomorphic`. In mathematics, an isomorphism between two sets, each equipped with some structure is a set correspondence transporting the structures, in this case the ordered field structures.

Existence As we'll see, there is a complete ordered field. Because it's unique, you call it the *real number field* and denote it by $\mathbb{R}$.

`What about the rationals?`

Dedekind's incompleteness theorem

$(\mathbb{Q},<)$ *is not complete.*

Proof

We show that $A_< := \{x \in \mathbb{Q}_+ :\ x^2 < 2\}$ is (a) bounded above but (b) does not have a LUB in $\mathbb{Q}$. By Dedekind's proposition,

$$\{x^2 :\ x \in A_<\} = (0,2)$$

so $A_<$ is bounded above and if $s \in \mathbb{Q}_+$ is a $\mathtt{LUB}$ for $A_<$ then $s^2 = 2$ contradicting Pythagoras' irrationality theorem. ☑

IRRATIONAL NUMBERS, D-CUTS AND DECIMALS

Dedekind Cuts (חתכי דדקינד)

A *Dedekind cut* (D-cut) is a nonempty set $A \subsetneqq \mathbb{Q}$ so that

(i) $(-\infty, a) := \{q \in \mathbb{Q} :\ q < a\} \subset A\ \forall\ a \in A$;

(ii) $\nexists$ maximal element in A (i.e. $\nexists\ \mathtt{LUB}\, A \in A$).

Dedekind's construction of the real numbers (המספרים ממשיים).

$$\mathbb{R} := \{\mathtt{D\text{-}cuts}\}.$$

Remarks `aka exercises`

- D-cuts are bounded above.

Proof If $A \in \mathbb{R}$, then $a \le M\ \forall\ a \in A,\ M \in \mathbb{Q} \setminus A$. ☑

- $(-\infty, a)$ is a D-cut $\forall\ a \in \mathbb{Q}$. A D-cut of this form is called *rational.*
- A D-cut A is rational iff $\exists\ \mathtt{LUB}\, A \in \mathbb{Q}$ in which case $A = (-\infty, \mathtt{LUB}\, A)$.

- A D-cut which is not rational is called *irrational.*
- As shown in the proof of that $\mathbb{Q}$ is not complete,

$$A := \{x \in \mathbb{Q} :\ x < 0 \ \mathtt{or}\ x \ge 0\ \&\ x^2 < 2\}$$

is an irrational D-cut.

- If A is a D-cut, then $A = \bigcup_{a \in A}(-\infty, a] = \bigcup_{a \in A}(-\infty, a)$.
- For any $C \subset \mathbb{Q}$, bounded above: $\bigcup_{a \in C}(-\infty, a)$ is a D-cut, but $\bigcup_{a \in C}(-\infty, a]$ is a D-cut iff $\nexists\ \mathtt{LUB}\, C \in C$.

$\mathbb{R}$ is a complete, ordered set

Let $\mathbb{R} := \{\texttt{D-cuts}\}$ and order $\mathbb{R}$ by `inclusion` (i.e. for $A,\ B \in \mathbb{R}$, $A < B$ if $A \subsetneqq B$).

Proposition $(\mathbb{R}, <)$ *is an ordered set.*

Proof We must show that

(i) if $A,\ B$ are D-cuts, then either $A \subsetneqq B$ or $B \subsetneqq A$ or $A = B$;
and

(ii) if $A,\ B,\ C$ are D-cuts and $A \subseteqq B,\ B \subseteqq C$ then $A \subseteqq C$.

Since (ii) is true for sets in general, we only prove (i). To this end, it suffices to show that $A \setminus B \neq \varnothing \Rightarrow\ B \subset A$.

Suppose that $q \in A \setminus B$, then $(-\infty, q) \subset A$ and $B \subset (-\infty, q) \because q \in \mathbb{Q} \setminus B$ is an upper bound for B. Thus $B \subset A$.☑

Proposition: `Density of the rationals.`

Suppose that $A,\ B \in \mathbb{R}$ and that $A < B$, then $\exists\ \ q \in \mathbb{Q}$ so that $A < (-\infty, q) < B$.

Proof

If $A = (-\infty, a)$ & $B = (-\infty, b)$ are both rational then $q := \frac{a+b}{2}$ is as required.

If A is irrational, then any $q \in B \setminus A$ is as required since then $A \subseteq (-\infty, q)$ but $A \neq (-\infty, q)$ by irrationality of A. Moreover $(-\infty, q) \subsetneqq B$ since $q \in B$.

If $A = (-\infty, a)$ is rational and B is irrational, then $\exists\ q \in \{x \in B :\ x > a\}$. It follows that $A \subsetneqq (-\infty, q) \subsetneqq B$. ☑

Theorem

$(\mathbb{R}, <)$ *is a complete ordered set.*

Proof

To check completeness, let $\mathcal{A} \subset \mathbb{R}$ be a collection of D-cuts which is `bounded above`, i.e. $\exists$ a D-cut $M \in \mathbb{R}$ so that $A \subset M\ \forall\ A \in \mathcal{A}$, then

- $L := \bigcup_{A \in \mathcal{A}} A$ is a cut.

Proof $L \neq \mathbb{Q}\ \because\ L \subset M \neq \mathbb{Q}$. If $a \in L$ then $\exists\ a \in A \in \mathcal{A}$ whence $(-\infty, a) \subset A \subset L$. If $a \in L$ is maximal and $a \in A \in \mathcal{A}$ then $a \in A$ is maximal contradicting $A \in \mathbb{R}$.☑

- L is an upper bound for $\mathcal{A}\ \because\ A \subset L\ \forall\ A \in \mathcal{A}$;
- if $M \in \mathbb{R}$ is also an upper bound for $\mathcal{A}$, then $M \supseteq A\ \forall\ A \in \mathcal{A}\ \Rightarrow M \supseteq L$.

Thus $L = \texttt{LUB}\ \mathcal{A}$ and $\mathbb{R}$ is complete.☑

Arithmetic of $\mathbb{R}$ & Dedekind's theorem

Algebra of positive parts.

A D-cut is *positive* if $A > 0^*$ $\big($`i.e.` $A \supsetneq (-\infty, 0)$ `or` $0 \in A\big)$.

The *positive part* of the positive D-cut $A \in \mathbb{R}_+$ is

$$A_+ := A \cap \mathbb{Q}_+.$$

Note that $A_+ = \bigcup_{x \in A_+} (0, x)$, thus for $A,\ B \in \mathbb{R}_+$.

Addition and multiplication of psitive parts are given by regular set addition and multiplication:

$$A_+ + B_+ := \{x + y :\ x \in A_+,\ y \in B_+\} = \bigcup_{x \in A_+,\ y \in B_+} (0, x + y);$$

$$A_+ B_+ := \{xy :\ x \in A_+,\ y \in B_+\} = \bigcup_{x \in A_+,\ y \in B_+} (0, xy).$$

Distributive Law for positive parts *For $A,\ B,\ C \in \mathbb{R}_+$:*

$$A_+(B_+ + C_+) = A_+B_+ + A_+C_+.$$

Proof

On the one hand

$$\begin{aligned} A_+(B_+ + C_+) &= \bigcup_{a \in A_+ +,\ b \in B_+,\ c \in C_+} (0, a(b + c)) \\ &= \bigcup_{a \in A_+ +,\ b \in B_+,\ c \in C_+} (0, ab + ac) \\ &\subseteq \bigcup_{a,\ a' \in A_+ +,\ b \in B_+,\ c \in C_+} (0, ab + a'c) \\ &= A_+B_+ + A_+C_+. \end{aligned}$$

On the other hand,

$$\begin{aligned} A_+B_+ + A_+C_+ &= \bigcup_{a,\ a' \in A_+ +,\ b \in B_+,\ c \in C_+} (0, ab + a'c) \\ &\subseteq \bigcup_{a,\ a' \in A_+ +,\ b \in B_+,\ c \in C_+} (0, a \vee a'(b + c)) \quad \text{where } a \vee a' := \max\{a, a'\} \\ &= \bigcup_{a \in A_+ +,\ b \in B_+,\ c \in C_+} (0, a(b + c)) \\ &= A_+(B_+ + C_+). \quad \boxtimes \end{aligned}$$

Addition.

Let $A,\ B \in \mathbb{R}$ be D-cuts. Define

$$A + B := \{a + b :\ a \in A,\ b \in\ B\},$$

then,

- $A + B$ is a cut.
- For $A,\ B \in \mathbb{R}_+$ we have

$$A + B = (-\infty, 0] \cup A_+ + B_+.$$

- a neutral element for addition is given by $0^* := (-\infty, 0)$.

Negative cuts.

The *negative* of the D-cut A is

$$\ominus A := \{-b :\ \exists\ r > 0 \text{ so that } b - r \notin A\} = \bigcup_{a \notin A} (-\infty, -a) \in \mathbb{R}.$$

This notation avoids confusion with the usual $-A := \{-x :\ x \in A\}$ for $A \subset \mathbb{Q}$.

- $A + \ominus A = 0^*$.

Proof

$$A + \ominus A := \bigcup_{x \in A,\ y \notin A} (-\infty, x - y) \subset (-\infty, 0)$$

because $x \in A,\ y \notin A \implies x < y$. On the other hand $\forall\ \epsilon > 0\ \exists\ x \in A,\ y \notin A$ with $0 < y - x < \epsilon$ and

$$A + \ominus A := \bigcup_{x \in A,\ y \notin A} (-\infty, x - y) \supset \bigcup_{\epsilon > 0} (-\infty, -\epsilon) = (-\infty, 0).\ ☑$$

- For $A, B \in \mathbb{R}$, $\ominus(A + B) = \ominus A + \ominus B$.

Proof For $x,\ y \in \mathbb{Q}$,

$$(-\infty, x)^c + (-\infty, y)^c = [x, \infty) + [y, \infty) = [x + y, \infty) = (-\infty, x + y)^c$$

whence

$$\ominus(-\infty, x) + \ominus(-\infty, y) = \ominus(-\infty, x + y)^c.$$

It follows that

$$A^c + B^c = (A + B)^c$$

and

$$\begin{aligned}\ominus A + \ominus B &= \bigcup_{x \notin A,\ y \notin B} (\ominus(-\infty, x) + \ominus(-\infty, y)) \\ &= \bigcup_{x \notin A,\ y \notin B} (\ominus(-\infty, x + y) \\ &= \bigcup_{z \notin A + B} (\ominus(-\infty, z) \\ &= \ominus(A + B).\ ☑\end{aligned}$$

Positivity and order.

Let $\mathbb{R}_+ := \{\texttt{positive D-cuts}\}$, then

- $\forall\ A \in \mathbb{R}$, either $A = 0^*$, $A \in \mathbb{R}_+$ or $\ominus A \in \mathbb{R}_+$;
- if $A,\ B \in \mathbb{R}_+$, then $A + B \in \mathbb{R}_+$;
- let $A,\ B \in \mathbb{R}$, then $A < B$ iff $B + \ominus A \in \mathbb{R}_+$, in particular (!) $(-\infty, b) + (-(-\infty, a)) = (-\infty, b - a)$.

Absolute value.

The *absolute value* of the D-cut A is $|A| \in \mathbb{R}_+$ defined by

$$|A| = \begin{cases} 0 & A = 0^*, \\ A & A \in \mathbb{R}_+, \\ -A & -A \in \mathbb{R}_+. \end{cases}$$

In particular (!) $|(-\infty, a)| = (-\infty, |a|)$.

Multiplication.

We first define multiplication on $\mathbb{R}_+$.

For $A,\ B \in \mathbb{R}_+$, define *multiplication* by

$$AB := (-\infty, 0] \cup A_+ B_+.$$

and for $A,\ B \in \mathbb{R}$ define multiplication by

$$A\ B := \begin{cases} 0 & A = 0 \texttt{ or } B = 0, \\ |A||B| & A,\ B \in \mathbb{R}_+ \texttt{ or } \ominus A,\ \ominus B \in \mathbb{R}_+ \\ \ominus(|A||B|) & \texttt{else}. \end{cases}$$

In particular,

$$(\ominus 1^*) A = \ominus A.$$

Multiplicative inverse.

Set $1^* := (-\infty, 1)$; and for $A \in \mathbb{R}_+$ set

$$A^{-1} := \bigcup_{a \notin A} (-\infty, a^{-1}).$$

- $AA^{-1} = 1^*$.

Proof that $A_+ A_+^{-1} = (0,1)$ Since $a < b\ \forall\ a \in A,\ b \notin A$,

$$A_+ A_+^{-1} = \bigcup_{a \in A,\ b \notin A} (0, ab^{-1}) \subseteq (0,1)$$

and since $\forall\ 0 < t < 1\ \exists\ a \in A,\ b \notin A$ so that $ab^{-1} > t$,

$$A_+ A_+^{-1} = \bigcup_{a \in A,\ b \notin A} (0, ab^{-1}) \supseteq \bigcup_{0<t<1} (0, t) = (0,1). \quad \boxtimes$$

Evidently, arithmetic on $\mathbb{R}$ is associative and commutative.

Distributive Law for D-cuts *For $A, B, C \in \mathbb{R}$,*
$$A(B+C) = AB + AC.$$

Proof Assume WLOG that $A, B, B+C \in \mathbb{R}_+$. It suffices that
$$(A(B+C))_+ = (AB+AC)_+.$$
If $C \in \mathbb{R}_+$, then by the Distributive Law for positive parts
$$(A(B+C))_+ = A_+(B_+ + C_+) = A_+B_+ + A_+C_+ = (AB+AC)_+$$
If $\ominus C \in \mathbb{R}_+$, then by the Distributive Law for positive parts
$$\begin{aligned}(A(B+C))_+ + (A(\ominus C))_+ &= A_+(B_+ + C_+) + A_+(\ominus C)_+ \\ &= A_+((B+C)_+ + \ominus C_+) = (AB)_+\end{aligned}$$
whence $A(B+C) + A \ominus C = AB$ and
$$A(B+C) = AB + (+\ominus A \ominus C) = AB + AC. \;☑.$$

Thus:

Dedekind's theorem $(\mathbb{R}, <, +, \cdot)$ *is a complete ordered field.*

Dedekind's proposition shows that
$$\sqrt{2} = \{x \in \mathbb{Q} : \; x \le 0 \text{ or } x > 0 \;\&\; x^2 < 2\} \in \mathbb{R}.$$

`FYI`**: Uniqueness.** Any two complete ordered fields are in correspondence by a bijection preserving ordered field structures. See theorem 6 in Birkhoff G., MacLane S. *A survey of modern algebra* (Macmillan, 4ed., 1977).

Decimal representation (הצגה עשרונית) of D-cuts

`Decimal representation` is a map $\pi : \mathbb{R} \to \mathbb{Z} \times \mathfrak{D}^{\mathbb{N}}$ where
$$\mathfrak{D} := \{\texttt{digits}\} = \{0,1,2,3,4,5,6,7,8,9\}.$$

To define $\pi(A) = (N; d_1, d_2, \dots)$ where A is a D-cut,

¶0 define $N := \max A \cap \mathbb{Z}$ whence $N \in A$, $N+1 \notin A$; and then

¶1 define $d_1 := \max\{d \in \mathbb{Z} : \; N + \frac{d}{10} \in A\}$;

- to see that $d_1 \in \mathfrak{D}$: $d_1 \ge 0 \;\because\; N \in A$; and $d_1 \le 9 \;\because\; N+1 \notin A$. Evidently $N + \frac{d_1}{10} \in A$ but $N + \frac{d_1}{10} + \frac{1}{10} \notin A$.

¶2 define $d_2 := \max\{d \in \mathbb{Z} : \; N + \frac{d_1}{10} + \frac{d}{100} \in A\}$;

- to see that $d_2 \in \mathfrak{D}$: $d_2 \ge 0 \because N + \frac{d_1}{10} \in A$; and $d_2 \le 9 \because N + \frac{d_1}{10} + \frac{1}{10} \notin A$.

Continuing, obtain $d_1, d_2, \dots \in \mathfrak{D}$ so that $\forall\ n \geq 1$,

$$d_n := \max\{d \in \mathbb{Z} :\ N + \sum_{k=1}^{n-1} \frac{d_k}{10^k} + \frac{d}{10^n} \in A\}.$$

It follows that

$$N + \sum_{k=1}^{n} \frac{d_k}{10^k} \in A \ \&\ N + \sum_{k=1}^{n-1} \frac{d_k}{10^k} + \frac{d_n + 1}{10^n} \notin A.$$

- The *decimal expansion* of the D-cut A is the sequence

$$\pi(A) = (N, d_1, d_2, \dots) \in \mathbb{Z} \times \mathfrak{D}^{\mathbb{N}}$$

sometimes written as

$$\pi(A) = N + 0.d_1 d_2 \dots.$$

Proposition

The decimal expansion map $\pi : \mathbb{R} \to \mathbb{Z} \times \mathfrak{D}^{\mathbb{N}}$ is an injection.

Proof

Suppose that $\pi(A) = N + 0.d_1 d_2 \dots,\ \ A' \leftrightsquigarrow N' + 0.d'_1 d'_2 \dots$ and $A \neq A'$. We show that $(N, d_1, d_2, \dots) \neq (N', d'_1, d'_2, \dots)$. To see this suppose WLOG that $A \smallsetminus A' \neq \varnothing$, then every element of $A \smallsetminus A' \subset \mathbb{Q} \smallsetminus A'$ is an upper bound for A' whence $A' \subsetneqq A$. It follows that $\exists\ n \geq 1,\ m \in \mathbb{Z}$ such that $\frac{m}{10^n} \in A \smallsetminus A'$, whence

$$N' + \sum_{k=1}^{n} \frac{d'_k}{10^k} < \frac{m}{10^n} \leq N + \sum_{k=1}^{n} \frac{d_k}{10^k}$$

and $(N, d_1, d_2, \dots) \neq (N', d'_1, d'_2, \dots)$. ☑

Examples of decimal expansions.

¶1 $(-\infty, 1) \leftrightsquigarrow 0.\overline{9}$

¶2 $(-\infty, \frac{1}{10}) \leftrightsquigarrow 0.0\overline{9}$;

¶3 $(-\infty, \frac{1}{3}) \leftrightsquigarrow 0.\overline{3}$.

`Next topics` Decimal expansions of cuts , uncountability of $\mathbb{R}$, $\mathbb{R}$-arithmetic, Archimedean property of $\mathbb{R}$, roots powers and logs, limits.

Lecture # 4

[4]

Correspondence of lower D-cuts with decimal expansions

Proposition *The decimal expansion map*

$$\pi : \mathbb{R} \to \mathbb{Z} \times \{a \in \mathfrak{D}^{\mathbb{N}} :\ \#\{k \ge 1 :\ a_k \ge 1\} = \infty\}$$

is a set correspondence (bijection).

Proof As shown above, $\pi : \mathbb{R} \to \mathbb{Z} \times \mathfrak{D}^{\mathbb{N}}$ is injective. We show that π is onto $\mathbb{Z} \times \{a \in \mathfrak{D}^{\mathbb{N}} :\ \#\{k \ge 1 :\ a_k \ge 1\} = \infty\}$.

To identify $\pi(\mathbb{R}) \subset \mathbb{Z} \times \{a \in \mathfrak{D}^{\mathbb{N}}$, note that if the D-cut A has decimal expansion $\pi(A) = N + 0.d_1 d_2 \dots$, then $\forall\ n \ge 1$

$$M_n(A) := \max\{a \in A :\ 10^n a \in \mathbb{Z}\} = N + \sum_{k=1}^{n} \frac{d_k}{10^k}.$$

Since cuts do not have maximal elements, no $M_n(A)$ is maximal and there are infinitely many $n \ge 1$ with $d_n \ge 1$. Thus $\pi(\mathbb{R}) \subset \mathbb{Z} \times \{a \in \mathfrak{D}^{\mathbb{N}} : \#\{k \ge 1 :\ a_k \ge 1\} = \infty\}$.

Let $N \in \mathbb{Z}$, $(d_1, d_2, \dots) \in \mathfrak{D}^{\mathbb{N}}$ such that $d_k \ge 1$ for infinitely many $k \ge 1$.

Let for $n \ge 1$,

$$\Delta_n := \sum_{k=1}^{n} \frac{d_k}{10^k}, \quad \overline{\Delta}_n := \sum_{k=1}^{n-1} \frac{d_k}{10^k} + \frac{d_n + 1}{10^n}.$$

Let

$$A := \bigcup_{n \ge 1} (-\infty, \Delta_n),$$

then A is a D-cut being a union of D-cuts (as shown in the proof of completeness of $\mathbb{R}$).

Evidently,

(i) $\Delta_n \le \Delta_{n+1}\ \forall\ n \ge 1$;

(ii) $\forall\ n \ge 1\ \exists\ k \ge 1,\ d_{n+k} \ge 1 \implies \Delta_n < \Delta_{n+k}$; &

(iii) $\Delta_k < \overline{\Delta}_n\ \forall\ k,\ n \ge 1$.

We claim that moreover

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¶1 $\mathit{\Delta}_n \in A \ \forall \ n \geq 1$; This follows from (ii).
¶2 $\overline{\mathit{\Delta}_n} \notin A \ \forall \ n \geq 1$. This follows from (iii).
Thus we have that

$$\max\{a \in A : \ 10^n a \in \mathbb{Z}\} = N + \sum_{k=1}^{n} \frac{d_k}{10^k} \ \forall \ n \geq 1$$

whence $\pi(A) = N + 0.d_1 d_2 \dots$. ☑

Exercise: `Ordering of decimal expansions`

Suppose that $A, \ B \in \mathbb{R}$ and that

$$\pi(A) = m + 0.a_1 a_2 \dots, \quad \pi(B) = n + 0.b_1 b_2 \dots$$

where $m, n \in \mathbb{Z}$ and $a_i, b_i = 0, 1, \dots, 9$ satisfy $\#\{i : a_i > 0\} = \#\{i : b_i > 0\} = \infty$; then $A > B$ iff either:
(i) $m > n$ or
(ii) $m = n$ and $\exists \ k \geq 1$ so that $a_i = b_i \ \forall \ 1 \leq i \leq k-1$ and $a_k > b_k$.
`Hint:` $A > B \iff (A \smallsetminus B) \cap \mathbb{Q}_{10} \neq \varnothing$.

Exercise: `The decimal expansion of rational D-cuts`

Let $\widetilde{\mathbb{Q}} := \{(-\infty, q) : \ q \in \mathbb{Q}\} = \{$`rational D-cuts`$\}$. Show that:
(i) $x \in \widetilde{\mathbb{Q}} \iff \pi(x) = m + 0.a_1 \dots a_k \overline{b_1 \dots b_\ell}$ for some $m \in \mathbb{Z}, \ k, \ell \in \mathbb{N}, \ a_1, \dots, a_k, b_1, \dots, b_\ell \in \mathfrak{D}$;
(ii) $x = (-\infty, \frac{p}{q}) \in \widetilde{\mathbb{Q}}$ with $p, q \in \mathbb{N}$ having no common divisor has

- a decimal expansion of form $\pi(x) = m + 0.a_1 \dots a_n \overline{9}$ iff $q = 2^k 5^\ell$ for some $k, \ \ell \geq 0$ in which case `LUB` $x = m + \sum_{k=1}^{n} \frac{a_k}{10^k} + \frac{1}{10^n}$;
- a purely periodic decimal expansion ($\pi(x) = m + 0.\overline{a_1 \dots a_n} \neq m + 0.\overline{9}$) iff q is neither divisible by 2 nor 5.

(iii) The D-cut with decimal expansion

$$0.1010010001 \dots 1 \underbrace{0 \dots 0}_{k\text{-times}} 1 \dots$$

is irrational.

Theorem

A non trivial interval in $\mathbb{R}$ is uncountable.

Proof

Let $J = (a, b) \subset \mathbb{R}$ where $a, b \in \mathbb{R}, \ a < b$. For some $N \in \mathbb{Z}, \ n \geq 1, \ d_1, \dots, d_n \in \mathcal{D}$, we have

$$[\mathit{\Delta}_n, \mathit{\Delta}_n + \tfrac{1}{10^n}] \subset J$$

where

$$\mathit{\Delta}_n := N + \sum_{k=1}^{n} \frac{d_k}{10^k}.$$

Now define $\psi : \{1, 2\}^{\mathbb{N}} \to \mathbb{R}$ by

$$\psi(a_1, a_2, \dots) := \pi^{-1}(N; d_1, \dots, d_n, a_1, a_2, \dots).$$

For each $(a_1, a_2, \dots) \in \{1,2\}^{\mathbb{N}}$,

$$\varDelta_n < \pi^{-1}(N; d_1, \dots, d_n, a_1, a_2, \dots) \le \pi^{-1}(N; d_1, \dots, d_n, \overline{9}) = \varDelta_n + \frac{1}{10^n}$$

whence

$$\psi(a_1, a_2, \dots) = \pi^{-1}(N; d_1, \dots, d_n, a_1, a_2, \dots) \in (\varDelta_n, \varDelta_n + \tfrac{1}{10^n}] \subset J.$$

Thus $\psi : \{1,2\}^{\mathbb{N}} \to J$. The set $\{1,2\}^{\mathbb{N}}$ is uncountable by Cantor's theorem and ψ is injective. So J is uncountable. ☑

Archimedean Property of $\mathbb{R}$

The ordered field $(F, <, +, \cdot)$ is called *archimedean* if $\forall\ x > 0,\ \exists\ n \in \mathbb{N}$ such that $nx := \underbrace{x + \cdots + x}_{n \text{ times}} > 1$ (i.e. there are no "*infinitesimals*").

Proposition. $\mathbb{Q}$ is archimedean.
Proof `DIY`.

Proposition *A complete, ordered field* $(F, <, +, \cdot)$ *is archimedean.*

Proof Suppose that $x \in F_+$. We must show that $\exists\ n \in \mathbb{N}$ such that $nx > 1$.

Suppose otherwise, then $nx \le 1\ \forall\ n \in \mathbb{N}$ and the set $A := \{nx : n \in \mathbb{N}\}$ is bounded above (by 1).

By completeness $\exists\ z \in F$ a LUB for A. This means that $nx \le z\ \forall\ n \in \mathbb{N}$ but, since $z - x < z$ and is not an upper bound for A, $\exists\ n_0 \in \mathbb{N}$ such that $n_0 x > z - x$. But then $(n_0 + 1)x \in A$ and $(n_0 + 1)x > z$ contradicting that z is an upper bound for A. ⊠

`BTW` $\exists$ non-Archimedean ordered fields.

`Next topics` real powers of positive numbers, logs, absolute value and distance in $\mathbb{C}$, limits.

Lecture # 5

5

General existence of real roots

We show that positive real numbers have roots of all orders. To do this we'll need

Bernoulli's inequality

$$(1+a)^n \geq 1+na \ \forall \ a > -1, \ n \in \mathbb{N}.$$

Proof Induction. $\square$

Theorem

$$\forall \ a \in \mathbb{R}_+ \ n \in \mathbb{N}, \ \exists \ ! \ s \in \mathbb{R}_+, \ s^n = a.$$

`Notation` For $s, \ a > 0$ write $s = a^{\frac{1}{n}}$ if $s^n = a$.

Proof

- Unicity follows from the ordered field properties of $\mathbb{R}$: if $0 < x < y$ then $x^n < y^n \ \forall \ n \in \mathbb{N}$.
- To prove existence, let $A := \{x \in \mathbb{R}_+ : \ x^n < a\}$. We prove that $A \neq \varnothing$ is bounded above and that $s := \mathtt{LUB}\ A$ does the job.

¶1 $A \neq \varnothing$.

We'll show that $y := \frac{1}{1+\frac{1}{a}} \in A$. Indeed, $y > 0$ and $\frac{1}{y} = 1 + \frac{1}{a} > 1$ whence $\frac{1}{y^n} = (\frac{1}{y})^n \geq \frac{1}{y} > \frac{1}{a}$, $y^n < a$ and $y \in A$.

¶2 $A \neq \mathbb{R}_+$ and any $M \in \mathbb{R}_+ \setminus A$ is an upper bound for A.

Proof By Bernoulli's inequality, $(1+a)^n \geq 1+na > a$ and so $1+a \notin A$.

Suppose $y \in A$, then $y^n < a \leq M^n$, whence $y < M$. In particular, A is bounded above and $\exists \ s := \mathtt{LUB}\ A \in \mathbb{R}_+$.

¶3 $s^n \geq a$.

Suppose otherwise, that $s^n < a$ and choose $\epsilon \in (0,1)$ so that $\epsilon < \frac{a-s^n}{na}$. It follows that $s^n < a(1-n\epsilon)$.

By Bernoulli's inequality, $(1-n\epsilon) < (1-\epsilon)^n$, so $s^n < a(1-\epsilon)^n$ but then $(\frac{s}{1-\epsilon})^n < a, \ s' := \frac{s}{1-\epsilon} \in A$ and s is not an upper bound for A $(\because s' > s)$. $\boxtimes$.

¶4 $s^n \leq a$.

Suppose otherwise, that $s^n > a$ and choose $\delta \in (0,1)$ so that $\delta < \frac{s^n - a}{ns^n}$. It follows that $s^n(1-n\delta) > a$ whence (using Bernoulli's inequality

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again), $a < s^n(1-n\delta) < (s(1-\delta))^n$ and $s(1-\delta) \notin A$. By ¶2, $s(1-\delta)$ is an upper bound for A, contradicting $s = \mathtt{LUB}\ A$.⊠.

□

The complex numbers (המספרים המרוכבים)

Define the *complex numbers* by

$$\mathbb{C} = \mathbb{R}(\sqrt{-1}) := \{x + iy : \ x, y \in \mathbb{R}\},$$

where $i = \sqrt{-1}$, with addition and multiplication to satisfy the normal laws of arithmetic (as with $\mathbb{Q}(\sqrt{-1})$).

The *complex conjugate* of $z = x + iy \in \mathbb{C}$ is

$$\overline{z} := x - iy \in \mathbb{C}.$$

Absolute value in $\mathbb{C}$. For $z := x + iy \in \mathbb{C}$ set $|z|_{\mathbb{C}} := \sqrt{x^2 + y^2}$. For $x \in \mathbb{R}$, we have that

$$|x|_{\mathbb{C}} = |ix|_{\mathbb{C}} = |x|$$

where $|\cdot|$ is absolute value in $\mathbb{R}$ and so (without danger of confusion) we can write $|z|_{\mathbb{C}} = |z|$.

Proposition

(i) $$|z|^2 = z\overline{z};$$

(ii) $$|zw| = |z||w|;$$

(iii) $$|z + w| \le |z| + |w|;$$

(iv) $$||z| - |w|| \le |z + w|.$$

Proof of (i) For $z := x + iy$,

$$z\overline{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2 = |z|^2. \quad ☑$$

Proof of (ii) For $z := x + iy$ & $w = a + ib$,

$$zw = (x + iy)(a + ib) = ax - by + i(ay + bx)$$

whence

$$\begin{aligned}
|zw|^2 &= (ax - by)^2 + (ay + bx)^2 \\
&= a^2x^2 + b^2y^2 - 2abxy + a^2y^2 + b^2x^2 + 2aybx \\
&= a^2x^2 + b^2y^2 + a^2y^2 + b^2x^2 \\
&= a^2(x^2 + y^2) + b^2(y^2 + x^2) \\
&= |z|^2|w|^2. \quad ☑
\end{aligned}$$

Proof of (iii) in case $z + w = 1$. Here, $x + a = 1$ & $y + b = 0$, whence
$$|z| + |w| \geq |x| + |a| \geq |x + a| = 1 = |z + w|. \quad \boxtimes.$$

Proof of (iii) in in case $z + w = a \neq 0$.
$$\begin{aligned}|z + w| &\overset{\text{(ii)}}{=} |a|\left|\tfrac{z}{a} + \tfrac{w}{a}\right| \\ &\leq |a|\left(\left|\tfrac{z}{a}\right| + \left|\tfrac{w}{a}\right|\right) \quad \left(\tfrac{z}{a} + \tfrac{w}{a} = 1\right) \\ &= |a|\left|\tfrac{z}{a}\right| + |a|\left|\tfrac{w}{a}\right| \\ &\overset{\text{(ii)}}{=} |z| + |w|. \quad \boxtimes\end{aligned}$$

Proof of (iv). Exercise.

Exercise

Let $\mathbb{R}^d = \mathbb{R} \underbrace{\times \ldots \times}_{d\text{-times}} \mathbb{R} = \{\underline{x} = (x_1, x_2, \ldots, x_d) : x_1, x_2, \ldots, x_d \in \mathbb{R}\}$ and for $\underline{x},\ \underline{y} \in \mathbb{R}^d$ define
$$\langle \underline{x}, \underline{y} \rangle := \sum_{k=1}^{d} x_k y_k \quad \& \quad \|\underline{x}\| := \sqrt{\langle \underline{x}, \underline{x} \rangle}.$$

(i) Prove the `Cauchy-Schwartz inequality`: $|\langle \underline{x}, \underline{y} \rangle| \leq \|\underline{x}\| \cdot \|\underline{y}\|$.

(ii) Using this (or otherwise) prove the `triangle inequality`: $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$.

When is there equality?

Bounded set in $\mathbb{C}$. A set $A \subset \mathbb{C}$ is *bounded* if $\exists\ M$ so that
$$|a| \leq M \ \forall\ a \in A.$$

Limit of a sequence

Definition of convergence. Suppose $b_n \in \mathbb{C}$ $(n \in \mathbb{N})$. We say that
$$b_n \ \texttt{tends to}\ (\text{שואף ל-})\ B \in \mathbb{C} \text{ as } n \to \infty$$
written
$$b_n \to B \text{ as } n \to \infty; \quad \text{or} \quad b_n \xrightarrow[n\to\infty]{} B$$
if for every $\epsilon > 0$, b_n is *ϵ-close* to B for large enough n. Here the numbers x and y are called *ϵ-close* if $|x - y| < \epsilon$.

In symbols:
$$\forall\ \epsilon > 0,\ \exists\ n_\epsilon \text{ such that } |b_n - B| < \epsilon\ \forall\ n \geq n_\epsilon.$$

- The number $B \in \mathbb{C}$ is called *the limit of the sequence* (הגבול של הסדרה) $(b_1, b_2, \ldots)$ and is denoted
$$B = \lim_{n\to\infty} b_n.$$

• In the same situation, the sequence is also said to *converge* (מתכנס) (to its limit) and a sequence is a *convergent sequence* (סדרה מתכנסת) if it converges to some limit.

• Suppose $a_n + ib_n,\ a + ib \in \mathbb{C}$, then

$$a_n + ib_n \xrightarrow[n\to\infty]{} a + ib \iff a_n \xrightarrow[n\to\infty]{} a \ \&\ b_n \xrightarrow[n\to\infty]{} b.$$

Proof of $\Leftarrow$ Use $|(a_n + ib_n) - (a + ib)| \le |a_n - a| + |b_n - b|$.

Proof of $\Rightarrow$ Use $|a_n - a|,\ |b_n - b| \le |(a_n + ib_n) - (a + ib)|$.

• A convergent sequence has precisely one limit.
Proof If $a_n \xrightarrow[n\to\infty]{} L$ and $a_n \xrightarrow[n\to\infty]{} L'$, then $\forall\ \epsilon > 0$ and n large, a_n is ϵ-close both to L and to L', whence (!) L and L' are 2ϵ-close. Thus the numbers L and L' are 2ϵ-close $\forall\ \epsilon > 0$, whence $L = L'$. ☑

Exercise

Write the above proof in symbols.

• A sequence which is not convergent is called a *divergent sequence* (סדרה מתבדרת).

Examples of limits.

• $\frac{1}{n} \xrightarrow[n\to\infty]{} 0$.
Proof Let $\epsilon > 0$. By the Archimedean property of $\mathbb{R}$, $\exists\ N = N_\epsilon \in \mathbb{N}$ so that $N\epsilon > 1$. It follows that $n\epsilon > N\epsilon > 1\ \forall\ n > N$ whence

$$0 < \frac{1}{n} < \epsilon \quad \forall\ n > N. \ ☑$$

• For $b > 1$, $b^{\frac{1}{n}} \xrightarrow[n\to\infty]{} 1$.
Proof

Proof Evidently, $b^{\frac{1}{n}} > 1\ \forall\ n \ge 1$ and it suffices to show that $\forall\ \epsilon > 0$, $b^{\frac{1}{n}} < 1 + \epsilon$ for sufficiently large n. To see this, fix $\epsilon > 0$ & let $n > \frac{b-1}{\epsilon}$, then by Bernoulli's inequality,

$$b - 1 > n(b^{\frac{1}{n}} - 1) > \tfrac{b-1}{\epsilon}(b^{\frac{1}{n}} - 1),$$

whence $b^{\frac{1}{n}} < 1 + \epsilon$ ☑

• $\frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$.

• for $|a| < 1$, $a^n \xrightarrow[n\to\infty]{} 0$.

Monotone sequences.

A sequence $(a_1, a_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ is called

- increasing if $a_{n+1} > a_n$, decreasing if $a_{n+1} > a_n$
- *monotone non-decreasing* (לא יורדת) if $a_n \le a_{n+1}$ and *monotone non-increasing* (לא עולה) if $a_n \ge a_{n+1}$.

A *monotone sequence* is one which is either monotone non-decreasing or monotone non-increasing.

Bounded sequences.

A sequence $(a_1, a_2, \dots) \in \mathbb{C}^{\mathbb{N}}$ is called *bounded* if $A = \{a_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{C}$; i.e. $\exists\ M > 0$ so that $|a_n| \le M\ \forall\ n \in \mathbb{N}$.

Theorem (Convergence of bounded monotone sequences)

Suppose that $(a_1, a_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ *is bounded, monotone, then* $\exists\ \lim_{n\to\infty} a_n \in \mathbb{R}$.

Proof We show that if $(a_1, a_2, \dots)$ is non-decreasing, then $a_n \to \mathtt{LUB}\,\{a_n : n \ge 1\}$ as $n \to \infty$. The other case where $(a_1, a_2, \dots)$ is non-increasing and $a_n \to \mathtt{GLB}\,\{a_n : n \ge 1\}$ as $n \to \infty$ is similar.

Proof of the theorem when $(a_1, a_2, \dots)$ is non-decreasing

Let $L := \mathtt{LUB}\,A$. Since L is an upper bound for A, we have

$$a_n \ \le\ L\ \forall\ n \ge 1.$$

Since L is the least upper bound for A, we have that $\forall\ \epsilon > 0,\ L - \epsilon$ is not an upper bound for A and so

$$\exists\ n_\epsilon \text{ such that } a_{n_\epsilon} > L - \epsilon.$$

Since $a_n \le a_{n+1}$, we have that $\forall\ n \ge n_\epsilon$,

$$L - \epsilon\ <\ a_{n_\epsilon}\ \le\ a_n\ \le\ L.\ ☑$$

Corollary *For each $x = 0.a_1a_2\cdots \in \mathbb{R}$, we have $x_n \le x_{n+1} \xrightarrow[n\to\infty]{} x$ where $x_n = 0.a_1a_2\cdots a_n\overline{0}$.*

Proof

As above, $x_n \xrightarrow[n\to\infty]{} \mathtt{LUB}\,\{x_n : n \in \mathbb{N}\} = x$. □

Proposition

A convergent sequence is bounded.

Proof

Suppose that $a_n \to L$, then $\exists\ N_1 \ge 1$ so that $|a_n - L| < 1\ \forall\ n \ge N_1$, whence

$$|a_n| \le \begin{cases} \max\{|a_k|:\ 1 \le k \le N_1\} & 1 \le n \le N_1, \\ |L| + 1 & n > N_1 \end{cases}$$

$$\le M$$

where

$$M := \max\{|a_1|, |a_2|, \dots, |a_{N_1}|, |L| + 1\}.$$

$\square$

Note that the converse is false: we'll see that the sequence $a_n := (-1)^n$ is bounded, but not convergent.

Example of a divergent, bounded sequence

If $a_n := (-1)^n$, then $a_n \nrightarrow$.

Proof Otherwise, for some L, $a_n \underset{n\to\infty}{\longrightarrow} L$, and $a_n - a_{n+1} \underset{n\to\infty}{\longrightarrow} 0$. But $|a_n - a_{n+1}| = 2$. $\boxtimes$

Next topics

CONDITIONS FOR CONVERGENCE, DIVERGENCE TO ∞, ARITHMETIC OF LIMITS, LIPSCHITZ FUNCTIONS & CONTINUOUS FUNCTIONS

Lecture # 6
6

Example of a bounded, divergent sequence a_n w. $a_n - a_{n+1} \to 0$

Construct $a : \mathbb{N} \to \mathbb{Q} \cap (0,1)$ by

$$(a_1, a_2, \dots) = (\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{4}{5}, \dots) = (B^{(2)}, B^{(3)}, \dots)$$

where

$$B_k^{(2m)} = \frac{k}{2m} \quad (1 \le k \le 2m-1) \;\&\; B_j^{(2m+1)} = 1 - \frac{j}{2m+1} \quad (1 \le j \le 2m).$$

¶1 $a_n - a_{n+1} \to 0$.

Proof For $N \ge 1$ & $1 \le j \le N-2$,

$$|B_{j+1}^{(N)} - B_j^{(N)}| = \frac{1}{N}$$

and

$$|B_1^{(N+1)} - B_{N-1}^{(N)}| = \frac{1}{N(N+1)} < \frac{1}{N}.$$

For each $n \ge 1$, $\exists\ N_n \ge 1$ & $1 \le j_n \le N_n - 1$ so that $a_n = B_{j_n}^{(N_n)}$. Indeed,

$$\frac{(N_n-2)(N_n-1)}{2} \textstyle\sum_{j=2}^{N_n-1}(j-1) < n \le \displaystyle\sum_{j=2}^{N_n}(j-1) = \frac{(N_n-1)N_n}{2} \le N_n^2.$$

Thus (!)

$$|a_{n+1} - a_n| \le \frac{1}{N_n} \le \frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0. \quad ☑$$

¶2 For each $r \in \mathbb{Q} \cap (0,1)$

$$\#\{n \ge 1 :\ a_n = r\} = \infty.$$

Proof Let $r = \frac{p}{q} \in \mathbb{Q} \cap (0,1)$, then for each $n \ge 1$,

$$B_{2qn}^{(2nq)} = r. \quad ☑$$

¶3 $a_n \not\to$.

Proof Suppose otherwise, that $a_n \xrightarrow[n\to\infty]{} L \in [0,1]$ and choose N_0 so that

$$|a_n - L| < \frac{1}{10^6}\ \forall\ n \ge N_0.$$

Choose $r \in \mathbb{Q} \cap (0,1) \setminus (L - \frac{1}{10^6}, L + \frac{1}{10^6})$, then $|r - L| \ge \frac{1}{10^6}$ and by ¶2 $\exists\ n > N_0$ so that $a_n = r$ contradicting $|a_n - L| < \frac{1}{10^6}$. ⊠

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Conditions for convergence.

Comparison Theorem

Suppose that $a_n \geq 0$, $a_n \xrightarrow[n\to\infty]{} 0$ *and that* $L \in \mathbb{R}$, $M > 0$, $b_n \in \mathbb{R}$, $|b_n - L| \leq M a_n \ \forall\ n \geq 1$, *then* $b_n \xrightarrow[n\to\infty]{} L$.

Proof

Let $\epsilon > 0$. We show that $\exists\ N_\epsilon$ so that $|b_n - L| < \epsilon\ \forall\ n \geq N_\epsilon$.

Since $a_n \to 0$ as $n \to \infty$, $\exists\ N_\epsilon$ so that $a_n < \frac{\epsilon}{M}\ \forall\ n \geq N_\epsilon$. It follows that for $n \geq N_\epsilon$,

$$|b_n - L| \leq M a_n < M \cdot \frac{\epsilon}{M} = \epsilon.$$

$\square$

Sandwich principle

Suppose that $a_n \leq x_n \leq b_n\ \forall\ n \geq 1$ *and that* $a_n \to L$, $b_n \to L$ *as* $n \to \infty$, *then* $x_n \xrightarrow[n\to\infty]{} L$.

Proof By assumption $\forall\ \epsilon > 0$, $\exists\ N_\epsilon$ so that $|a_n - L|$, $|b_n - L| < \epsilon\ \forall\ n \geq N_\epsilon$. For such n,

$$L - \epsilon < a_n \leq x_n \leq b_n < L + \epsilon$$

and $x_n \to L$. $\square$

Exercise

Suppose that $a_n \leq b_n\ \forall\ n \geq 1$ and that $a_n \to L$, $b_n \to M$ as $n \to \infty$. Show that $L \leq M$.

Exercises: More examples of convergence

a) $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.
b) $\frac{(n+1)}{n} \to 1$ as $n \to \infty$.
c) $a^n \to 0$ as $n \to \infty\ \forall\ 0 < a < 1$.
e) For $|a| < 1$, $\sum_{k=0}^{n} a^k \xrightarrow[n\to\infty]{} \frac{1}{1-a}$.

Limits in $\mathbb{C}$.

Let $z_n = x_n + iy_n$, $z = x + iy \in \mathbb{C}$. We say

$$z_n \xrightarrow[n\to\infty]{\mathbb{C}} z \ \text{ if } |x - z_n| \to 0$$

It is not hard to check $z_n \xrightarrow[n\to\infty]{\mathbb{C}} z$ iff $x_n \xrightarrow[n\to\infty]{\mathbb{R}} x$ & $y_n \xrightarrow[n\to\infty]{\mathbb{R}} y$.

Neighborhoods. For ϵ-*neighborhood* of a point is the collection $N(x, \epsilon)$ of all those points which are ϵ-close to the point.

In $\mathbb{R}$, for $x \in \mathbb{R}$, this is

$$N(x, \epsilon) = I(x, \epsilon) = \{y \in \mathbb{R} : |y - x| < \epsilon\} = (x - \epsilon, x + \epsilon)$$

an interval around x of length 2ϵ.

In $\mathbb{C}$, for $w = a + ib \in \mathbb{C}$, this is

$$N(w,\epsilon) = D(w,\epsilon) = \{z = x + iy \in \mathbb{C} : \ |z - w| = \sqrt{(x-a)^2 + (y-b)^2} < \epsilon\}$$

a disk in $\mathbb{C} \cong \mathbb{R}^2$ with center w and radius ϵ.

Divergence to infinity

Suppose that $(x_1, x_2, \dots)$ is an increasing sequence in $\mathbb{R}$ and that $\{x_n : n \in \mathbb{N}\}$ is unbounded. Let $M > 0$. Since M is not an upper bound for $\{x_n : n \in \mathbb{N}\}$, $\exists\ N_M \in \mathbb{N}$ so that $x_{N_M} > M$. But since $x_n < x_{n+1}$, then $x_n > M\ \ \forall\ n \geq N_M$.

We say that the sequence $(x_1, x_2, \dots)$ *diverges* (מתבדרת) *to* ∞ (as $n \to \infty$) if for each $M > 0$, $\exists\ N_M$ such that $x_n > M\ \ \forall\ n \geq N_M$ (and write this $x_n \to \infty$).

Proposition *Let* $(x_1, x_2, \dots)$ *be an increasing sequence, then either* $(x_1, x_2, \dots)$ *is convergent, or* $x_n \to \infty$.

Proof The dichotomy is based on the boundedness (or not) of the sequence. Either $x_n \underset{n\to\infty}{\longrightarrow} \mathtt{LUB}\,\{x_n : \ n \in \mathbb{N}\}$, or $x_n \underset{n\to\infty}{\longrightarrow} \infty$. The proof is immediate from the definitions.

Examples.

¶1 As shown above,

$$\sum_{k=0}^{n} \frac{1}{2^k} = 2\left(1 - \frac{1}{2n+1}\right) \underset{n\to\infty}{\longrightarrow} 2.$$

¶2

$$\sum_{k=1}^{n} \frac{1}{k} \underset{n\to\infty}{\longrightarrow} \infty.$$

Proof We contradict boundedness of $a_n := \sum_{k=1}^{n} \frac{1}{k}$ by proving that $a_{2^n} \geq \frac{n}{2}$ $(n \geq 1)$. This is because of the

Claim $a_{2^{n+1}} - a_{2^n} > \frac{1}{2}$.

Proof

$$a_{2^{n+1}} - a_{2^n} = \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n} > 2^n \cdot \frac{1}{2^n+2^n} = \frac{1}{2}$$

whence

$$a_{2^n} = 1 + \sum_{k=0}^{n-1} (a_{2^{k+1}} - a_{2^k}) \geq 1 + \frac{n}{2}. \ \boxtimes$$

Exercises

(i) Show that $\sum_{k=1}^n \frac{1}{k(k+1)} \xrightarrow[n\to\infty]{} 1$.

(ii) Show that $\exists\ L \in \mathbb{R}$ such that $\sum_{k=1}^n \frac{1}{k^2} \xrightarrow[n\to\infty]{} L$.

(iii) Show that $\sum_{k=1}^n \frac{1}{\sqrt{k}} \xrightarrow[n\to\infty]{} \infty$.

`Hint`: Comparison.

Theorem (arithmetic of limits)

Suppose that $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then

$$a_n + b_n \to a + b \text{ as } n \to \infty; \tag{1}$$

$$a_n b_n \to ab \text{ as } n \to \infty; \tag{2}$$

and in case $b \ne 0$:

$$\frac{a_n}{b_n} \to \frac{a}{b} \text{ as } n \to \infty; \tag{3}$$

Proof

1) Using the absolute value proposition: $|a_n - a|,\ |b_n - b| \to 0$ whence $\forall\ \epsilon > 0, \exists\ N_\epsilon$ so that $|a_n - a|,\ |b_n - b| < \frac{\epsilon}{2}$ for $n \ge N_\epsilon$. It follows that

$$|(a_n + b_n) - (a + b)| \le |a_n - a| + |b_n - b| < \epsilon$$

for $n \ge N_\epsilon$ and $a_n + b_n \to a + b$.

2) Suppose that $|a_n|,\ |b_n|,\ |b| \le M\ \forall\ n \ge 1$, then

$$|a_n b_n - ab| \le |a_n b_n - a_n b| + |a_n b - ab| \le M(|a_n - a| + |b_n - b|) \to 0$$

by the comparison theorem.

3) We'll show that $\frac{1}{b_n} \to \frac{1}{b}$ in case $b \ne 0$. To see this note that $\exists\ N_0$ such that $0 < \frac{|b|}{2} < |b_n| < 2|b|\ \forall\ n \ge N_0$ whence for such n,

$$|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b_n - b|}{|b_n b|} \le \frac{2}{|b|^2}|b_n - b| \to 0$$

and $|\frac{1}{b_n} - \frac{1}{b}| \to 0$ by the comparison theorem. ☐

For example

$$\frac{n^2 - n + 1}{3n^2 + 2n + 1} = \frac{1 - \frac{1}{n} + \frac{1}{n^2}}{3 + \frac{2}{n} + \frac{1}{n^2}} \to \frac{1}{3}.$$

Continuous functions

A continuous function $f : (a,b) \to \mathbb{R}$ is one that maps convergent sequences onto convergent sequences.

Let $f : (a,b) \to \mathbb{R}$ and fix $L \in (a,b)$. We call f

- *continuous* **at** L if

$$x_n \in (a,b),\ x_n \xrightarrow[n\to\infty]{} L \implies f(x_n) \xrightarrow[n\to\infty]{} f(L);$$

- and *continuous* **on** (a,b) if it is continuous at each $L \in (a,b)$.

It follows from "`arithmetic of limits`" that $f(z) := z^2$ is continuous on $\mathbb{R}$ and that $f(z) = \frac{1}{z}$ is continuous on $\mathbb{R} \smallsetminus \{0\}$, also sums and products of continuous functions are continuous.

For more examples, we consider:

Lipschitz functions. Let $f : (a,b) \to \mathbb{R}$ and let $(c,d) \subset (a,b)$.

We say that f is *Lipschitz* (`Lip`) **on** (c,d) (`Lip`) if $\exists\ M \in \mathbb{R}_+$ such that

(`Lip`) $$|f(x) - f(y)| \le M|x-y| \ \forall\ x,y \in (a,b)$$

and that f is *Lipschitz* (`Lip`) **at** $z \in (a,b)$ if f is `Lip` on some (c,d) where $z \in (c,d) \subset (a,b)$.

Lipschitz's Theorem *If $f : (a,b) \to \mathbb{R}$ is `Lip` at $L \in (a,b)$, then f is continuous at z.*

Proof

Let $L \in (c,d) \subset (a,b)$ satisfy $|f(x) - f(y)| \le M|x-y| \ \forall\ x,y \in (c,d)$.

Suppose that $x_n \in (a,b)$ & $x_n \to L$, then for large n, $x_n \in (c,d)$ and

$$|f(x_n) - f(L)| \le M|x_n - L| \xrightarrow[n\to\infty]{} 0. \quad ☑$$

Proposition *A polynomial is `Lip` on any bounded interval.*

Proof

We'll show first that for $x,\ y \in \mathbb{C}$ & $n \ge 1$,

(⊕) $$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^k y^{n-k-1}.$$

This is clear for $y = 0$. For $n \ge 1$, $x \in \mathbb{C}$ and $y = 1$, we have

$$(x-1)\sum_{k=0}^{n-1} x^k = \sum_{k=0}^{n-1}(x^{k+1} - x^k) = x^n - 1$$

which is (⊕).

For $x,\ y \in \mathbb{C},\ y \neq 0,$

$$\begin{aligned} x^n - y^n &= y^n\big((\frac{x}{y})^n - 1\big) \\ &= y^n\big((\frac{x}{y}) - 1\big)\sum_{k=0}^{n-1}(\frac{x}{y})^k \\ &= (x-y)\sum_{k=0}^{n-1}x^k y^{n-k}. \quad \boxtimes \quad (\oplus) \end{aligned}$$

It follows that for $x, y \in [-M, M]$,

$$|x^n - y^n| \le |x-y|nM^n.$$

Let $P(x) = \sum_{k=0}^N a_k x^k$, then for $x, y \in [-M, M]$,

$$\begin{aligned} |P(x) - P(y)| &\le \sum_{k=0}^N |a_k||x^k - y^k| \\ &\le |x-y|\sum_{k=0}^N k|a_k|M^k. \quad \boxtimes \end{aligned}$$

Exercises

Show that

(i) if $f.g : (a,b) \to \mathbb{R}$ are continuous, then so are $\alpha f + \beta g : (a,b) \to \mathbb{R}$ (defined by $(\alpha f + \beta g)(x) := \alpha f(x) + \beta g(x)$) and $f \cdot g : (a,b) \to \mathbb{R}$ (defined by $(f \cdot g)(x) := f(x)g(x)$);

(ii) $x \mapsto \frac{1}{x} + \frac{x}{2}$ satisfies (`Lip`) on $(a, \infty)\ \ \forall\ a > 0$;

(iii) $x \mapsto \sqrt{x}$ is locally Lipschitz at each $z \in (0,1)$ but does not satisfy (`Lip`) on $(0,1)$.

Next topics

DIOPHANTINE APPROXIMATION, DIRICHLET & LIOUVILLE THEOREMS, D'ALEMBERT'S RATIO THEOREM, e, `Lip` OF log & exp.

חג שמח!

Lecture # 7 [7]

Number theory

Approximation by rationals.

We saw before that any real number can be approximated by rationals. The question arose as to the "playoff" between the approximation error and the denominator (~ "complexity") of the approximating rational. This kind of approximation is called *Diophantine* after the Alexandrian number theorist `Diophantus`.

Dirichlet's theorem *Let $x \in (0,1) \setminus \mathbb{Q}$, then $\exists\ p_n,\ q_n \in \mathbb{N},\ q_n \uparrow \infty$ so that*

🚂 $$|x - \frac{p_n}{q_n}| < \frac{1}{q_n(q_n+1)}.$$

Proof

We'll prove:

⚽ $$\forall\ Q \in \mathbb{N},\ \exists\ p,q \in \mathbb{N} \text{ such that } 0 \le p \le q \le Q \ \ \& \\ |x - \frac{p}{q}| < \frac{1}{q(Q+1)}.$$

Proof of Dirichlet's theorem given ⚽ Suppose that for $p_n,\ q_n \in \mathbb{N}\ \ (1 \le n \le N,\ q_1 < q_2 < \cdots < q_N$ satisfy 🚂. Let

$$\epsilon_N := \min\{q|x - \tfrac{p}{q}| :\ 1 \le p \le q \le q_N\}$$

and choose $Q > \frac{1}{\epsilon_N}$. Let $1 \le p_{N+1} \le q_{N+1} \le Q$ be so that

$$|x - \frac{p_{N+1}}{q_{N+1}}| < \frac{1}{q_{N+1}(Q+1)}.$$

Since $\frac{1}{Q+1} < \epsilon_N$, it follows that $q_{N+1} > q_N$. Dirchlet's theorem follows by induction. ☑

Proof of ⚽ Let $\lfloor qx \rfloor := \max\{n \in \mathbb{N} :\ n \le qx\}$ and let $\{qx\} := qx - \lfloor qx \rfloor$ and for $1 \le k \le Q+1$, let

$$I_k := [\frac{k-1}{Q+1}, \frac{k}{Q+1}].$$

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If $\{qx\} \in I_1$ for $1 \le q \le Q$ then

$$0 < qx - \lfloor qx \rfloor < \frac{1}{Q+1} \implies 0 < x - \frac{\lfloor qx \rfloor}{q} < \frac{1}{q(Q+1)}.$$

If $\{qx\} \in I_{Q+1}$ for $1 \le q \le Q$ then

$$\begin{aligned}\frac{Q}{Q+1} &< qx - \lfloor qx \rfloor < 1 \\ \implies 0 &< 1 + \lfloor qx \rfloor - qx < \frac{1}{Q+1} \\ \implies 0 &< \frac{\lfloor qx \rfloor + 1}{q} - x < \frac{1}{q(Q+1)}.\end{aligned}$$

If neither of the above holds, then $\{qx\}_{q=1}^{Q} \subset \bigcup_{k=2}^{Q} I_k$ and

$$\exists\ k, r, s,\ 2 \le k \le Q,\ 1 \le r \le s \le q$$

so that $\{rx\},\ \{sx\} \in I_k$. Set $q := s - r$ and $p := \lfloor sx \rfloor - \lfloor rx \rfloor$, then

$$\frac{1}{Q+1} > |\{sx\} - \{rx\}| = |qx - p| \implies |x - \frac{p}{q}| < \frac{1}{q(Q+1)}. \quad ☑⚽$$

Bad approximation by algebraic numbers.

- A number $x \in \mathbb{R}$ is said to have *bad approximation of order* $N \in \mathbb{N}$ if

$$\underset{\frac{p}{q} \in \mathbb{Q}}{\mathtt{GLB}}\ q^N |\alpha - \frac{p}{q}| > 0;$$

equivalently $\exists\ c > 0$ so that

$$|\alpha - \frac{p}{q}| \ge \frac{c}{q^N} \quad \forall\ \frac{p}{q} \in \mathbb{Q}.$$

Algebraic numbers.

- Let $N \in \mathbb{N}$. The number $x \in \mathbb{R}$ is called *algebraic of degree* $\le\ N$ if $\exists\ a_0, a_1, \dots, a_N \in \mathbb{Z}$ with $\sum_{k=0}^{N} a_k x^k = 0$.

The *degree* of the algebraic number $x \in \mathbb{R}$ is N if x is of degree $\le\ N$ but not of degree $\le\ N - 1$.

The collection numbers of degree 1 is $\mathbb{Q}$.

There are countably many algebraic numbers. This is because of the proposition:

¶ If $P(x) = \sum_{k=0}^{N} a_k x^k$ where $(a_0, \dots, a_N) \in \mathbb{R}^N \setminus \{0\}$, then

$$\#\{z \in \mathbb{R} :\ P(z) = 0\} \le N.$$

Liouville's theorem

Suppose that $\alpha \in \mathbb{R}$ is algebraic and has degree $N \ge 2$, then α has bad approximation of order N.

Proof We show that $\exists\ c > 0$ so that

$$|\alpha - \frac{p}{q}| \geq \frac{c}{q^N} \quad \forall\ \frac{p}{q} \in \mathbb{Q}.$$

We can assume WLOG that $|\alpha - \frac{p}{q}| < 1$ since otherwise

$$|\alpha - \frac{p}{q}| \geq 1 \geq \frac{c}{q^N}\ \forall\ c \leq 1.$$

- Let $a_0, a_1, \ldots, a_N \in \mathbb{Z}$ with $P(\alpha) := \sum_{k=0}^{N} a_k \alpha^k = 0$.
- As above, P is `Lip` on $[\alpha - 1, \alpha + 1]$: `i.e.`: $\exists\ K > 0$ so that
  $K := \texttt{LUB}_{x,y\in[\alpha-1,\alpha+1]} |P(y) - P(x)| \leq K|y - x|\ \forall\ x, y \in [\alpha - 1, \alpha + 1]$.

- $P(r) \neq 0 \quad \forall\ r \in \mathbb{Q}$, else if $P(r) = 0$ then

$$\begin{aligned} P(x) = P(x) - P(r) &= \sum_{k=0}^{N} a_k (x^k - r^k) \\ &= (x - r) \sum_{k=0}^{N} a_k \sum_{j=0}^{k-1} x^j r^{k-1-j} \\ &= (x - r) \sum_{k=0}^{N-1} b_k x^k \\ &=: (x - r) R(x) \end{aligned}$$

where $R(x) = \sum_{k=0}^{N-1} b_k x^k = 0$ where $b_0, b_1, \ldots, b_{N-1} \in \mathbb{Q}$. Thus $R(\alpha) = 0$ and the degree of α is $\leq N - 1$ contradicting the assumption that the degree of α is N.

- $|P(\frac{p}{q})| \geq \frac{1}{q^N} \quad \forall\ \frac{p}{q} \in \mathbb{Q}$, since $P(\frac{p}{q}) \in \frac{1}{q^N}\mathbb{Z}$, $P(\frac{p}{q}) \neq 0$.
  To finish the proof,

$$\frac{1}{q^N} \leq |P(\frac{p}{q})| = |P(\frac{p}{q}) - P(\alpha)| \leq |\frac{p}{q} - \alpha| K. \ \boxtimes$$

Liouville numbers. • A *Liouville number* is a number $x \in \mathbb{R}$ which does not have bad approximation of any order, equivalently (!)

$$\forall\ n \geq 1,\ \exists\ \frac{p}{q} \in \mathbb{Q}, \quad |x - \frac{p}{q}| < \frac{1}{q^n}.$$

Exercise

(i) Show that $\alpha := 0.a_1a_2\ldots$ where $a_{n!} := 1 \quad (n \geq 1)$ and $a_k = 0$ if $k \notin \{J! : \ J \geq 1\}$ is a Liouville number.

`Hint`: $0 < \alpha - 0.a_1a_2\dots a_{N!}\overline{0} = \alpha - 0.a_1a_2\dots a_{(N+1)!-1}\overline{0} < \frac{1}{10^{(N+1)!}}$.

(ii) Show that for any $\epsilon_q > \epsilon_{q+1} \downarrow 0$ there is an irrational number $\alpha \in (0,1)$ and a sequence of rationals $\frac{p_n}{q_n}$ so that $q_n \to \infty$ and $|\alpha - \frac{p_n}{q_n}| < \epsilon_{q_n}$.

`Hint`: Try with $q_n = 10^{k_n}$ with $k_n \to \infty$ fast enough.

Convergence of Averages

Theorem (`convergence of arithmetic means`)
Suppose that $x_n \xrightarrow[n\to\infty]{} L$, *then*

$$\frac{1}{n}\sum_{k=1}^{n} x_k \xrightarrow[n\to\infty]{} L.$$

Proof *when* $L = 0$:
Suppose that $x_n \xrightarrow[n\to\infty]{} 0$, then $\exists\ M > 0$ such that $|x_n| \le M\ \forall\ n \ge 1$.
Let $\epsilon > 0$. We show that $\exists\ N_\epsilon$ such that $|\frac{1}{n}\sum_{k=1}^n x_k| < \epsilon\ \forall\ n > N_\epsilon$. To see this,

- $\exists\ N_0$ such that $|x_n| < \frac{\epsilon}{2}\ \forall\ n > N_0$;
- $\exists\ N_\epsilon > N_0$ such that $\frac{MN_0}{N_\epsilon} < \frac{\epsilon}{2}$.

For $n > N_\epsilon$

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^{n} x_k\right| &\le \frac{1}{n}\sum_{k=1}^{N_0}|x_k| + \frac{1}{n}\sum_{k=N_0+1}^{n}|x_k| \\ &< \frac{N_0M}{n} + \frac{n-N_0}{n}\cdot\frac{\epsilon}{2} \\ &< \epsilon. \quad ☑\end{aligned}$$

Theorem (`convergence of geometric means`)
Suppose that $x_n > 0$ *and* $x_n \xrightarrow[n\to\infty]{} L > 0$, *then*

$$\Big(\prod_{k=1}^{n} x_k\Big)^{\frac{1}{n}} \xrightarrow[n\to\infty]{} L.$$

Proof of `convergence of geometric means` **when** $L = 1$:
We'll show that $\forall\ r > 1,\ \exists\ N_r$ such that

$$\frac{1}{r} < \Big(\prod_{k=1}^{n} x_k\Big)^{\frac{1}{n}} < r\ \ \forall\ n > N_r.$$

To see this,

- $\exists\ M > 1$ so that $\frac{1}{M} < x_n < M\ \forall\ n \ge 1$.
- Fix $r > 1$ then $\exists\ N_0 \ge 1$ such that $\frac{1}{\sqrt{r}} < x_n < \sqrt{r}\ \forall\ n > N_0$.
- by the lemma, $\exists\ N_r > N_0$ so that $(M^{N_0})^{\frac{1}{n}} < \sqrt{r}\ \forall\ n > N_r$.

It follows that for $n > N_r$,

$$\begin{aligned}(\prod_{k=1}^{n} x_k)^{\frac{1}{n}} &= (\prod_{k=1}^{N_0} x_k)^{\frac{1}{n}} \cdot (\prod_{k=N_0+1}^{n} x_k)^{\frac{1}{n}} \\ &< (M^{N_0})^{\frac{1}{n}} \cdot ((\sqrt{r})^{n-N_0})^{\frac{1}{n}} \\ &< \sqrt{r} \cdot \sqrt{r} = r\end{aligned}$$

and

$$\begin{aligned}(\prod_{k=1}^{n} x_k)^{\frac{1}{n}} &= (\prod_{k=1}^{N_0} x_k)^{\frac{1}{n}} \cdot (\prod_{k=N_0+1}^{n} x_k)^{\frac{1}{n}} \\ &> (\frac{1}{M^{N_0}})^{\frac{1}{n}} \cdot ((\frac{1}{\sqrt{r}})^{n-N_0})^{\frac{1}{n}} \\ &> \frac{1}{\sqrt{r}} \cdot \frac{1}{\sqrt{r}} = \frac{1}{r}. \qquad ☑\end{aligned}$$

Exercise

Suppose that $x_n \geq 0$ and that $x_n \underset{n\to\infty}{\longrightarrow} 0$. Show that $(\prod_{k=1}^{n} x_k)^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} 0$.

D'Alembert's ratio theorem

Suppose that $a_n > 0 \quad (n \in \mathbb{N})$ and that $\frac{a_{n+1}}{a_n} \underset{n\to\infty}{\longrightarrow} L$, then $a_n^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} L$.

Proof when $L \in (0, \infty)$:

Let $a_0 := 1,\ x_n := \frac{a_n}{a_{n-1}} \quad (n \geq 1)$, then $x_n \underset{n\to\infty}{\longrightarrow} L$ and

$$a_n = \frac{a_1}{a_0}\frac{a_2}{a_1}\cdots\frac{a_n}{a_{n-1}} = x_1 x_2 \cdots x_n.$$

By convergence of geometric means

$$a_n^{\frac{1}{n}} = (x_1 x_2 \cdots x_n)^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} L. \quad ☑$$

Corollary(i) $n^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} 1$; (ii) $\binom{2n}{n}^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} 4$.

Proof $\frac{n+1}{n} \underset{n\to\infty}{\longrightarrow} 1$; $\binom{2n+2}{n+1}/\binom{2n}{n} \underset{n\to\infty}{\longrightarrow} 4$. ☑

Exercise

For $\kappa \geq 2$, find $\lim_{n\to\infty} \binom{\kappa n}{n}^{\frac{1}{n}}$.

Proposition

Assume that $a_n > 0$ and $a_n \underset{n\to\infty}{\longrightarrow} a$.

(i) *If $a > 1$, then $(a_n)^n \underset{n\to\infty}{\longrightarrow} \infty$.*

(ii) *If* $a < 1$, *then* $(a_n)^n \xrightarrow[n\to\infty]{} 0$.

Proof We claim first that

¶1 $\lambda^n \xrightarrow[n\to\infty]{} \infty\ \forall\ \lambda > 1$ and $\lambda^n \xrightarrow[n\to\infty]{} 0\ \forall\ 0 \le \lambda < 1$.

Proof : For $\lambda > 1$, $A_\lambda := \{\lambda^n :\ n \ge 1\}$ is not bounded above and for $0 < \lambda < 1$, $\mathtt{GLB}\ A_\lambda = 0$. ☑ ¶1

Proof of (i) We show that $\forall\ M > 0,\ \exists\ N_M \in \mathbb{N}$ so that $a_n^n > M\ \forall\ n > N_M$.

To this end fix $M > 0$ and $1 < \lambda < a$, then (since $a_n \to a$)

- $\exists\ N_\lambda$ so that $a_n > \lambda\ \forall\ n \ge N_\lambda$.
- By ¶1, $\exists\ N_M \ge N_\lambda$ so that $\lambda^n > M\ \forall\ n \ge N_M$.

  It follows that for $n \ge N_M$, $a_n^n > \lambda^n > M$.

Proof of (ii) If $a < 1$, then $\frac{1}{a} > 1$, $\frac{1}{a_n} \xrightarrow[n\to\infty]{} \frac{1}{a}$ whence by (i)

$$\frac{1}{a_n^n} \xrightarrow[n\to\infty]{} \infty.$$

Now fix $\epsilon > 0$, then $\exists\ N_\epsilon$ so that $\frac{1}{a_n^n} > \frac{1}{\epsilon}\ \forall\ n > N_\epsilon$ whence $a_n^n < \epsilon\ \forall\ n > N_\epsilon$. ☑

Proposition e

$$\exists\ \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n =: e \in (2,3).$$

Proof $A_n := (\frac{n+1}{n})^{n+1}$. $A_n \sim a_n := (\frac{n+1}{n})^n$. $A_n \ge 1$.

$$\begin{aligned}\frac{A_{n-1}}{A_n} &= \left(\frac{n}{n-1}\right)^n\left(\frac{n}{n+1}\right)^{n+1} = \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^n \\ &= \frac{n}{n+1}\left(1+\frac{1}{n^2-1}\right)^n \ge \frac{n}{n+1}\left(1+\frac{n}{n^2-1}\right) \\ &> \frac{n}{n+1}\left(1+\frac{1}{n}\right) = 1.\end{aligned}$$

$\therefore A_n > A_{n+1},\ A_n \xrightarrow[n\to\infty]{} \mathtt{GLB}\,\{A_n :\ n \ge 1\}$. Since $A_n \sim a_n,\ a_n \xrightarrow[n\to\infty]{} \mathtt{GLB}\,\{A_n :\ n \ge 1\} =: e$.

- To see that $e < 3$, $e < A_5 < 3$.

Next topics

e ctd., Lip OF log & exp, ACCUMULATION POINTS, SUBSEQUENCES, BOLZANO-WEIERSTRASS THEOREM, UPPER AND LOWER LIMITS, CAUCHY SEQUENCES.

Lecture # 8 [8]

- To see that $e > 2$ we show that $a_n < a_{n+1}$ whence $e > a_n = (1+\frac{1}{n})^n > 2\ \forall\ n \geq 2$.

Proof that $a_n < a_{n+1}$:

By the binomial theorem,

$$\begin{aligned} a_n &= (1+\frac{1}{n})^n \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{1}{n^k} \\ &= 1 + \sum_{k=1}^{n} \frac{n(n-1)\dots(n-k+1)}{n^k}\frac{1}{k!} \\ &= 1 + 1 + \sum_{k=2}^{n}(1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{k-1}{n})\frac{1}{k!} \\ &=: \sum_{k=0}^{n} b_{n,k} \end{aligned}$$

where

$$b_{n,k} := \begin{cases} 1 & k = 0,1; \\ (1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{k-1}{n})\frac{1}{k!} & 2 \leq k \leq n. \end{cases}$$

Now for each $0 \leq k \leq n$, $b_{n,k} \leq b_{n+1,k}$ (equality iff $k = 0,1$).

For $k \leq n$ this follows from $(1-\frac{j}{n}) < (1-\frac{j}{n+1})$.

Since there is strong inequality for some k, it follows that

$$a_n = \sum_{k=0}^{n} b_{n,k} < \sum_{k=0}^{n+1} b_{n+1,k} < a_{n+1}. \ ☑$$

Next topics

ACCUMULATION & LIMIT POINTS, BOLZANO-WEIERSTRASS THEOREM, SUBSEQUENCES, CLOSED SETS, UPPER AND LOWER LIMITS, CAUCHY SEQUENCES, INFINITE SERIES.

[8] 23/3/2017

Log functions

Landau's log theorem *There is a continuous, increasing bijection* $\ell : \mathbb{R}_+ \to \mathbb{R}$ *so that*

(⚔) $\ell_n(x) := 2^n(x^{\frac{1}{2^n}} - 1) \downarrow \ell(x)$ *as* $n \uparrow \infty\ \forall\ x > 0$;

(‡) $1 - \frac{1}{x} \le \ell(x) \le x - 1\ \forall\ x > 0$;

(†) $\ell(xy) = \ell(x) + \ell(y)\ \forall\ x, y > 0$,

(♗) $|\ell(x) - \ell(y)| \le \frac{1}{a} \cdot |x - y|\ \forall\ a > 0,\ x, y \in [a, \infty)$;

(⚗) $\ell(e) = 1$.

Proof

For $x > 0$, $\ell_0(x) = x - 1$ & for $n \in \mathbb{N}_0$,

(i) $\ell_n(x) > 0\ \forall\ x > 1$, $\ell_n(x) < 0\ \forall\ x \in (0,1)$ & $\ell_n(1) = 0$;

(ii) $\ell_n(x) = 2^n(x^{\frac{1}{2^n}} - 1) = -x^{\frac{1}{2^n}}\ell_n(\frac{1}{x})$;

(iii) $\ell_n(xy) = 2^n(x^{\frac{1}{2^n}}y^{\frac{1}{2^n}} - 1) = y^{\frac{1}{2^n}}\ell_n(x) + \ell_n(y)$.

Next,

$$\begin{aligned}\ell_n(x) &= 2^n(x^{\frac{1}{2^n}} - 1) = 2^n((x^{\frac{1}{2^{n+1}}})^2 - 1)\\ &= 2^n(x^{\frac{1}{2^{n+1}}} + 1)(x^{\frac{1}{2^{n+1}}} - 1)\\ &\ge 2^n \cdot\ 2 \cdot\ (x^{\frac{1}{2^{n+1}}} - 1) = \ell_{n+1}(x)\end{aligned}$$

(where the last inequality is established considering the cases $x > 1$ & $0 < x < 1$ separately); whence $\ell_n(x) \xrightarrow[n\to\infty]{} \ell(x) \ge -\infty$.

Proof of (‡)

By monotonicity, $\forall\ x > 0$, $\ell(x) \le \ell_n(x) \le \ell_0(x) = x - 1 < \infty$ and recalling that $x^{\frac{1}{2^n}} \xrightarrow[n\to\infty]{} 1$,

$$\ell(x) \xleftarrow[n\to\infty]{} \ell_n(x) \overset{\text{(ii)}}{=} -x^{\frac{1}{2^n}}\ell_n(\tfrac{1}{x}) \xrightarrow[n\to\infty]{} -\ell(\tfrac{1}{x}) \ge -\ell_0(\tfrac{1}{x}) = 1 - \tfrac{1}{x}. \ ☑$$

In particular, $\ell(x) \in \mathbb{R}\ \forall\ x > 0$.

Proof of (†) For x, $y > 0$,

$$\ell(xy) \xleftarrow[n\to\infty]{} \ell_n(xy) \overset{\text{(iii)}}{=} y^{\frac{1}{2^n}}\ell_n(x) + \ell_n(y) \xrightarrow[n\to\infty]{} \ell(x) + \ell(y). \ ☑$$

To see that $\ell : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, let $0 < x < y$, then $\frac{y}{x} > 1$ whence

$$\ell(y) = \ell(x \cdot \tfrac{y}{x}) \overset{(\dagger)}{=} \ell(x) + \ell(\tfrac{y}{x}) \overset{(\ddagger')}{>} \ell(x) + 1 - \tfrac{x}{y} > \ell(x).$$

Proof of (📐) Fix $a > 0$. For $x > y > a$,

$$|\ell(x) - \ell(y)| = \ell(\tfrac{x}{y}) \overset{(\ddagger)}{\le} \tfrac{x}{y} - 1 \le \tfrac{1}{a} \cdot |x - y|. \ \boxtimes$$

Proof of (⚗)

Write $x_n := 1 + \frac{1}{n}$.

On the one hand, $x_n^n \to e$ and by continuity of ℓ,

$$\ell(x_n^n) \xrightarrow[n\to\infty]{} \ell(e).$$

On the other hand, by ($\ddagger$)

$$\tfrac{1}{n+1} = 1 - \tfrac{1}{x_n} < \ell(x_n) < x_n - 1 = \tfrac{1}{n}$$

whence

$$1 \xleftarrow[n\to\infty]{} n\ell(x_n) \overset{(\dagger)}{=} \ell(x_n^n) \xrightarrow[n\to\infty]{} \ell(e). \ \boxtimes$$

Natural logarithm.

The function $\ell = \log = \ln : \mathbb{R}_+ \to \mathbb{R}$ is called the *natural logarithm function.*

Real powers of positive real numbers

Theorem `exp`

$\exists$ *a continuous, increasing bijection* $\exp : \mathbb{R} \to \mathbb{R}_+$ *so that* $\exp(1) = e$ *and*

(📖) $$\exp(x + y) = \exp(x)\exp(y) \ \forall \ x, \ y \in \mathbb{R}.$$

Proof Define $\exp : \mathbb{R} \to \mathbb{R}_+$ by $\exp := \log^{-1}$, then exp is an increasing bijection, $\exp(1) = e$ and

$$\exp(x + y) = \exp(x)\exp(y) \ \forall \ x, \ y \in \mathbb{R}.$$

Also, by ($\ddagger$),

(💉) $$1 - \exp(-x) < x < \exp(x) - 1 \quad \forall \ x \in \mathbb{R}$$

whence, for $x, y \in \mathbb{R}, \ x < y$,

$$\begin{aligned}\exp(y) - \exp(x) &= \exp(y)(1 - \exp(-(y - x))) \\ &\le \exp(y)(y - x).\end{aligned}$$

It follows that for any $a, b \in \mathbb{R},\ a < b$, $\exp : (a, b) \to \mathbb{R}_+$ satisfies

(Lip) $\quad |\exp(x) - \exp(y)| \le \exp(b)|x - y| \ \forall\ x, y \in (a, b).$

Thus $\exp : \mathbb{R} \to \mathbb{R}_+$ is continuous. ☑

Powers and logs.

- If $a = \exp(t)$ and $q \in \mathbb{N}$, then

$$\exp(qt) = \exp(\underbrace{t + \cdots + t}_{q \text{ times}}) = a^q \quad \& \quad \exp(-qt) = a^{-q}$$

whence

$$\exp(\tfrac{t}{q})^q = \exp(t) = a \quad \Longrightarrow \quad \exp(\tfrac{t}{q}) = a^{\frac{1}{q}}$$

and for $\frac{p}{q} = r \in \mathbb{Q}$,

$$(a^{\frac{1}{q}})^p = \exp(rt) = (a^p)^{\frac{1}{q}}.$$

Accordingly, we define for $a > 0$

$$a^r := \exp(r \log(a)) \ \forall\ r \in \mathbb{R}.$$

It follows that for $a,\ b > 0,\ r,\ s \in \mathbb{R}$

$$(ab)^r = a^r b^r,\ \&\ a^{r+s} = a^r a^s.$$

Moreover

$$(a^r)^s := \exp(s \log(a^r)) = \exp(sr \log(a)) =: a^{rs}.$$

Write $E_a(x) := a^x \quad (a > 0,\ x \in \mathbb{R})$,

(i) if $a > 1$, then $E_a : \mathbb{R} \to \mathbb{R}_+$ is a strictly increasing, continuous bijection and;

(ii) if $0 < b < 1$, then $E_b : \mathbb{R} \to \mathbb{R}_+$ is a strictly decreasing, continuous bijection.

The inverse function to E_a is known as *logarithm base* a, is denoted $\log_a := E_a^{-1}$ and satisfies $z = a^{\log_a z}$, equivalently $\log_a(z) = \frac{\log(z)}{\log(a)}$.

Exponential continuity proposition

Suppose that $a > 0,\ t \in \mathbb{R}$ and $a_n, x_n \in \mathbb{R},\ a_n > 0$ satisfy $x_n \xrightarrow[n\to\infty]{} t \in \mathbb{R}$ and $a_n \xrightarrow[n\to\infty]{} a > 0$, then $a_n^{x_n} \xrightarrow[n\to\infty]{} a^t$.

Proof By assumption and continuity of log, $x_n \log(a_n) \xrightarrow[n\to\infty]{} t \log(a)$, whence by continuity of exp,

$$a_n^{x_n} = \exp(\log(a_n^{x_n})) = \exp(x_n \log(a_n)) \xrightarrow[n\to\infty]{} \exp(t \log(a)) = a^t. \qquad ☑$$

Corollary e

(*e*) $$(1+\tfrac{x}{n})^n \longrightarrow e^x \ \forall\ x \in \mathbb{R}.$$

We need

Lemma e

$$(1+\tfrac{1}{a_n})^{a_n} \to e \ \forall\ a_n \to \infty.$$

Proof Recall that by proposition e, $(1+\frac{1}{n})^n \to e$. It follows (!) that $(1+\frac{1}{n+1})^n \to e$ and $(1+\frac{1}{n})^{n+1} \to e$. Lemma e now follows from monotonicity and the sandwich principle:

$$e \ \leftarrow\ (1+\tfrac{1}{[a_n]+1})^{[a_n]} < (1+\tfrac{1}{a_n})^{a_n} < (1+\tfrac{1}{[a_n]})^{[a_n]+1} \ \to\ e.$$

$\square$

Proof of corollary (*e*) For $x > 0$, by lemma e, $(1+\frac{x}{n})^{\frac{n}{x}} \to e$ whence using the exponential continuity proposition,

$$(1+\tfrac{x}{n})^n = \left((1+\tfrac{x}{n})^{\frac{n}{x}}\right)^x \to e^x.$$

To complete the proof, we show that $(1-\frac{x}{n})^n \to e^{-x} \ \forall\ x > 0$. To see this, note first that

$$(1-\tfrac{x}{n})^n(1+\tfrac{x}{n})^n = (1-\tfrac{x^2}{n^2})^n.$$

Using Bernoulli's inequality and the sandwich principle,

$$1 \geq (1-\tfrac{x^2}{n^2})^n \geq 1-\tfrac{x^2}{n} \to 1$$

whence

$$(1-\tfrac{x}{n})^n = \frac{(1-\frac{x^2}{n^2})^n}{(1+\frac{x}{n})^n} \to e^{-x}.$$

$\square$

Corollary

(i) *For $\alpha \in \mathbb{R}$, the function $P_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $P_\alpha(x) := x^\alpha$ is continuous.*

(ii) *For $\alpha > 0$, the function $P_\alpha : [0,\infty) \to [0,\infty)$ defined by $P_\alpha(0) = 0$ & $P_\alpha(x) := x^\alpha$ for $x > 0$ is continuous.*

Proof (i) follows directly from the exponential continuity proposition.

(ii) To see continuity at 0 assume otherwise, then (!) $\exists\ \epsilon > 0,\ 0 < x_n < 1$ so that $x_n \to 0$ and $x_n^\alpha \geq \epsilon \ \forall\ n \geq 1$. Fix $Q \geq 1$ so that $Q\alpha \geq 1$, then

$$\epsilon^Q < x_n^{Q\alpha} \leq x_n \to 0. \quad \boxtimes \quad ☑$$

Lecture # 9 [9]

Accumulation points and isolated points

Neighborhoods in $\mathbb{R}$ & $\mathbb{C}$. For $\epsilon > 0$, the *ϵ-neighborhood* of x is the set

$$N(x,\epsilon) := \{y: \ |y-x| < \epsilon\}.$$

In $\mathbb{R}$, $N(x,\epsilon) = (x-\epsilon, x+\epsilon)$ and in $\mathbb{C}$, for $z = u+iv$,

$$N(z,\epsilon) = \{w = a+ib: \ |w-z| = \sqrt{(u-a)^2 + (v-b^2)} < \epsilon\},$$

the interior of the disc with center z and radius ϵ.

Isolated points. Let $E \subset \mathbb{V}$ ($\mathbb{V} = \mathbb{R}$ or $\mathbb{V} = \mathbb{C}$). A point $x \in E$ is called an *isolated point* of E if for some $\epsilon > 0$, $E \cap N(x,\epsilon) = \{x\}$.
If $x \in E$ is not isolated, then $\forall\ \epsilon > 0$, $\#E \cap N(x,\epsilon) \ge 2$.
Here $\#A :=$ `the number of elements in the set` A.

Accumulation points. For $\mathbb{V} = \mathbb{R}$ or $\mathbb{V} = \mathbb{C}$, an *accumulation point* of E is a point $x \in \mathbb{V}$ so that $\forall\ \epsilon > 0$, $\#E \cap N(x,\epsilon) \ge 2$.

So a non-isolated point of E is an accumulation point of E, but there may be accumulation points of E outside E (see examples below). The collection of accumulation points of E is denoted E' and is called the *derived set* of E (קבוצה הנגזרת של).

Examples

1) $E := \{0\}$, $E' = \varnothing$;

2) $E := [0,1]$, $E' = E$;

3) $E := (0,1) \cap \mathbb{Q}$, $E' = [0,1]$;

4) $E := \{\frac{1}{n}: \ n \in \mathbb{N}\}$, $E' = \{0\}$.

5) If E is bounded and `LUB` $\notin E$, then `LUB` $\in E'$.

Proposition

For $\mathbb{V} = \mathbb{R}$ or $\mathbb{V} = \mathbb{C}$, the following are equivalent for $E \subset \mathbb{V}$ and $x \in \mathbb{V}$:
(i) *x is an accumulation point of E;*
(ii) $\forall\ \epsilon > 0$, $\#E \cap N(x,\epsilon) = \infty$*;*
(iii) $\exists\ (z_1, z_2, \dots) \in E^{\mathbb{N}}$ *such that* $z_k \ne z_\ell\ \forall\ k \ne \ell$ & $z_n \xrightarrow[n\to\infty]{} x$.

Proof of (i) $\Rightarrow$ (iii):

[9] 27/4/2017

Assume that $x \in \mathbb{V}$ is such that $\forall\ \epsilon > 0,\ \exists\ y \in E \cap N(x,\epsilon),\ y \neq x$. We show (iii). By assumption, $\exists\ y_1 \in E,\ y_1 \neq x,\ |y_1 - x| =: \epsilon_1 < 1$. Similarly, $\exists\ y_2 \in E,\ y_2 \neq x,\ |y_2 - x| =: \epsilon_2 < \frac{\epsilon_1}{2}$. Evidently $y_2 \neq y_1$ since $|y_2 - x| < |y_1 - x|$. Continuing, we obtain $y_n \in E\quad (n \geq 1)$ such that $|y_{n+1} - x| < |y_n - x| < \frac{1}{2^n} < \frac{1}{n}$, whence

- $y_n \neq y_{n'}$ for $n \neq n'$ and $y_n \underset{n\to\infty}{\longrightarrow} x$.

Proof of (iii) $\Rightarrow$ (ii):

Suppose that $(z_1, z_2, \dots) \in E^{\mathbb{N}}$ is such that $z_k \neq z_\ell\ \forall\ k \neq \ell\ \&\ z_n \underset{n\to\infty}{\longrightarrow} x$. Let $\epsilon > 0$ and let N_ϵ be such that $|z_n - x| < \epsilon$ for $n \geq N_\epsilon$, then

$$E \cap N(x,\epsilon) \supset \{z_k :\ k \geq N_\epsilon\} \implies \#\, E \cap N(x,\epsilon) = \infty.$$

$\square$

Bolzano-Weierstrass theorem `(accumulation points)` *Let* $\mathbb{V} = \mathbb{R}$ *or* $\mathbb{V} = \mathbb{C}$*. If* $E \subset \mathbb{V}$ *is an infinite, bounded set, then* $E' \neq \varnothing$*.*

The proof of the Bolzano-Weierstrass theorem. uses:

"`Chinese boxes`" AND CANTOR'S LEMMA.

A *nested sequence* (סידרה מקוננת) of intervals (aka *Chinese box* (תיבה סינית)) is a sequence of closed intervals $\{I_n :\ n \in \mathbb{N}\}$ such that $I_n \supset I_{n+1}$.

Cantor's Lemma `(or the Chinese box theorem)`

A nested sequence of non-empty, closed intervals in $\mathbb{R}$ *has a non-empty intersection.*

Proof Since $I_n = [a_n, b_n] \supset I_k = [a_k, b_k]\ \forall\ k > n$, we have $a_n \leq a_k \leq b_k \leq b_n$ whence the sets $\{a_n :\ n \geq 1\}$ and $\{b_n :\ n \geq 1\}$ are bounded; and $a := \mathtt{LUB}\ \{a_n :\ n \geq 1\} \leq b =: \mathtt{GLB}\ \{b_n :\ n \geq 1\}$.

Thus $[a,b] \subseteqq \bigcap_{n=1}^\infty I_n \neq \varnothing$.

To see $[a,b] \supseteqq \bigcap_{n=1}^\infty I_n$ suppose that $x \in I_n = [a_n, b_n]\ \forall\ n \geq 1$, then $a_n \leq x \leq b_n\ \forall\ n \geq 1$. Equivalently, x is an upper bound for $\{a_n :\ n \geq 1\}$ and a lower bound for $\{b_n :\ n \geq 1\}$;

whence $a := \mathtt{LUB}\ \{a_n :\ n \geq 1\} \leq x \leq b := \mathtt{GLB}\ \{b_n :\ n \geq 1\}$ and $x \in [a,b]$. ☑

Proof of the Bolzano-Weierstrass theorem for $\mathbb{V} = \mathbb{R}$:

Suppose that $E \subset I$ a closed, finite interval. For $I = [a,b]$, write $I^- := [a, \frac{a+b}{2}]$ and $I^+ := [\frac{a+b}{2}, b]$. Evidently, $I = I^- \cup I^+$ and $\exists\ I_1 = I^\pm$ with $\#\, E \cap I_1 = \infty$.

Similarly $\exists\ I_2 = I_1^\pm$ with $\#\, E \cap I_2 = \infty$ and continuing, we obtain closed intervals $I_n \supset I_{n+1}$ so that

- $I_{n+1} = I_n^\pm$ and $\#\, E \cap I_n = \infty\ \forall\ n \geq 1$.

Since $|I_n| = \frac{|I|}{2^n} \xrightarrow[n\to\infty]{} 0$, by Cantor's lemma, $\bigcap_{n=1}^{\infty} I_n = \{Z\}$ for some $Z \in \mathbb{R}$.

To see that $Z \in E'$ note that for $\epsilon > 0$ & n large,

$$E \cap N(Z,\epsilon) \supset\supset E \cap I_n$$

whence $\# E \cap N(Z,\epsilon) \ge \# E \cap I_n = \infty$.☑

Proof of BW theorem for $\mathbb{V} = \mathbb{C} \cong \mathbb{R}^2$

Proof Suppose that $E \subset R = I \times J$, a closed, finite box (i.e. I, J are closed, finite intervals).

We have

$$I \times J = \bigcup_{\epsilon,\delta=\pm} R(\epsilon,\delta)$$

where $R(\epsilon,\delta) := I^\epsilon \times J^\delta$ with I^ϵ & J^δ as above. and $\exists\ R_1 \in \{R(\epsilon,\delta) : \epsilon,\delta = \pm\}$ with $\# E \cap R_1 = \infty$.

Similarly $\exists\ R_2 \in \{R_1(\epsilon,\delta) :\ \epsilon,\delta = \pm\}$ with $\# E \cap R_2 = \infty$ and continuing, we obtain closed intervals $R_n \supset R_{n+1}$ so that

- $R_{n+1} \in \{R_1(\epsilon,\delta) :\ \epsilon,\delta = \pm\}$ and $\# E \cap R_n = \infty\ \forall\ n \ge 1$.

Writing $R_n = I_n \times J_n$ we have $|I_n| = \frac{|I|}{2^n} \xrightarrow[n\to\infty]{} 0$ and $|J_n| = \frac{|J|}{2^n} \xrightarrow[n\to\infty]{} 0$. By Cantor's lemma, $\exists\ a, b \in \mathbb{R}$ so that

$$\bigcap_{n=1}^{\infty} I_n = \{a\}\ \&\ \bigcap_{n=1}^{\infty} J_n = \{b\}.$$

It follows that

$$\bigcap_{n=1}^{\infty} R_n = \{(a,b)\}.$$

To see that $(a,b) \in E'$ note that for $\epsilon > 0$ & n large

$$E \cap N((a,b),\epsilon) \supset E \cap R_n$$

whence $\# E \cap N((a,b),\epsilon) = \infty$ and $(a,b) \in E'$.☑

Limit points.

Let $E \subset \mathbb{V}$ (for $\mathbb{V} = \mathbb{R}$ or $\mathbb{V} = \mathbb{C}$). A point $x \in \mathbb{V}$ is called a *limit point* (נקודת גבול) of E if $\exists\ y_n \in E\ \ (n \ge 1)$ such that $y_n \to x$.

- As shown above, x is an accumulation point of E iff $\exists\ y_n \in E\ \ (n \ge 1)$ such that $y_n \ne y_{n'}$ for $n \ne n'$ and $y_n \to x$. In particular, an accumulation point is a limit point.
- The converse is false. Each $x \in E$ is necessarily a limit point of E, but not necessarily an accumulation point of E (e.g. if E is finite).

The collection of limit points of E is denoted $\overline{E}$ and is called the *closure* (סגור) of E.

Proposition

$$\overline{E} = E \cup E'$$

Proof of $\overline{E} \subseteq E \cup E'$:

If $x \in \overline{E}$, $\exists\ (x_1, x_2, \dots),\ x_n \in E,\ x_n \to x$. Either $\#\{x_n :\ n \in \mathbb{N}\} < \infty$ whence $x_n = x\ \forall n$ large and $x \in E$; or $\#\{x_n :\ n \in \mathbb{N}\} = \infty$. In this case, $\forall\ \epsilon > 0,\ \exists\ N(\epsilon)$ such that $|x_n - x| < \epsilon\ \forall\ n \geq N(\epsilon)$ whence

$$\#E \cap (x - \epsilon, x + \epsilon) \geq \#\{x_n :\ n \geq N(\epsilon)\} = \infty$$

and $x \in E'$. ☑

Closed sets and open sets.

A set $E \subset \mathbb{V}$ (where $\mathbb{V} = \mathbb{R},\ \mathbb{C}$) is *closed* if $\overline{E} = E$ and *open* if $\forall\ x \in E\ \exists\ \epsilon > 0,\ N(x, \epsilon) \subset E$.

Proposition

The set $E \subset \mathbb{V}$ is closed if and only if its complement E^c is open.

Proof

Suppose that E is closed and let $x \in E^c$. If there is no $\epsilon > 0$ with $N(x, \epsilon) \subset E$, then

$$\forall\ n \geq 1\ \exists\ x_n \in E \cap N(x, \tfrac{1}{n}).$$

It follows that $x_n \xrightarrow[n\to\infty]{} x$ and $x \in \overline{E} = E$. This contradicts $x \in E^c$.⊠

Now suppose that E^c is open and let $x \in \overline{E}$. We must show that $x \in E$.

Let $x_n \in E,\ x_n \to x$. If $x \in E^c$ then, for some $\epsilon > 0,\ N(x, \epsilon) \subset E^c$. On the other hand $x_n \to x$ and $\exists\ n$ so that $x_n \in N(x, \epsilon)$. Thus $x_n \in E \cap N(x, \epsilon) \subset E \cap E^c = \varnothing$. ⊠

Exercise

Show that a closed subset of $\mathbb{R}$ which is bounded above has a maximal element.

Subsequences.

An *integer subsequence* (תת־סידרה של שלמים) is an infinite subset $K \subset \mathbb{N},\ K = \{n_1, n_2, \dots\}$ arranged in increasing order $n_1 < n_2 < \dots \to \infty$.

A *subsequence* of the sequence $\{a_1, a_2, \dots\}$ is a sequence of form $\{a_{n_1}, a_{n_2}, \dots\}$ where $n_k \to \infty$ is an integer subsequence.

For a bounded, non-convergent sequence, different subsequences may have different limits. For example if $\{r_n\}_{n=1}^{\infty}$ is defined by

$$r_N = \begin{cases} -\frac{1}{n+1} & N = 3n, \\ \frac{1}{2} + \frac{1}{4(n+1)^2} & N = 3n+1, \\ 1 + \frac{1}{n+1} & N = 3n+2 \end{cases}$$

so that

$$(r_1, r_2, r_3, \dots) = (\frac{3}{4}, 2, -\frac{1}{2}, \dots),$$

then

$$r_{3n} \to 0,\ r_{3n+1} \to \frac{1}{2},\ \&\ r_{3n+2} \to 1.$$

Bolzano-Weierstrass Theorem `(convergent subsequences)`

Every bounded sequence has a convergent subsequence.

Proof Suppose that $(x_1, x_2, \dots)$ is a bounded sequence.

- If $E := \{x_n :\ n \in \mathbb{N}\}$ is infinite then by the BW theorem (accumulation points), it has an accumulation point $x \in E'$, which is a limit point and $\exists\ n_k \to \infty,\ x_{n_k} \xrightarrow[k\to\infty]{} x$.
- If E is finite then $\exists\ n_k \to \infty,\ x \in E$ so that $x_{n_k} = x\ \forall\ k \ge 1$, whence $x_{n_k} \xrightarrow[k\to\infty]{} x$. ☑

Partial limits of a sequence

Let $(a_1, a_2, \dots)$ be a bounded sequence, and let $\mathtt{PL}(a_1, a_2, \dots)$, the *partial limit set* (קבוצת גבולות החלקיים) (of the sequence) be the collection of limits of its subsequences:

$$\mathtt{PL}(a_1, a_2, \dots) := \{a \in \mathbb{R} :\ \exists\ n_k \to \infty,\ a_{n_k} \xrightarrow[k\to\infty]{} a\} \ne \varnothing.$$

Proposition *For* $(a_1, a_2, \dots)$ *a bounded sequence,*

$\#\mathtt{PL}(a_1, a_2, \dots) = 1 \iff \exists\ \lim_{n\to\infty} a_n$.

Proof of $\Leftarrow$

If a sequence converges, then every subsequence converges to the same limit.

Proof of $\Rightarrow$: Fix $a \in \mathtt{PL}(a_1, a_2, \dots)$.

If $a_n \not\to a$, then for some $\epsilon > 0$:

- $\forall\ k \ge 1,\ \exists\ n_k > k,\ |a_{n_k} - a| \ge \epsilon.$

In other words,

- $\exists$ a subsequence $n_k \to \infty$ with $|a_{n_k} - a| \ge \epsilon\ \forall\ k \ge 1$.

By the Bolzano-Weierstrass theorem, $\exists$ a subsequence (of the subsequence) $m_\ell = n_{k_\ell} \to \infty$ and $b \in \mathbb{R}$ so that $a_{m_\ell} \to b$. Evidently, $b \in \mathtt{PL}(a_1, a_2, \dots)$ and $|b - a| \leftarrow |a_{m_\ell} - a| \ge \epsilon$ so $b \ne a$. □

Example. The above proposition fails for an unbounded sequence. Let $a_{2n} = 1$ &$a_{2n+1} = n$, then (!) $\mathtt{PL}(a_1, a_2, \dots) = \{1\}$ but $x \not\to 1$.

Proposition *Let* $(a_1, a_2, \dots)$ *be a bounded sequence, then* $\mathtt{PL}(a_1, a_2, \dots)$ *is closed and bounded.*

Proof

`Bounded`: Suppose that $|a_n| \le M$. If $x \in \mathtt{PL}(a_1, a_2, \dots)$ then $\exists\ a_{n_k} \to x$. For k large, $|a_{n_k} - x| < 1$ whence

$$|x| \le |x - a_{n_k}| + |a_{n_k}| < 1 + M. \ \boxtimes$$

`Closed`: Suppose that $x \in \mathtt{PL}(a_1, a_2, \dots)'$, then $\forall\ k \ge 1,\ \exists$:

- $x^{(k)} \in \mathtt{PL}(a_1, a_2, \dots),\ |x^{(k)} - x| < \frac{1}{k}$; and
- $n_k \to \infty,\ |a_{n_k} - x^{(k)}| < \frac{1}{k}$ $(\because x^{(k)} \in \mathtt{PL}(a_1, a_2, \dots))$.

Thus $|a_{n_k} - x| < \frac{2}{k},\ a_{n_k} \xrightarrow[k\to\infty]{} x$ and $x \in \mathtt{PL}(a_1, a_2, \dots)$. $\boxtimes$

Exercise

Let $a : \mathbb{N} \to \mathbb{Q} \cap (0,1)$, $(a_1, a_2, \dots) = (\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{5}, \dots)$ be as on p. 35. Show that $\mathtt{PL}(a_1, a_2, \dots) = [0,1]$.

Exercise: Covers (כיסוים)

Let $Y \subset \mathbb{R}$. $\mathcal{S}$, a collection of subsets of $\mathbb{R}$ *covers* Y, if $Y \subset \bigcup \mathcal{S} := \{x \in \mathbb{R} :\ \exists\ S \in \mathcal{S},\ x \in S\}$. i.e. every point in Y belongs to some set in $\mathcal{S}$

(i) Let $Y \subset \mathbb{R}$. Show that $\mathcal{S}_1 := \{Y\}$ covers Y and so does $\mathcal{S}_2 := \{\{y\} :\ y \in Y\}$.

(ii) Show that $\mathcal{S}_3 := \{(-1, \frac{1}{2}), (\frac{1}{4}, 3)\}$ covers $[0,1]$ and so does $\mathcal{S}_4 := \{(x - \frac{1}{4}, x - \frac{1}{4}) :\ x \in (0,1)\}$.

(iii) Show that $\mathcal{S}_5 := \{(n-1, n+1) :\ n \in \mathbb{Z}\}$ covers $\mathbb{R}$, but no proper subset of $\mathcal{S}_5$ covers $\mathbb{R}$.

Exercise: Prove the Heine Borel theorem

☆

Show that if a collection $\mathcal{S}$ of open sets covers a closed, bounded set E, then $\exists\ \mathcal{S}_0 \subset \mathcal{S}$ finite, which covers E.

`Hint`: Suppose otherwise, and show there is an infinite Chinese box of closed intervals (`boxes` if you're proving it in $\mathbb{C}$) $I_n \supset I_{n+1},\ |I_n| = \frac{|I|}{2^n}$, no one of which is covered by a finite subcollection of $\mathcal{S}$. Then use Cantors lemma to obtain a contradiction.

Next topics

UPPER AND LOWER LIMITS, CAUCHY SEQUENCES, CANTOR'S CONSTRUCTION OF $\mathbb{R}$, CONVERGENCE OF SERIES.

Lecture # 10
[10]

Upper and lower limits

The *upper limit* (גבול עליון) of the bounded sequence $(a_1, a_2, \dots)$ is

$$\overline{\lim_{n\to\infty}}\, a_n := \max \mathtt{PL}(a_1, a_2, \dots)$$

and the *lower limit* (גבול תחתון) of the sequence $(a_1, a_2, \dots)$ is

$$\underline{\lim_{n\to\infty}}\, a_n := \min \mathtt{PL}(a_1, a_2, \dots).$$

Note that $\overline{\lim}$ denotes "`limsup`" which means "`upper limit`" in Latin (and $\underline{\lim}$ denotes "`liminf`" which means "`lower limit`").

Another interpretation of "`limsup`" & "`liminf`" is given via the the *tails* of the bounded sequence $a = (a_1, a_2, \dots)$ which are defined by

$$\mathcal{T}_n = \{a_k : \ k \ge n\} = \{a_n, a_{n+1}, \dots\}.$$

Evidently $\mathtt{LUB}\, \mathcal{T}_{n+1} \le \mathtt{LUB}\, \mathcal{T}_n \ \forall\ n \ge 1$ and $\mathtt{GLB}\, \mathcal{T}_{n+1} \ge \mathtt{GLB}\, \mathcal{T}_n \ \forall\ n \ge 1$.

Tails proposition

(a) $$\overline{L}_n := \mathtt{LUB}\, \mathcal{T}_n \xrightarrow[n\to\infty]{} \overline{\lim_{n\to\infty}}\, a_n \quad \&$$

(b) $$\underline{L}_n := \mathtt{GLB}\, \mathcal{T}_n \xrightarrow[n\to\infty]{} \underline{\lim_{n\to\infty}}\, a_n.$$

Proof of (a)

Since $(\overline{L}_n)_{n\ge1}$ is bounded and non-decreasing, $\exists\ \lim_{n\to\infty} \overline{L}_n =: \overline{L}$ and we must show that

$$\overline{L} = \overline{\lim_{n\to\infty}}\, a_n.$$

- If $a_{n_k} \to J \in \mathtt{PL}(a_1, a_2, \dots)$, then

$$J \longleftarrow a_{n_k} \le \overline{L}_{n_k} \to \overline{L}.$$

Thus $\overline{L}$ is an upper bound for $\mathtt{PL}(a_1, a_2, \dots)$ and

$$\overline{L} \ge \max \mathtt{PL}(a_1, a_2, \dots) =: \overline{\lim_{n\to\infty}}\, a_n.$$

- $\exists\ n_k \ge k$ so that $|a_{n_k} - \overline{L}| < \frac{1}{k}$ whence $a_{n_k} \to \overline{L}$ & $\overline{L} \in \mathtt{PL}(a_1, a_2, \dots)$.

Consequently

$$\overline{L} \le \ \max \mathtt{PL}(a_1, a_2, \dots) =: \overline{\lim_{n\to\infty}}\, a_n. \quad ☑$$

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Corollary

A bounded sequence $(a_1, a_2, \dots)$ *converges iff* $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n$.

Proof $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n$ iff $\#\mathtt{PL}(a_1, a_2, \dots) = 1$. ☑

Proposition *Let* $a = (a_1, a_2, \dots)$ *be a bounded sequence, then*

$\underline{\text{(i)}}$ $\forall\ \alpha < \underline{\lim}_{n\to\infty} a_n,\ \exists\ N_\alpha$ *such that* $a_n > \alpha\ \forall\ n > N_\alpha$*;*

$\underline{\text{(ii)}}$ $\forall\ \beta > \underline{\lim}_{n\to\infty} a_n,\ K \geq 1,\ \exists\ N > K$ *such that* $a_N < \beta$*;*

$\overline{\text{(i)}}$ $\forall\ \omega > \overline{\lim}_{n\to\infty} a_n,\ \exists\ N_\omega$ *such that* $a_n < \omega\ \forall\ n > N_\omega$*;*

$\overline{\text{(ii)}}$ $\forall\ \xi < \overline{\lim}_{n\to\infty} a_n,\ K \geq 1,\ \exists\ N > K$ *such that* $a_N > \xi$*;*

Proof Follows from the tails proposition.

Exercise

Let $(a_1, a_2, \dots)$ & $(b_1, b_2, \dots)$ be bounded sequences.
Show that if $b_n - a_n \xrightarrow[n\to\infty]{} 0$, then $\mathtt{PL}(a_1, a_2, \dots) = \mathtt{PL}(b_1, b_2, \dots)$.

Exercises

1) Show that every sequence $a : \mathbb{N} \to \mathbb{R}$ has a monotonic subsequence.

2) Show that if $a : \mathbb{N} \to \mathbb{R}$ is bounded and $a_n - a_{n+1} \xrightarrow[n\to\infty]{} 0$, then $\mathtt{PL}(a_1, a_2, \dots)$ is a (possibly trivial) closed interval.

3) Are there a bounded sequences $a : \mathbb{N} \to \mathbb{R}$ with $a_n - a_{n+1} \xrightarrow[n\to\infty]{} 0$ and $\mathtt{PL}(a_1, a_2, \dots) = [0, 1]$?

Exercise: Alternative proof of the Bolzano-Weierstrass theorem in $\mathbb{R}$.

Here you show that if $E \subset \mathbb{R}$ is bounded and infinite, then $E' \neq \varnothing$.
Let $F := \{y \in \mathbb{R} :\ \#(E \cap (y, \infty)) = \infty\}$. Show that:

(i) $F \neq \varnothing$; (ii) $t \in F$ & $s < t \implies s \in F$; (iii) F is bounded above and (iv) $\mathtt{LUB}\, F \in E'$.

Cauchy sequences

`Or how to prove a sequence converges without knowing the limit.`

Definition. A sequence $(a_1, a_2, \dots)$ is called a *Cauchy sequence* if $\forall\ \epsilon > 0,\ \exists\ N_\epsilon \geq 1$ such that

$$|a_n - a_{n'}| < \epsilon \quad \forall\ n, n' \geq N_\epsilon.$$

- It follows directly from the definitions that any convergent sequence is a Cauchy sequence.

Cauchy's Theorem

A sequence converges $\iff$ *it is a Cauchy sequence.*

Proof of $\Leftarrow$: Suppose that $(a_1, a_2, \dots)$ is a Cauchy sequence. We'll show

(i) $(a_1, a_2, \dots)$ is bounded and (ii) $\exists \ \lim_{n\to\infty} a_n$.
Proof of (i): Let $N_1 \ge 1$ be such that $|a_J - a_K| < 1 \ \ \forall \ J, K \ge N_1$ and let $M := \max_{1\le k\le N_1} |a_k|$. We claim that $|a_n| \le M + 1 \ \forall \ n \ge 1$. To see this, if $n \le N_1$, then $|a_n| \le M < M + 1$ and if $n \ge N_1$, then

$$|a_n| \le |a_n - a_{N_1}| + |a_{N_1}| < 1 + M. \boxtimes$$

Proof of (ii): By the Bolzano-Weierstrass theorem $\exists \ n_k \to \infty, \ a \in \mathbb{R}$ so that $a_{n_k} \to a$. We show that $a_n \to a$. To this end fix $\epsilon > 0$, then $\exists \ N_\epsilon$ so that $|a_J - a_K| < \frac{\epsilon}{2} \ \ \forall \ J, K \ge N_\epsilon$. Also $\exists \ k_\epsilon$ so that $n_{k_\epsilon} > N_\epsilon$ and $|a_{n_{k_\epsilon}} - a| < \frac{\epsilon}{2}$. It follows that for $n \ge N_\epsilon$,

$$|a_n - a| < |a_n - a_{n_{k_\epsilon}}| + |a_{n_{k_\epsilon}} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \boxtimes$$

Sketch of Cantor's construction of $\mathbb{R}$.

Let

$$\begin{aligned}\widehat{\mathbb{R}} &:= \{\text{rational Cauchy sequences}\}\\ &= \{\underline{q} = (q_1, q_2, \dots) \in \mathbb{Q}^{\mathbb{N}} : \ \forall \ r \in \mathbb{Q}_+ \ \exists \ N_r \text{ st } |q_n - q_{n'}| < r \ \forall \ n, n' > N_r\}.\end{aligned}$$

Define a relation $\sim$ on $\widehat{\mathbb{R}}$ by

$$(q_1, q_2, \dots) \sim (q_1', q_2', \dots) \text{ iff } \ |q_n - q_n'| \xrightarrow[n\to\infty]{} 0,$$

`then` **(!)** $\sim$ is an equivalence relation.

Let $q \in \mathbb{Q}$. Call the Cauchy sequence $(q_1, q_2, \dots)$ *q-rational* if

$$(q_1, q_2, \dots) \sim (q, q, q, \dots) =: \overline{q}$$

i.e. $q_n \xrightarrow[n\to\infty]{} q$.

Now define another (order) relation $\prec$ on $\widehat{\mathbb{R}}$ by

$$(q_1, q_2, \dots) \prec (q_1', q_2', \dots) \text{ iff } \ \exists \ \epsilon > 0 \ \& \ N \text{ st } q_n' - q_n \ge \epsilon \ \forall \ n > N,$$

`then` **(!)**

(i) $\forall \ \underline{q}, \ \underline{r} \in \widehat{\mathbb{R}}$, either $\underline{q} \prec \underline{r}$, or $\underline{q} \succ \underline{r}$, or $\underline{q} \sim \underline{r}$ and

(ii) that if $\underline{q} \prec \underline{r}, \ \underline{q} \sim \underline{q}' \ \& \ \underline{r} \sim \underline{r}'$ then $\underline{q}' \prec \underline{r}'$.

The rational Cauchy sequences are *dense* in $\widehat{\mathbb{R}}$: If $\underline{x}, \ \underline{y} \in \widehat{\mathbb{R}}$ and $\underline{x} \prec \underline{y}$ then $\exists \ q \in \mathbb{Q}$ so that $\underline{x} \prec \overline{q} \prec \underline{y}$.

The *Cantor reals* are defined by

$$\widetilde{\mathbb{R}} := \widehat{\mathbb{R}}/\sim.$$

Define a relation $\ll$ on $\widetilde{\mathbb{R}}$ by

$$[\underline{q}] \ \ll \ [\underline{r}] \text{ if } \exists \ \underline{q} \in [\underline{p}] \ \& \ \underline{s} \in [\underline{r}] \text{ st } \underline{q} \prec \underline{s},$$

`then` **(!)** $(\widetilde{\mathbb{R}}, \ll)$ is an ordered set.

To see that the ordered set $(\widetilde{\mathbb{R}}, \ll)$ is complete, let $E \subset \widetilde{\mathbb{R}}$ be bounded. For each $n \geq 1$, set

$$k_n := \min\{k \in \mathbb{N} : \ \overline{\tfrac{k}{2^n}} \text{ is an upper bound for } E\}.$$

Define the sequence $\underline{z} \in \mathbb{Q}^{\mathbb{N}}$ by $z_n := \frac{k_n}{2^n}$.

To see that $\underline{z} \in \widetilde{\mathbb{R}}$, note that $|z_n - z_{n+1}| < \frac{1}{2^n}$ (since (!) $z_n - \frac{1}{2^n} < z_{n+1} \leq z_n$). Thus (!) $|z_n - z_{n+k}| \leq \sum_{j=0}^{k-1} \frac{1}{2^{n+k-1}} \leq \frac{1}{2^{n-1}}$ and $\underline{z}$ is Cauchy.

Since $\forall\ n \geq 1$, $\overline{z_n}$ is an upper bound for E we have that $\underline{z}$ is an upper bound for E. On the other hand, $\forall\ n \geq 1$, $\overline{z_n - \frac{1}{2^n}}$ is not an upper bound for E. Consequently if $\underline{U}$ is an upper bound for E, then $\overline{z_n - \frac{1}{2^n}} \ll \underline{U}$ $\forall\ n \geq 1$, whence

$$\underline{z}' := (z_n - \tfrac{1}{2^n} :\ n \geq 1) \ll \underline{U}.$$

But $\underline{z}' \sim \underline{z}$ so $\underline{z} \ll \underline{U}$ and $\underline{z} = \mathtt{LUB}\, E$.

Define addition and multiplication on $\widehat{\mathbb{R}}$ by

$$(q_1, q_2, \ldots) \oplus (r_1, r_2, \ldots) := (q_1 + q_1', q_2 + q_2', \ldots)\ \&\ (q_1, q_2, \ldots) \odot (r_1, r_2, \ldots) := (q_1 r_1, q_2 r_2, \ldots),$$

then (!)

(i) $(\widehat{\mathbb{R}}, \oplus, \odot)$ satisfies the associative, commutative and distributive laws;

(ii) if $\underline{q} \sim \underline{q}'$ & $\underline{r} \sim \underline{r}'$ then $\underline{q}' \oplus \underline{r}' \sim \underline{q} \oplus \underline{r}$ & $\underline{q}' \odot \underline{r}' \sim \underline{q} \odot \underline{r}$.

Now define addition and multiplication on $\widetilde{\mathbb{R}}$ by

$$[\underline{q}] \boxplus [\underline{r}] := [\underline{q} \oplus \underline{r}]\ \&\ [\underline{q}] \boxdot [\underline{r}] := [\underline{q} \odot \underline{r}].$$

Cantor's theorem. $(\widetilde{\mathbb{R}}, \ll, \boxplus, \boxdot)$ is a complete, ordered field.

As mentioned above, any two complete ordered fields are in correspondence by a bijection preserving ordered field structures. In particular, Cantor's reals and Dedekind's.

Series (טורים)

A *series* (טור) is a sequence $(s_1, s_2, \ldots)$ of form $s_n = \sum_{k=1}^{n} a_k$ where $a_k \in \mathbb{R}$ $(k \geq 1)$.

Note that any sequence is of this form. We'll study convergence properties of s_n in terms of the a_n's.

Notation. The series $\sum_{k=1}^{\infty} a_k$ *converges* if

$$\exists \lim_{n\to\infty} \sum_{k=1}^{n} a_k =: \sum_{k=1}^{\infty} a_k \in \mathbb{R}.$$

Proposition (tails of a series) *If* $\sum_{k=1}^{\infty} a_k$ *converges, then*

(i) $\forall\ N \geq 1$ *so does* $\sum_{k=N}^{\infty} a_k$ *and*

(ii)
$$\sum_{k=N}^{\infty} a_k \underset{N\to\infty}{\longrightarrow} 0.$$

Proof (i) Fix $N \geq 1$, then

$$\begin{aligned}\sum_{k=N}^{N+n} a_k &= \sum_{k=1}^{N+n} a_k - \sum_{k=1}^{N-1} a_k \\ &\xrightarrow[n\to\infty]{} \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{N-1} a_k \\ &=: \sum_{k=N}^{\infty} a_k.\end{aligned}$$

(ii) Fix $\epsilon > 0$ then $\forall\ N \geq 1$ large

$$\epsilon > |\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{N-1} a_k| = |\sum_{k=N}^{\infty} a_k|. \ \boxtimes.$$

Proposition: (`linearity of series`)

Suppose that the series $\sum_{k=1}^{\infty} a_k$ & $\sum_{k=1}^{\infty} b_k$ both converge, then for any $s, t \in \mathbb{R}$, the series $\sum_{k=1}^{\infty}(sa_k + tb_k)$ also converges and

$$\sum_{k=1}^{\infty}(sa_k + tb_k) = s\sum_{k=1}^{\infty} a_k + t\sum_{k=1}^{\infty} b_k.$$

Series with non-negative terms (טורים עם אברים אי־שליליים)

If $a_n \geq 0\ \forall\ n \geq 1$ then $\sum_{k=1}^{n} a_k\ \uparrow$. As proved before, either the sequence $\sum_{k=1}^{n} a_k$ is bounded and

$$\sum_{k=1}^{n} a_k\ \uparrow \sum_{k=1}^{\infty} a_k \in [0, \infty),$$

or

$$\sum_{k=1}^{n} a_k\ \uparrow \infty =: \sum_{k=1}^{\infty} a_k.$$

Thus, for a series with non-negative terms, we have the shorthand:
$\sum_{k=1}^{\infty} a_k < \infty$ for $\sum_{k=1}^{\infty} a_k$ converges; and
$\sum_{k=1}^{\infty} a_k = \infty$ for $\sum_{k=1}^{\infty} a_k$ diverges.

Examples.

- $\sum_{n=1}^{\infty} r^n = \infty\ \forall\ r \geq 1$,
- $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, $\because\ \sum_{k=1}^{2^n} \frac{1}{k} \geq \frac{n}{2}$.
- $\sum_{n=0}^{\infty} r^n < \infty\ \forall\ 0 \leq r < 1\ \because\ \sum_{n=0}^{N} r^n + \frac{r^{N+1}}{1-r} = \frac{1}{1-r}\ \forall\ N \geq 1$.
- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1\ \because\ \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

- $\sum_{n=1}^{\infty} \frac{1}{n^2} \le 2$ (i.e. $\sum_{n=1}^{N} \frac{1}{n^2} \xrightarrow[N\to\infty]{} L \in [0,2]$) because $\frac{1}{n^2} \le \frac{2}{n(n+1)}$ $\forall\ n \ge 1$ whence

$$\sum_{n=1}^{N} \frac{1}{n^2} \le 2\sum_{n=1}^{N} \frac{1}{n(n+1)} \le 2.$$

Exercise (comparison of positive term series)

Suppose that $a_n, b_n \ge 0$ $(n \ge 1)$ and that $M > 0$, $N \ge 1$ are such that $a_n \le M b_n$ $\forall\ n \ge N$.

Show that if $\sum_{n=1}^{\infty} b_n < \infty$, then $\sum_{n=1}^{\infty} a_n < \infty$.

Exercise: Prove the Heine Borel theorem

☆

Show that if a collection $\mathcal{S}$ of open sets covers a closed, bounded set E, then $\exists\ \mathcal{S}_0 \subset \mathcal{S}$ finite, which covers E.

`Corrected Hint`: Suppose otherwise, and show there is an infinite Chinese box of closed intervals (`boxes` if you're proving it in $\mathbb{C}$) $I_n \supset I_{n+1}$, $|I_n| = \frac{|I|}{2^n}$, each one of which intersects with E and no one of which is covered by a finite subcollection of $\mathcal{S}$. Show that the "Chinese box intersection" is a singleton subset of E and obtain a contradiction.

Next topics

ABSOLUTE CONVERGENCE, EXPONENTIAL SERIES, ROOT TEST, CONDENSATION TEST, POWER SERIES.

Lecture #11[11]

Absolute convergence of series (התכנסות בהחלט של טורים)

The series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* (הטור מתכנס בהחלט) if $\sum_{n=1}^{\infty} |a_n| < \infty$. This implies convergence.

Theorem

If $\sum_{n=1}^{\infty} |a_n| < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges .

Proof We prove that $(s_1, s_2, \dots)$ is a Cauchy sequence where $s_n := \sum_{j=1}^{n} a_j$.

The assumptions imply that $(t_1, t_2, \dots)$ is a Cauchy sequence where $t_n := \sum_{j=1}^{n} |a_j|$. Given $\epsilon > 0$, let $N_\epsilon \in \mathbb{N}$ be such that

$$t_{n+k} - t_n = \sum_{j=n+1}^{n+k} |a_j| < \epsilon \ \forall \ n \geq N_\epsilon, \ k \geq 1.$$

It follows that

$$|s_{n+k} - s_n| = |\sum_{j=n+1}^{n+k} a_j| \leq \sum_{j=n+1}^{n+k} |a_j| = t_{n+k} - t_n < \epsilon \ \forall \ n \geq N_\epsilon, \ k \geq 1$$

and $(s_1, s_2, \dots)$ is indeed a Cauchy sequence. □

Exercises

(i) Show that if $a_n \in \mathbb{R}$ $(n \geq 1)$ and $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$ with equality iff all nonzero a_n's have the same sign;

(ii) Now show that if $a_n \in \mathbb{C}$ $(n \geq 1)$ and $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$.

(iii)☆ Show equality occurs in in (ii) iff $\exists \ \lambda \in \mathbb{C}, \ \lambda = 1$ so that $a_n = \lambda |a_n| \ \forall \ n \geq 1$.

Proposition (convergence of exponential series)

For $x \in \mathbb{C}$, the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely.

Proof It suffices to show that

$$\forall \ x \in \mathbb{R}, \ \exists \ M, Q \geq 1 \ \text{ such that } \ |\frac{x^k}{k!}| \leq \frac{M}{2^k} \ \forall \ k \geq Q.$$

To see this, fix $Q > 2|x|$. For $n \geq Q$, we have

$$|\frac{x^n}{n!}| = \frac{|x|}{1} \dots \frac{|x|}{(Q-1)} \frac{|x|}{Q} \dots \frac{|x|}{n} \leq \frac{|x|^{Q-1}}{(Q-1)!} \frac{1}{2^{n-Q+1}} = \frac{(2|x|)^{Q-1}}{(Q-1)!} \frac{1}{2^n} =: \frac{M}{2^n}.$$

□

[11] 4/5/2017

Exponential series

Theorem e

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \;\; \forall \;\; x \in \mathbb{R}.$$

Proof By corollary e,

$$(1 + \frac{x}{n})^n \underset{n\to\infty}{\longrightarrow} e^x$$

and so it suffices to show that

$$(e) \qquad (1 + \frac{x}{n})^n \underset{n\to\infty}{\longrightarrow} \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall \;\; x \in \mathbb{R}.$$

Proof of (e) : Fix $x \in \mathbb{R}$ and let $L := \sum_{n=0}^{\infty} \frac{x^n}{n!} < \infty$ which converges absolutely by the proposition.

By the binomial theorem,

$$\begin{aligned}
\left(1 + \frac{x}{n}\right)^n &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{x^k}{n^k} \\
&= 1 + x + \sum_{k=2}^{n} \frac{n(n-1)\ldots(n-k+1)}{n^k} \frac{x^k}{k!} \\
&= 1 + x + \sum_{k=2}^{n} (1 - \frac{1}{n})(1 - \frac{2}{n})\ldots(1 - \frac{k-1}{n}) \frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty} b_{n,k} \frac{x^k}{k!}
\end{aligned}$$

where

$$b_{n,k} := \begin{cases} 1 & k = 0,\; 1 \\ (1 - \frac{1}{n})(1 - \frac{2}{n})\ldots(1 - \frac{k-1}{n}) & 2 \le k \le n \\ 0 & k > n. \end{cases}$$

We must show that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} - \sum_{k=0}^{\infty} b_{n,k} \frac{x^k}{k!} = \sum_{k=0}^{\infty} (1 - b_{n,k}) \frac{x^k}{k!} \underset{n\to\infty}{\longrightarrow} 0.$$

Fix $\epsilon > 0$. Since $\sum_{k \ge N} \frac{|x|^k}{k!} \underset{N\to\infty}{\longrightarrow} 0$, $\exists \; K = K_\epsilon$ so that

$$\sum_{k \ge K} \frac{|x|^k}{k!} < \frac{\epsilon}{2}.$$

- Since $0 \le b_{n,k} \le 1$, it follows that

$$|\sum_{k\ge K}(1-b_{n,k})\frac{x^k}{k!}| \le \sum_{k\ge K}(1-b_{n,k})\frac{|x|^k}{k!} < \frac{\epsilon}{2} \quad \forall\ n \ge 1.$$

- Since $b_{n,k} \xrightarrow[n\to\infty]{} 1\ \ \forall\ k \ge 1$, we have

$$\sum_{k=0}^{K}(1-b_{n,k})\frac{|x|^k}{k!} \xrightarrow[n\to\infty]{} 0$$

and

- $\exists\ Q_\epsilon > K$ so that

$$\sum_{k=0}^{K}(1-b_{n,k})\frac{|x|^k}{k!} < \frac{\epsilon}{2}\ \forall\ n \ge Q_\epsilon.$$

¶It follows that for $n > Q_\epsilon$:

$$\sum_{k=0}^{\infty}(1-b_{n,k})\frac{|x|^k}{k!}| \le \sum_{k=0}^{K}(1-b_{n,k})\frac{|x|^k}{k!} + \sum_{k=K+1}^{\infty}\frac{|x|^k}{k!} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad ☑$$

Corollary $e \notin \mathbb{Q}$.

Proof Suppose otherwise that $e \in \mathbb{Q}$ then since $2 < e < 3$, $e = \frac{p}{q}$ where $p,\ q \in \mathbb{N},\ q \ge 2$. It follows that $e = \frac{P}{q!}$ (with $P = p(q-1)!$), whence $e - \sum_{j=0}^{q}\frac{1}{j!} \ge \frac{1}{q!}$ (being a positive fraction with denominator $q!$).

Thus

$$\begin{aligned}\frac{1}{q!} &\le e - \sum_{j=0}^{q}\frac{1}{j!} \\ &= \sum_{j=q+1}^{\infty}\frac{1}{j!} \\ &= \frac{1}{(q+1)!}\left(1 + \sum_{j=q+2}^{\infty}\frac{1}{(q+2)\dots(j-1)j}\right) \\ &< \frac{1}{(q+1)!}\left(1 + \sum_{j=q+2}^{\infty}\frac{1}{4^{j-q-1}}\right) \\ &= \frac{1}{(q+1)!}\sum_{j=0}^{\infty}\frac{1}{4^j} \\ &= \frac{4}{3}\frac{1}{(q+1)!} \le \frac{4}{9}\frac{1}{q!}. \qquad \boxtimes\end{aligned}$$

Tests for convergence of series

Cauchy's Root test

Suppose that $a_n \geq 0$.

1) If $\limsup_{n\to\infty} a_n^{\frac{1}{n}} < 1$, then $\sum_{n=1}^{\infty} a_n < \infty$.

2) If $\limsup_{n\to\infty} a_n^{\frac{1}{n}} > 1$, then $a_n \nrightarrow 0$.

Proof

1) If $\limsup_{n\to\infty} a_n^{\frac{1}{n}} < 1$, then $\exists\ q \in (0,1),\ N_0$ such that $a_n^{\frac{1}{n}} \leq q\ \forall\ n \geq N_0$, whence $a_n \leq q^n\ \forall\ n \geq N_0$ and $\sum_{n=1}^{\infty} a_n < \infty$ by comparison with $\sum_{n=1}^{\infty} q^n$

2) If $\limsup_{n\to\infty} a_n^{\frac{1}{n}} > 1$, then $\exists\ R > 1$ and $n_k \to \infty$ such that $a_{n_k}^{\frac{1}{n_k}} > R\ \forall\ k$, whence $a_{n_k} > R^{n_k} \to \infty$. □

D'Alembert's ratio theorem ($\overline{\lim}$ `version`). *Suppose that $a_n > 0$ for large $n \in \mathbb{N}$.*

(i) $\limsup_{n\to\infty} a_n^{\frac{1}{n}} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$.

(ii) $\liminf_{n\to\infty} a_n^{\frac{1}{n}} \geq \liminf_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Proof

(i) If $a_N > 0$ and $\frac{a_{n+1}}{a_n} \leq q$ for $n \geq N$ then

$$a_n = a_N \frac{a_{N+1}}{a_N} \dots \frac{a_n}{a_{n-1}} \leq a_N q^{n-N} = Mq^n$$

with $M > 0$, whence $a_n^{\frac{1}{n}} \leq qM^{\frac{1}{n}} \to q$.

(ii) If $a_N > 0$ and $\frac{a_{n+1}}{a_n} \geq r$ for $n \geq N_0$ then

$$a_n = a_N \frac{a_{N+1}}{a_N} \dots \frac{a_n}{a_{n-1}} \geq a_N r^{n-N} = Mr^n$$

with $M := a_N r^{-N} > 0$; whence $a_n^{\frac{1}{n}} \geq rM^{\frac{1}{n}} \to r$. □

Cauchy's condensation test

Suppose that $a_n \geq a_{n+1} \downarrow 0$, then

$$\sum_{n=1}^{\infty} a_n < \infty \iff \sum_{n=1}^{\infty} 2^n a_{2^n} < \infty.$$

Proof

⇒)

$$\infty > \sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k-1} a_n \geq \sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k-1} a_{2^k} = \frac{1}{2} \sum_{k=1}^{\infty} 2^k a_{2^k}.$$

⇐)

$$\sum_{n=1}^{\infty} a_n = \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} a_n \leq \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} a_{2^k} = \sum_{k=0}^{\infty} 2^k a_{2^k} < \infty.$$

□

Corollary

Let $t > 0$, then

$$\sum_{n=1} \frac{1}{n^t} < \infty \iff t > 1.$$

Exercise

For which $t > 0$ is it true that $\sum_{n\geq 1} \frac{1}{n \log(n+e)^t} < \infty$?

Next topics

POWER SERIES, RADIUS OF CONVERGENCE & LIPSCHITZ PROPERTY, LEIBNIZ'S THEOREM ON CONDITIONAL CONVERGENCE, CAUCHY & HEINE DEFINITIONS OF LIMITS OF FUNCTIONS OF A REAL VARIABLE.

Lecture # 12
12

Power series and radius of convergence

Cauchy-Hadamard Theorem
Let $a_n \in \mathbb{C}$ $(n \geq 0)$ *and set*

$$R := \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}} \in [0, \infty].$$

a) If $|x| < R$*, then the series* $\sum_{n=1}^{\infty} a_n x^n$ *converges absolutely, and*

b) if $|x| > R$*, then the series* $\sum_{n=1}^{\infty} a_n x^n$ *diverges.*

Proof Root test. □

The series $\sum_{n=1}^{\infty} a_n x^n$ is known as a *power series* and

$$R := \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}} \in [0, \infty]$$

is known as its *radius of convergence*.

By the Cauchy-Hadamard theorem, the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $|x| < R$ and diverges for $|x| > R$.

Examples.

- The radius of convergence of $\sum_{n=1}^{\infty} n^n x^n$ is $R = 0$, since $a_n^{\frac{1}{n}} = n \to \infty$. Consequently $\sum_{n=1}^{\infty} n^n x^n$ converges only for $x = 0$.
- By (e), $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges $\forall\ x \in \mathbb{C}$ and the radius of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is $R = \infty$.

To calculate R directly, we use D'Alembert's ratio theorem:
if $b_n := \frac{1}{n!}$, then $\frac{b_{n+1}}{b_n} = \frac{1}{n+1} \to 0$, whence $b_n^{\frac{1}{n}} \to 0$ and $R = \infty$.

- The radius of convergence of $\sum_{n=1}^{\infty} \binom{2n}{n} x^n$ is $R = \frac{1}{4}$.

Set $A_n := \binom{2n}{n}$, then $\frac{A_{n+1}}{A_n} = \frac{(2n+1)(2n+2)}{(n+1)^2} \to 4$ whence

$$R := \frac{1}{\limsup A_n^{\frac{1}{n}}} = \frac{1}{4}.$$

The convergence of $\sum_{n=1}^{\infty} a_n (\pm R)^n$ depends on a_n.

[12] 7/5/2017

If $a_n = \frac{1}{n^t}$ where $t > 0$, then $R = 1$ but the convergence of $\sum_{n=1}^{\infty} a_n$ depends on t. To see this use condensation. See exercises below.

Proposition (`Radius of convergence of derived power series`)

For $\theta \in \mathbb{R}$, the radius of convergence R' of the power series $\sum_{n=1}^{\infty} n^{\theta} a_n x^n$ is the same as R, the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$.

Proof Write $A_n := |a_n|$ & $B_n := n^{\theta}|a_n|$, then $n^{\frac{\theta}{n}} \to 1$ and so

$$\frac{B_n^{\frac{1}{n}}}{A_n^{\frac{1}{n}}} = n^{\frac{\theta}{n}} \xrightarrow[n\to\infty]{} 1$$

whence

$$\limsup A_n^{\frac{1}{n}} = \limsup B_n^{\frac{1}{n}}$$

and

$$R := \frac{1}{\limsup A_n^{\frac{1}{n}}} = \frac{1}{\limsup B_n^{\frac{1}{n}}} =: R'. \quad ☑$$

Functions defined by power series

Proposition (`Lipschitz property of power series`) *Suppose that the power series $S(x) := \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. For each $0 < r < R$,*

$$M_r := \sum_{n=1}^{\infty} n|a_n| r^{n-1} < \infty$$

and

(`Lip`) $$|S(x) - S(y)| \le M_r |x - y| \quad \forall\ x,\ y \in \overline{N(0,r)}.$$

Note that $\overline{N(a,R)} = \{z :\ |z - a| \le R\}$ – the closure of $N(a,R)$.

Proof The power series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ also has radius of convergence R and so $M_r < \infty$.

To see (`Lip`), for $x, y \in \overline{N(0,r)}$,

$$|x^n - y^n| = |x - y| \left|\sum_{k=0}^{n-1} x^k y^{n-1-k}\right| \le n r^{n-1} |x - y|$$

whence

$$\begin{aligned}|S(x)-S(y)| &= |\sum_{n=1}^{\infty} a_n(x^n-y^n)| \\ &\le \sum_{n=1}^{\infty}|a_n|\cdot|x^n-y^n| \\ &\le |x-y|\sum_{n=1}^{\infty}|a_n|nr^{n-1} \\ &= M_r|x-y|. \quad ☑\end{aligned}$$

Corollary (`continuity of power series`) *Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R>0$, then $S: N(0,R)\to\mathbb{R}$ defined by $S(x) := \sum_{n=0}^{\infty} a_n x^n$ is continuous.*

Proof To see continuity at $x\in N(0,R)$, fix $r\in(|x|,R)$. By the proposition, f is `Lip` on $\overline{N(0,r)}$, whence, by Lipschitz's theorem, continuous at $x\in N(0,r)$. ☑

Conditional convergence. The series $\sum_{n=1}^{\infty} a_n$ is said to *converge conditionally* if (i) $\sum_{n=1}^{\infty} a_n$ converges and (ii) $\sum_{n=1}^{\infty}|a_n|=\infty$.

Leibniz's Theorem

Suppose that $a_n\ge a_{n+1}\downarrow 0$, then the series $\sum_{n=1}^{\infty} a_n(-1)^{n+1}$ converges and $0<\sum_{n=1}^{\infty} a_n(-1)^{n+1}<a_1$.

Proof Set $S_n := \sum_{k=1}^{n} a_n(-1)^{n+1}$, then

(a) $Z_n := S_{2n} = \sum_{k=1}^{n}(a_{2k-1}-a_{2k})$ and so $0\le Z_n\le Z_{n+1}$; and

(b) $Z_n = S_{2n} = a_1-\sum_{k=1}^{n-1}(a_{2k}-a_{2k+1})-a_{2n}\le a_1\ \forall\ n\ge 1$.

It follows from (a) and (b) that

$$S_{2n}\underset{n\to\infty}{\longrightarrow} S\in[0,a_1] \tag{☎}$$

Since $a_k\to 0$ we have that $S_{2n+1}=S_{2n}+a_{2n+1}\to S$. This proves the theorem. □

Leibniz's theorem shows that the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^t}$$

converges $\forall\ t>0$. Using the condensation test, we see that the convergence is absolute for $t>1$ and conditional for $0<t\le 1$.

Exercise: Convergence of power series at the endpoints of the interval of convergence

The radius of convergence of the power series $S(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^t}$ is $R = 1\ \ \forall\ t \in \mathbb{R}$. Show that

(i) for $t > 1$, $S(x)$ converges absolutely for $x = \pm 1$;

(ii) for $t \in (0,1]$ $S(x)$ converges conditionally for $x = -1$ and diverges for $x = 1$;

(iii) for $t \in (-\infty, 0]$ $S(x)$ diverges for $x = \pm 1$.

Exercises

¶1 Show that

(i) $\sum_{n=1}^{\infty} \frac{1}{n(\log(1+n))^\beta} < \infty$ iff $\beta > 1$;

(ii) If $a_n \ge a_{n+1} \to 0$ and $\sum_{n=1}^{\infty} a_n = \infty$, then $\sum_{n=1}^{\infty} \min\{a_n, \frac{1}{n}\} = \infty$.

(iii)* If $a_n > 0$ and $S_n := \sum_{k=1}^{n} a_k \underset{n\to\infty}{\longrightarrow} \infty$ then $\sum_{n=1}^{\infty} \frac{a_n}{S_n^\beta} < \infty$ iff $\beta > 1$.

¶2

(a) Show that in the situation of Leibniz's theorem
$\forall\ m \ge 1,\ 0 < (-1)^m \sum_{n=m+1}^{\infty} a_n (-1)^{n+1} < a_{m+1}$.

(b) Using Leibniz's theorem `(or otherwise)`, show that:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \ge \sum_{k=0}^{2N-1} \frac{x^k}{k!}\ \ \forall\ |x| \le 2N+1,\ N \in \mathbb{N}.$$

Limits of values of functions

Let $f : (a,b) \to \mathbb{R}$ and suppose that $A \subset (a,b),\ c \in A'$.

Heine's definition of limit (sequences).

We say that $f(x) \xrightarrow[x\to c,\ x\in A]{\text{Heine}} L$ if

$$x_n \in A,\ x_n \to c \implies f(x_n) \to L.$$

Cauchy's definition of limit ($\epsilon - \delta$).

We say that $f(x) \xrightarrow[x\to c,\ x\in A]{\text{Cauchy}} L$ if

$\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in A,\ |x - c| < \delta$.

Equivalence Theorem

Let $f : (a,b) \to \mathbb{R}$ and suppose that $A \subset (a,b),\ c \in A'$, then

$$f(x) \xrightarrow[x\to c,\ x\in A]{\text{Heine}} L \iff f(x) \xrightarrow[x\to c,\ x\in A]{\text{Cauchy}} L.$$

Proof of $\Leftarrow$)

Suppose that $f(x) \xrightarrow[x\to c,\ x\in A]{\text{Cauchy}} L$, and let $x_n \in A,\ x_n \to c$, fixing $\epsilon > 0$. By assumption, $\exists\ \delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in A,\ |x - c| < \delta$. Since $x_n \to c$, $\exists\ N_\delta$ so that $|x_n - c| < \delta\ \forall\ n \ge N_\delta$. It follows that $|f(x_n) - L| < \epsilon\ \forall\ n \ge N_\delta$.

proof of ⇒)

Suppose that $f(x) \underset{x\to c,\ x\in A}{\overset{\text{Heine}}{\longrightarrow}} L$ but not $f(x) \underset{x\to c,\ x\in A}{\overset{\text{Cauchy}}{\longrightarrow}} L$, i.e. that it is not true that

$$\forall\ \epsilon > 0,\ \exists\ \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ when } x \ni A,\ |x - c| < \delta;$$

then

$\exists\ \epsilon > 0$ such that $\forall\ \delta > 0,\ \exists\ x(\delta) \in (c-\delta, c+\delta) \cap A$ with $|f(x(\delta)) - L| \geq \epsilon$. In particular, if $x_n := x(\frac{1}{n})$, then $x_n \in A,\ x_n \to c$ (since $|x_n - c| < \frac{1}{n}$), and $|f(x_n) - L| \geq \epsilon\ \forall\ n \geq 1$ contradicting $f(x) \underset{x\to c,\ x\in A}{\overset{\text{Heine}}{\longrightarrow}} L$. $\square$

Accordingly, we write $f(x) \underset{x\to c,\ x\in A}{\longrightarrow} L$, and $L = \lim_{x\to c,\ x\in A} f(x)$.

The equivalence theorem provides us with useful tools.

Continuity proposition *The function* $f : N(a,r) \to \mathbb{C}$ *is continuous at* $z \in N(a,r)$ *iff* $f(x) \xrightarrow[x\to z,\ x\in N(a,c)]{} f(z)$.

Proof By definition, f is continuous at $z \in N(a,r)$ iff

$$f(x) \xrightarrow[x\to z,\ x\in N(a,c)]{\texttt{Heine}} f(z).\ ☑$$

Proposition *Suppose that* $f_1, f_2, \dots, f_K : N(c,r) \to \mathbb{R}$ *are continuous on* $N(c,r)$. *Define* $g : N(c,r) \to \mathbb{R}$ *by* $g(x) := \min\{f_j(x) :\ 1 \le j \le K\}$, *then* $g : N(c,r) \to \mathbb{R}$ *is continuous on* $N(c,r)$.

Proof

To show continuity of g at $x \in N(c,r)$ it suffices, given $\epsilon > 0$ to find $\delta > 0$ so that

$$z \in N(c,r),\ |z - x| < \delta \implies |g(z) - g(x)| < \epsilon.$$

Let $\Sigma(x) := \{1 \le j \le K :\ f_j(x) = g(x)\}$. By possibly shrinking $\epsilon > 0$, we can ensure that

$$f_\nu(x) > g(x) + \epsilon \quad \forall\ \nu \notin \Sigma(x)$$

(even if there are no such ν). Next, by continuity (Cauchy version), $\exists\ \delta > 0$ so that

$$z \in N(c,r),\ |z - x| < \delta \implies |f_j(z) - f_j(x)| < \frac{\epsilon}{4} \quad \forall\ 1 \le j \le K.$$

Fix $z \in N(c,r)$, $|z - x| < \delta$. We claim that $g(z) = f_j(z)$ for some $j \in \Sigma(x)$.

To see this, fix $k \in \Sigma(x)$ & $\nu \notin \Sigma(x)$, then

$$\begin{aligned} f_\nu(z) &> f_\nu(x) - \frac{\epsilon}{4} \quad \text{by continuity} \\ &> g(x) + \frac{3\epsilon}{4} \quad \text{because } \nu \notin \Sigma(x) \\ &= f_k(x) + \frac{3\epsilon}{4} \quad \text{because } k \in \Sigma(x) \\ &> f_k(z) - \frac{\epsilon}{4} + \frac{3\epsilon}{4} > f_k(z) \end{aligned}$$

showing that $f_\nu(z) > g(z)$.

Accordingly, suppose that $g(z) = f_j(z)$ where $j \in \Sigma(x)$, then

$$|g(z) - g(x)| = |f_j(z) - f_j(x)| < \frac{\epsilon}{4} < \epsilon. \quad \boxtimes$$

`Next topics`

Limits as $x \to \infty$, arithmetic operations and limits, continuity points, continuity of series, Takagi-Rudin function.

Lecture #13[13]

Proposition *Let $a < b$ & $f : (a,b) \to \mathbb{R}$. Suppose that $A_k \subset (a,b)$ $(1 \le k \le K)$ and let $A := \bigcup_{k=1}^{K} A_k$. Suppose that $t \in A_k' \setminus (a,b)$ $\forall\ 1 \le k \le K$.*

If $L \in \mathbb{R}$ and

$$f(x) \xrightarrow[x \to\ t,\ x \in A_k]{} L\ \forall\ 1 \le k \le K,$$

then

$$f(x) \xrightarrow[x \to\ t,\ x \in A]{} L.$$

Proof Fix $\epsilon > 0$. By assumption, for each $1 \le k \le K$, $\exists\ \delta_k > 0$ so that

$$x \in A_k,\ |x - t| < \delta_k \implies |f(x) - L| < \epsilon.$$

Now suppose that

$$x \in A\ \&\ |x - t| < \varDelta := \min\{\delta_k :\ 1 \le k \le K\}.$$

For some $1 \le k \le K$,

$$x \in A_k,\ |x - t| < \varDelta \le \delta_k \implies |f(x) - L| < \epsilon.\ \boxtimes$$

One sided limits. Let $f : (a,b) \to \mathbb{R}$ and let $c \in (a,b]$.

Write $f(x) \underset{x \to c-}{\longrightarrow} L$ if $f(x) \underset{x \to c,\ x \in (a,c)}{\longrightarrow} L$. In this case, write $L = f(c-) := \lim_{x \to c-} f(x)$.

Similarly for $c \in [a,b)$, write $f(x) \underset{x \to c+}{\longrightarrow} L$ if $f(x) \underset{x \to c,\ x \in (c,b)}{\longrightarrow} L$ and in this case, write $L = f(c+) := \lim_{x \to c+} f(x)$.

Two sided and unrestricted limits. Let $f : (a,b) \to \mathbb{R}$ and suppose $c \in (a,b)$.

Write $f(x) \underset{x \to c}{\longrightarrow} L$ if $f(x) \underset{x \to c,\ x \in (a,b)}{\longrightarrow} L$. In this case, write $L := \lim_{x \to c} f(x)$.

Analogously, $f(x) \underset{x \to c,\ x \ne c}{\longrightarrow} L$ if $f(x) \underset{x \to c,\ x \in (a,c) \cup (c,b)}{\longrightarrow} L$.

Proposition

Let $f : (a,b) \to \mathbb{R}$ and suppose $c \in (a,b)$, then

$$f(x) \underset{x \to c,\ x \ne c}{\longrightarrow} L \iff f(x) \underset{x \to c-}{\longrightarrow} L\ \&\ f(x) \underset{x \to c+}{\longrightarrow} L.$$

Proof Follows from the proposition above.

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[13] 11/5/2017

Note that

$$f(x) \xrightarrow[x\to c]{} L \iff f(x) \xrightarrow[x\to c,\ x\ne c]{} L \ \ \& \ L = f(c).$$

$\lim_{x\to\infty}$.

Suppose $f:(a,\infty)\to\mathbb{R},\ A\subset(a,\infty)$ is unbounded and $L\in\mathbb{R}$.

- $f(x) \xrightarrow[x\to\infty,\ x\in A]{\text{Heine}} L$ if $x_n\in A,\ x_n\to\infty \implies f(x_n)\to L$;
- $f(x) \xrightarrow[x\to\infty,\ x\in A]{\text{Cauchy}} L$ if $\forall\ \epsilon>0,\ \exists\ M>0$ such that $|f(x)-L|<\epsilon\ \forall\ x>M,\ x\in A$.

In this situation, we have

Proposition

$$f(x) \xrightarrow[x\to\infty,\ x\in A]{\text{Heine}} L \iff f(x) \xrightarrow[x\to\infty,\ x\in A]{\text{Cauchy}} L.$$

Proof Exercise.

Proposition

(i) $$(1+\tfrac{1}{x})^x \xrightarrow[x\to\infty]{} e.$$

(ii) $$(1+\tfrac{1}{x})^x \xrightarrow[x\to-\infty]{} e.$$

(iii) $$(1+x)^{\frac{1}{x}} \xrightarrow[x\to 0,\ x\ne 0]{} e.$$

Proof

Statement (i) is lemma e on p. 51.

We use (i) to show that $(1+x)^{\frac{1}{x}} \xrightarrow[x\to 0,\ x>0]{} e$.

Suppose that $x_n>0,\ x_n\to 0$ and let $a_n := \frac{1}{x_n}$, then $a_n\to\infty$ and

$$(1+x_n)^{\frac{1}{x_n}} = (1+\frac{1}{a_n})^{a_n} \to e. \ \ ☑$$

It suffices to show that $(1-x)^{\frac{1}{x}} \xrightarrow[x\to 0+]{} e^{-1}$.

To establish (ii)

$$(1-\tfrac{1}{x})^x \xrightarrow[x\to\infty]{} e^{-1},$$

we show that

$$a_n\to\infty \implies \left(a_n-\frac{1}{a_n}\right)^{a_n} \xrightarrow[n\to\infty]{} e^{-1}.$$

Indeed $(1-\frac{1}{n})^{n\pm 1} \underset{n\to\infty}{\longrightarrow} e^{-1}$ whence, constructing a sandwich (as in lemma e),

$$e^{-1} \underset{n\to\infty}{\longleftarrow} \left(1-\frac{1}{[a_n]}\right)^{[a_n]+1} < \left(1-\frac{1}{a_n}\right)^{a_n} < \left(1-\frac{1}{[a_n]+1}\right)^{[a_n]} \underset{n\to\infty}{\longrightarrow} e^{-1}$$

establishing (ii). ☑

Lastly, we use (ii) to show that $(1+x)^{\frac{1}{x}} \underset{x\to 0,\ x<0}{\longrightarrow} e$.

Suppose that $x_n < 0,\ x_n \to 0$ and let $a_n := \frac{1}{x_n}$, then $a_n \to -\infty$ and

$$(1+x_n)^{\frac{1}{x_n}} = \left(1+\frac{1}{a_n}\right)^{a_n} \to e$$

by (ii). ☑

Statement (iii) now follows from
$(1+x)^{\frac{1}{x}} \underset{x\to 0,\ x>0}{\longrightarrow} e$ and $(1+x)^{\frac{1}{x}} \underset{x\to 0,\ x<0}{\longrightarrow} e$. □

Arithmetic operations on limits

Let $f, g : (a,b) \to \mathbb{R},\ c \in (a,b)$ and let $A \subset (a,b),\ c \in A'$. Suppose that $f(x) \underset{x\to c,\ x\in A}{\longrightarrow} L$ and $g(x) \underset{x\to c,\ x\in A}{\longrightarrow} M$, then

$$(f+g)(x) \underset{x\to c,\ x\in A}{\longrightarrow} L+M,$$

$$(fg)(x) \underset{x\to c,\ x\in A}{\longrightarrow} LM,$$

and, in case $M \neq 0$:

$$\frac{f}{g}(x) \underset{x\to c,\ x\in A}{\longrightarrow} \frac{L}{M}.$$

Proof Follows from the Heine definition of limit and analogous propositions for limits of sequences.

Continuity on an interval. Let J be an interval. The function $f : J \to \mathbb{R}$ is *continuous at $L \in J$* if $f(x) \underset{x\to L,\ x\in[a,b]}{\longrightarrow} f(L)$.

This definition corresponds with previous definitions in case L is in the interior of J (i.e. not an endpoint of J).

- The collection of continuous functions on the interval J is denoted by $C(J)$.

Example.

Define $f : (-1,1) \to \mathbb{R}$ by $f(x) = (1+x)^{\frac{1}{x}}$ when $x \neq 0$ and define $f(0) := e$. By the exponential continuity proposition f is continuous at each $0 \neq x \in (-1,1)$ and (iii) above shows that f is continuous at 0. Thus $f \in C((-1,1))$.

Proposition

Let $f, g : (a,b) \to \mathbb{R},\ c \in (a,b)$. If f and g are continuous at c, then so are $f+g$ and fg. If in addition $g(c) \neq 0$, then $\frac{f}{g}$ is also continuous at c.

Continuity points of functions

Suppose that $f : (a,b) \to \mathbb{R}$. A *continuity point of f* is a point $c \in (a,b)$ satisfying $f(x) \xrightarrow[x\to c]{} f(c)$. Let $C_f := \{\texttt{continuity points of } f\}$.

Example: No continuity points.

Let $D : \mathbb{R} \to \mathbb{R}$ be defined by

$$D(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Here, $C_f = \varnothing$ as $\forall\ c \in \mathbb{R}$,

$$\nexists \lim_{x\to c} f(x).$$

To see this, $\exists\ x_n \to c,\ x_n \in \mathbb{Q}$ and $f(x_n) \equiv 1 \to 1$ but also $\exists\ y_n \to c,\ y_n \notin \mathbb{Q}$ and $f(y_n) \equiv 0 \to 0$.

Example: Continuity points. $= \mathbb{R} \setminus Q$

Let $\mathfrak{q} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\mathfrak{q}(x) = \begin{cases} 1 & x = 0, \\ \frac{1}{q} & x = \frac{p}{q},\ p \in \mathbb{Z} \setminus \{0\},\ q \in \mathbb{N},\ \gcd(p,q) = 1, \\ 0 & x \notin \mathbb{Q}, \end{cases}$$

Proposition

$$C_{\mathfrak{q}} = \mathbb{R} \setminus \mathbb{Q}.$$

Proof

If $c \in \mathbb{Q}$, then $\mathfrak{q}(c) > 0$ and $\exists\ x_n \notin \mathbb{Q},\ x_n \to c$ whence $0 = \mathfrak{q}(x_n) \not\to \mathfrak{q}(c)$ and $\mathfrak{q}$ is not continuous at c.

If $c \notin \mathbb{Q}$, then $\mathfrak{q}(c) = 0$ and if $x_n = \frac{p_n}{q_n} \in \mathbb{Q},\ \gcd(p_n, q_n) = 1,\ x_n \to c$, then $q_n \to \infty$ and $\mathfrak{q}(x_n) = \frac{1}{q_n} \to 0 = \mathfrak{q}(c)$. Thus if $x_n \to c$, then $\mathfrak{q}(x_n) \to 0 = \mathfrak{q}(c)$ (as above for the rational subsequence, and $\mathfrak{q}(x_n) = 0$ on the irrational subsequence). This shows that $\mathfrak{q}$ is continuous at c. $\square$

Exercises

(i) Show that $\forall\ x \in \mathbb{R},\ \mathfrak{q}(y) \xrightarrow[y\to x,\ y\neq x]{} 0$.

(ii) Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a bounded function with the property that $\forall\ x \in \mathbb{R},\ F(y) \xrightarrow[y\to x,\ y\neq x]{} 0$. Show that $\exists\ x \in \mathbb{R}$ with $F(x) = 0$.

`Hint` $\forall\ \epsilon > 0,\ \{x \in \mathbb{R} :\ |F(x)| \geq \epsilon\}' = \varnothing$.

(iii)* Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a bounded function with the property that $\forall\ x \in \mathbb{R},\ \exists\ \lim_{y\to x,\ y\neq x} F(y)$. Is there a point at which F is continuous?

Convergence of sequences of series

Example 1. For $N,\ k \geq 1$, let

$$a_k(N) := (1 - \tfrac{1}{N}) \cdot \frac{1}{2^k},$$

then

$$a_k(N) \xrightarrow[N\to\infty]{} a_k := \frac{1}{2^k}\ \forall\ k \geq 1$$

and

$$\sum_{k=1}^{\infty} a_k(N) = (1 - \tfrac{1}{N}) \sum_{k=1}^{\infty} \frac{1}{2^k} = (1 - \tfrac{1}{N})$$
$$\xrightarrow[N\to\infty]{} 1 = \sum_{k=1}^{\infty} a_k.$$

This is a situation where $a_k(N),\ a_k \geq 0,\ a_k(N) \xrightarrow[N\to\infty]{} a_k\ \forall\ k \geq 1$, all the series

$$\sum_{k=1}^{\infty} a_k(N) \quad (N \geq 1)\ \&\ \sum_{k=1}^{\infty} a_k$$

converge to finite limits and

$$\sum_{k=1}^{\infty} a_k(N) \xrightarrow[N\to\infty]{} \sum_{k=1}^{\infty} a_k.$$

This is not always the case:

Example 2. Let

$$a_k(N) := \begin{cases} \frac{1}{N} & 1 \leq k \leq N; \\ 0 & k \geq N+1, \end{cases}$$

then

$$a_k(N) \xrightarrow[N\to\infty]{} a_k = 0\ \forall\ k \geq 1,$$

all the series converge to finite limits, but for each $N \geq 1$,

$$\sum_{k=1}^{\infty} a_k(N) = \sum_{k=1}^{N} \tfrac{1}{N} = 1 \nrightarrow 0 = \sum_{k=1}^{\infty} a_k.$$

Without more assumptions, one cannot ensure that

$$\sum_{k=1}^{\infty} a_k(N) \xrightarrow[N\to\infty]{} \sum_{k=1}^{\infty} a_k.$$

However,

Discrete Fatou Lemma

Suppose that for each $N,\ k \geq 1,\ a_k(N) \geq 0$ and that

$$a_k(N) \xrightarrow[N\to\infty]{} a_k \ \forall\ k \geq 1,$$

then

$$\varliminf_{N\to\infty} \sum_{k=1}^{\infty} a_k(N) \geq \sum_{k=1}^{\infty} a_k.$$

Note that here, the series, having non-negative terms, are not required to converge and may diverge to ∞.

Proof

WLOG, $\varliminf_{N\to\infty} \sum_{k=1}^{\infty} a_k(N) < \infty$. Fix $K \geq 1$, then

$$\begin{aligned}\sum_{k=1}^{K} a_k \xleftarrow[N\to\infty]{} & \sum_{k=1}^{K} a_k(N) \\ & \leq \sum_{k=1}^{\infty} a_k(N).\end{aligned}$$

It follows that

$$\mathtt{PL}\,((\sum_{k=1}^{\infty} a_k(N) :\ N \geq 1)) \subset [\sum_{k=1}^{K} a_k, \infty).$$

But $K \geq 1$ was arbitrary and

$$\varliminf_{N\to\infty} \sum_{k=1}^{\infty} a_k(N) \geq \sum_{k=1}^{K} a_k \xrightarrow[K\to\infty]{} \sum_{k=1}^{\infty} a_k. \quad ☑$$

`Next topics`

CONVERGENCE OF SEQUENCES OF SERIES, CONTINUITY OF SERIES, TAKAGI-RUDIN FUNCTIONS, INTERMEDIATE VALUE THEOREM, CONTINUITY OF COMPOSITIONS AND INVERSE FUNCTIONS, CONTINUOUS

FUNCTIONS ON CLOSED, BOUNDED INTERVALS; UNIFORM CONTINUITY, DIFFERENTIABILITY.

Lecture #14[14]

Dominated convergence theorem for series

Suppose that for each $N,\ k \geq 1,\ a_k(N) \in \mathbb{C}$, the series $\sum_{k=1}^{\infty} a_k(N)$ converges absolutely and that

$$\text{(a)} \qquad \sum_{k=1}^{\infty} \sup_{N \geq 1} |a_k(N)| < \infty \quad \&$$

$$\text{(b)} \qquad a_k(N) \xrightarrow[N\to\infty]{} a_k \ \forall\ k \geq 1,$$

then $\sum_{k=1}^{\infty} a_k$ converges absolutely and

$$\text{(c)} \qquad \sum_{k=1}^{\infty} a_k(N) \xrightarrow[N\to\infty]{} \sum_{k=1}^{\infty} a_k.$$

Proof

Set $M_k := \sup_{N\geq 1} |a_k(N)|$, then by the discrete Fatou lemma,

$$\sum_{k=1}^{\infty} |a_k| \leq \varliminf_{N\to\infty} \sum_{k=1}^{\infty} |a_k(N)| \leq \sum_{k=1}^{\infty} M_k < \infty.$$

To prove (c), we'll "break the error estimate" into a large initial finite sum and a tail sum. The finite sum is chosen so large so that the tail sum is small by the "`domination`" assumption (a). The finite sum converges by (b), wherever the "break"was made.

Let $\epsilon > 0$. Choose $K_\epsilon \geq 1$ so that

$$\sum_{k=K_\epsilon+1}^{\infty} M_k < \frac{\epsilon}{4}.$$

Now choose $N_\epsilon \geq K_\epsilon$ so that

$$|a_k(N) - a_k| < \frac{\epsilon}{2K} \quad \forall\ 1 \leq k \leq K_\epsilon,\ N \geq N_\epsilon.$$

[14] 14/5/2017

It follows that for $N \geq N_\epsilon$,

$$
\begin{aligned}
|\sum_{k=1}^{\infty} a_k(N) - \sum_{k=1}^{\infty} a_k| &\leq \sum_{k=1}^{K_\epsilon} |a_k(N) - a_k| + \sum_{k=K_\epsilon+1}^{\infty} (|a_k(N)| + |a_k|) \\
&\leq \sum_{k=1}^{K_\epsilon} |a_k(N) - a_k| + 2 \sum_{k=K_\epsilon+1}^{\infty} M_k \\
&< \sum_{k=1}^{K_\epsilon} \frac{\epsilon}{2K_\epsilon} + 2 \cdot \frac{\epsilon}{4} \\
&= \epsilon. \quad ☑
\end{aligned}
$$

Continuity of series

Theorem

Suppose that $u_n : [a, b] \to \mathbb{R}$ *are continuous, and that*

$$\sum_{n=1}^{\infty} \sup_{x \in [a,b]} |u_n(x)| < \infty$$

then
(i) *the series* $U(t) := \sum_{n=1}^{\infty} u_n(t)$ *converges absolutely* $\forall\ t \in [a, b]$;
(ii) *the function* $U : [a, b] \to \mathbb{R}$ *is continuous.*

Proof *of* (ii)

To see that U is continuous at $Z \in [a, b]$, let $z_N \in [a, b]$, $z_N \to Z$ and set $a_k(N) := u_k(z_N)$, then by continuity of each u_k,

$$a_k(N) = u_k(z_N) \to u_k(Z) =: a_k \quad \forall\ k \geq 1.$$

Next

$$\sum_{k \geq 1} \sup_{N \geq 1} |a_k(N)| \leq \sum_{k \geq 1} \sup_{x \in [a,b]} |u_k(x)| < \infty$$

and so by the dominated convergence theorem for series,

$$U(z_N) = \sum_{k \geq 1} a_k(N) \xrightarrow[N \to \infty]{} \sum_{k \geq 1} a_k = U(Z). \ ☑$$

In addition to power series, this also covers

Takagi-Rudin functions.

Let $\langle x \rangle := \min\{|x - 2n| : \ n \in \mathbb{Z}\}$.

Proposition (periodicity and $\texttt{Lip}$ **of** $x \mapsto \langle x \rangle$**)**

(i) $\langle x + 2n \rangle = \langle x \rangle \ \forall\ x \in \mathbb{R},\ n \in \mathbb{Z};$

(ii) $|\langle x \rangle - \langle y \rangle| \leq |x - y|.$

Proof of (ii) For $x, y \in \mathbb{R}$, $x < y \le x + 1$, there are two cases to consider:

$$\text{(a)} \quad (x, y) \cap \mathbb{Z} = \varnothing \;\&\; \text{(b)} \quad \exists\, n \in \mathbb{Z},\ x < n < y.$$

In case (a), $|\langle x\rangle - \langle y\rangle| = |x - y|$.

In case (b), write $x = n - \Delta$ & $y = n + \mathcal{E}$ where $\Delta,\ \mathcal{E} > 0$, $\Delta + \mathcal{E} = |y - x| \le 1$.

If n is even, then $\langle x\rangle = \Delta$, $\langle y\rangle = \mathcal{E}$ whence

$$|\langle x\rangle - \langle y\rangle| = |\mathcal{E} - \Delta| < \Delta + \mathcal{E} = |y - x|.$$

If n is odd, then $\langle x\rangle = x - (n-1) = 1 - \Delta$ and $\langle y\rangle = n + 1 - y = 1 - \mathcal{E}$ whence

$$|\langle x\rangle - \langle y\rangle| = |(1 - \Delta) - (1 - \mathcal{E})| = |\mathcal{E} - \Delta| < \Delta + \mathcal{E} = |y - x|.$$

For other $x, y \in \mathbb{R}$, $y = z + 2k$ where $|x - z| \le 1$ & $|k| \ge 1$.

We see that here

$$|z - x| \le |z + 2k - x|$$

whence

$$\begin{aligned}|\langle x\rangle - \langle y\rangle| &= |\langle x\rangle - \langle z\rangle|\\ &\le |z - x|\\ &\le |z + 2k - x| = |y - x|. \quad \boxtimes\end{aligned}$$

A `Takagi-Rudin function` is a function of form

$$x \mapsto \mathfrak{T}_{a,d}(x) = \sum_{n=1}^{\infty} a^n \langle d^n x\rangle$$

where $0 < a < 1$, $d > 0$.

Because of the shape of the graph of $T_{\frac{1}{2},2}$, Takagi-Rudin functions are aka `blancmange functions` (in English) and `courbes du pouding` (in French). For more information and graphics, see
`http://en.wikipedia.org/wiki/Blancmange_curve`.

Proposition

Any Takagi-Rudin function $\mathfrak{T}_{a,d}$ $(0 < a < 1,\ d > 0)$ *is continuous.*

Proof The functions $u_n : \mathbb{R} \to [0,1]$ $(n \ge 1)$ defined by $u_n(x) := a^n\langle d^n x\rangle$ are `LIP`, whence continuous. Moreover,

$$\sum_{n\ge1} \sup_{x\in\mathbb{R}} |u_n(x)| = \sum_{n\ge1} a^n < \infty.$$

Thus by the continuity of series theorem, $\mathfrak{T}_{a,d}\mathbb{R} \to \mathbb{R}$ is continuous. $\boxtimes$

`Takagi's function`[15] is $\mathfrak{T}_{\frac{1}{2},2}$ and `Rudin's function` [16] is $\mathfrak{T}_{\frac{3}{4},4}$.

Proposition R1

Rudin's function $T(x) = \mathfrak{T}_{\frac{3}{4},4}(x) = \sum_{n=1}^{\infty}(\frac{3}{4})^n\langle 4^n x\rangle$ *is continuous but not* `Lip` *on any subinterval of* $\mathbb{R}$.

Proof Continuity follows from the above proposition. To see that T is nowhere `LIP`, we'll need the following lemma.

Lemma R2 *For every* $Z \in \mathbb{R}$, $\exists\ \delta_N(Z) = \pm\frac{1}{2\cdot 4^N}$ *so that*

$$\frac{|T(Z+\delta_N) - T(Z)|}{|\delta_N|} \ \geq\ \frac{3^N}{2}. \tag{🚂}$$

Proof of (🚂)

For $N \geq 1$ write

$$T(x) = \sum_{n=1}^{N}(\frac{3}{4})^n\langle 4^n x\rangle + \sum_{n>N}(\frac{3}{4})^n\langle 4^n x\rangle =: T_N(x) + R_N(x). \tag{🛡}$$

For each $Z \in \mathbb{R}$, $\exists\ \omega(Z) = \pm 1$ so that there is no integer strictly between Z and $Z + \frac{\omega(Z)}{2}$.

This ensures that

$$|\langle Z + \tfrac{\omega(Z)}{2}\rangle - \langle Z\rangle| = \frac{1}{2}.$$

Now define

$$\delta_N(Z) := \frac{\omega(4^N Z)}{2\cdot 4^N} = \pm\frac{1}{2\cdot 4^N}$$

obtaining

$$|\langle 4^N(Z+\delta_N)\rangle - \langle 4^N Z\rangle| = |\langle 4^N Z + \tfrac{\omega(4^N Z)}{2}\rangle - \langle 4^N Z\rangle| = \frac{1}{2}.$$

For $n > N$,

$$\langle 4^n(Z+\delta_N)\rangle = \langle 4^n Z \pm 2^{2(n-N)-1}\rangle = \langle 4^n Z\rangle$$

(because $2(n-N)-1 \geq 1$ & $\langle x+2\rangle = \langle x\rangle$) whence

$$T(Z+\delta_N) = T_N(Z+\delta_N) + R_N(Z).$$

---

[15] Teiji Takagi, *A Simple Example of a Continuous Function without Derivative*, Proc. Phys. Math. Japan, (1903) Vol. 1, pp. 176-177.

[16] Walter Rudin, *Principles of mathematical analysis*, theorem 7.18.

It follows that

$$
\begin{aligned}
|T(Z+\delta_N)-T(Z)| &= |T_N(Z+\delta_N)-T_N(Z)| \\
&= \Big|\sum_{n=1}^{N}(\frac{3}{4})^n(\langle 4^n(Z+\delta_N)\rangle - \langle 4^n Z\rangle)\Big| \\
&\overset{\Delta \text{ inequality}}{\geq} (\frac{3}{4})^N|\langle 4^N(Z+\delta_N)\rangle - \langle 4^N Z\rangle| - \sum_{n=1}^{N-1}(\frac{3}{4})^n|\langle 4^n(Z+\delta_N)\rangle - \langle 4^n Z\rangle| \\
&\geq (\frac{3}{4})^N|4^N\delta_N| - \sum_{n=1}^{N-1}(\frac{3}{4})^n|4^n\delta_N| \quad \because \ |\langle x+h\rangle - \langle x\rangle| \leq |h|; \\
&= |\delta_N|(3^N - \sum_{n=1}^{N-1}3^n) \\
&> \frac{3^N}{2}|\delta_N|. \quad ☑
\end{aligned}
$$

Proof of nowhere `Lip` of T Suppose that $a, b \in \mathbb{R},\ a < b$ & $M > 0$ are such that

$$(⚔) \qquad |T(Z)-T(W)| \leq M|Z-W| \ \forall \ Z, W \in [a,b].$$

Fix $Z \in (a,b)$, then for large N, $Z + \delta_N(z) \in (a,b)$,

$$
\begin{aligned}
\infty \xleftarrow[N\to\infty]{} \ & \frac{3^N}{2} \\
&\leq \frac{|T(Z+\delta_N)-T(Z)|}{|\delta_N|} \quad \text{by lemma R2} \\
&\leq M \quad \text{by } (⚔). \quad \boxtimes
\end{aligned}
$$

Thus (⚔) leads to a contradiction and is therefore impossible. ☑

Remark. We'll see later from (🚂) that Rudin's function is `nowhere differentiable`.

Exercise: more on Rudin's function

Let $T = \mathfrak{T}_{\frac{3}{4},4}$, and for $N \in \mathbb{N}$, let $T_N,\ R_N$ be as in (⊘) (above).

(i) Show that $\exists\ M > 0$ so that

$$|T_N(x+h)-T_N(x)| \leq M3^N|h| =: M\mathcal{E}_{N,h} \ \& \ |R_N(x+h)-R_N(x)| \leq M(\frac{3}{4})^N =: M\Delta_{N,h} \ \forall \ x, h \in \mathbb{R},\ N \geq 1.$$

(ii) Show that if $h \neq 0$ & $N = N(h) := 2\log_2 \frac{1}{|h|}$, then

$$\mathcal{E}_{N,h} = \Delta_{N,h} = |h|^{1-\frac{\log 3}{\log 4}}.$$

(iii) Deduce that

$$|T(y)-T(x)| \leq 2M|y-x|^{1-\frac{\log 3}{\log 4}} \ \forall \ x,\ y \in \mathbb{R}.$$

(iv) Show that for every $x \in \mathbb{R}$, $N \in \mathbb{N}$ and for either $\delta_N = \frac{1}{2\cdot 4^N}$ or $\delta_N = -\frac{1}{2\cdot 4^N}$:

$$|T(x+\delta_N) - T(x)| \geq \frac{1}{4} \cdot |\delta_N|^{1-\frac{\log 3}{\log 4}}.$$

Exercise: `Lip` Rudin-Takagi functions

Let $0 < a < 1 < d < \frac{1}{a}$ and let $T(x) = \mathfrak{T}_{a,d}(x) = \sum_{n=0}^{\infty} a^n \langle d^n x\rangle$. Show that $\exists\ M = M_{a,d} \in \mathbb{R}_+$ so that

$$|T(x) - T(y)| \leq M|x-y| \quad \forall\ x,\ y \in \mathbb{R}.$$

For example, $\mathfrak{T}_{\frac{1}{4},2}(x) = 2x(1-x)$.

Next topics

INTERMEDIATE VALUE THEOREM; CONTINUOUS FUNCTIONS ON CLOSED, BOUNDED INTERVALS; UNIFORM CONTINUITY, DIFFERENTIABILITY.

Lecture #15 [17]

Continuity of inverse functions

Theorem
Suppose that $I \subset \mathbb{R}$ is a bounded, closed interval and that $f : I \to \mathbb{R}$ is continuous and strictly monotone, then $f : I \to f(I)$ is a bijection and $f^{-1} : f(I) \to I$ is continuous.

Proof
To check continuity of f^{-1} at $f(x)$ we must show that

$$x_n \in I, \quad f(x_n) \to f(x) \ \Rightarrow \ x_n \to x.$$

Suppose otherwise, that $f(x_n) \to f(x)$, $x_n \nrightarrow x$, then (!) $\exists\ \epsilon > 0$ and $n_k \to \infty$ so that $|x_{n_k} - x| \ge \epsilon$. By the Bolzano-Weierstrass theorem $\exists\ z \in \mathbb{R}$, $m_\ell = n_{k_\ell} \to \infty$ so that $x_{m_\ell} \to z$. Since I is closed, $z \in I$. Also $z \neq x$ since

$$|z - x| \ \leftarrow \ |x_{m_\ell} - x| \ \ge \ \epsilon.$$

By continuity, and assumption (respectively):

$$f(z) \ \leftarrow \ f(x_{m_\ell}) \ \to \ f(x).$$

Thus $f(x) = f(z)$ and since f is 1-1, $z = x$. Contradiction. ☐

The complex case. Let $F \subset \mathbb{C}$. A function $T : F \to \mathbb{C}$ is *continuous* if $z_n \in F,\ z_n \to z \in F \implies T(z_n) \to T(z)$.

Generalized Theorem *Let $F \subset \mathbb{C}$ a closed, bounded set and let $T : F \to \mathbb{C}$ continuous and 1-1, then $T^{-1} : T(F) \to F$ is continuous.*

Proof As above!

Theorem (Continuity of composition of functions)
Suppose that $I, J \subset \mathbb{R}$ are intervals, and that $f : I \to J$, $g : J \to \mathbb{R}$. Let $x \in I$.
If f is continuous at x and g is continuous at $f(x)$, then $g \circ f$ is continuous at x.

Proof We show that $g \circ f(y) \underset{y \to x}{\longrightarrow} g \circ f(x)$. Let $x_n \in I$, $x_n \to x$, then by continuity of f at x, $f(x_n) \to f(x)$. By continuity of g at $f(x)$,

$$g \circ f(x_n) = g(f(x_n)) \to g(f(x)) = g \circ f(x). \quad ☑$$

[17] 18/5/2017

Intermediate value property

We say that $f : (a,b) \to \mathbb{R}$ has the *intermediate value property (*`IVP`*)* if $f(J)$ is an interval $\forall$ intervals $J \subset (a,b)$

$\big($`i.e. if` $a < \alpha < \beta < b$ `and` x `is between` $f(\alpha)$ `and` $f(\beta)$`, then` $\exists\ c \in (\alpha,\beta)$ `with` $f(c) = x\big)$.

By Cauchy's theorem below, continuous functions have the `IVP`.

Cauchy's Intermediate value theorem **(IVT) (משפט ערך הביניים)**

The continuous image of an interval is an interval.

"`Explanation`" You can draw the graph of a function with the `IVP` without taking your pencil off the paper. (???)

Proof

Note that a bounded set $A \subset \mathbb{R}$ is an interval iff $(\texttt{GLB}\ A, \texttt{LUB}\ A) \subset A$.

Thus, it suffices to show that if $f : [a,b] \to \mathbb{R}$ is continuous, and that $f(a) < f(b)$, then

$$(\bigstar) \qquad \forall\ L \in (f(a), f(b)),\ \exists\ c \in (a,b) \text{ such that } f(c) = L.$$

Proof *of* $(\bigstar)$:

Let $f(a) < L < f(b)$. By $f(t) \underset{t\to a+}{\longrightarrow} f(a)$ and $f(t) \underset{t\to b-}{\longrightarrow} f(b)$, $\exists\ a < a' \le b' < b$ such that $f(t) < L\ \forall\ t \in (a,a')$ and $f(t) > L\ \forall\ t \in (b',b)$.

Define $c \in [a,b]$ by $c := \texttt{GLB}\{t \in [a,b] :\ f(t) > L\}$. By the above, $c \in [a',b'] \subset (a,b)$.

Proof that $f(c) = L$. On the one hand (by the properties of `GLB`): $\exists\ t_n \downarrow c$ such that $f(t_n) > L\ \forall\ n \ge 1$ whence by continuity

$$f(c) = \lim_{n\to\infty} f(t_n) \ge L;$$

whereas on the other hand, $\forall\ t_n \in (a,c),\ t_n \uparrow c,\ f(t_n) \le L\ \forall\ n \ge 1$. Again by continuity,

$$f(c) = \lim_{n\to\infty} f(t_n) \le L.$$

The conclusion is $f(c) = L$. ☐

Corollary *Every polynomial of odd degree has a real zero.*

Proof

Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N$ where $a_N \ne 0$ and $N = 2k+1$ for some $k \in \mathbb{Z}_* := \mathbb{N} \cup \{0\}$. We assume that $a_N > 0$ and write

$$P(x) = a_Nx^N + \cdots + a_1x + a_0 = x^N \cdot \frac{P(x)}{x^N}.$$

Now for $|x| > 1$,

$$\left|\frac{P(x)}{x^N} - a_N\right| = \left|a_{N-1}\frac{1}{x} + a_{N-2}\frac{1}{x^2} + \cdots + a_0\frac{1}{x^N}\right| \le \frac{M}{|x|}$$

where $M = \sum_{k=0}^{N-1} |a_k|$.

Consequently,

$$\frac{P(x)}{x^N} \xrightarrow[|x|\to\infty]{} 1$$

and for $M > 0$ large

$$\frac{P(x)}{x^N} \ge \frac{1}{2} \ \forall \ |x| \ge M.$$

For such $M > 0$ $P(-M) < 0 < P(M)$. By continuity of P and the IVT, $\exists\ z \in (-M, M)$ such that $P(z) = 0$. ☑

Note that a polynomial of even degree may have no real zeros (e.g. $P(x) = x^2 + 1$).

The `fundamental theorem of algebra` (which we'll prove later) says that *every polynomial of degree* $N \ge 1$ *has a complex* zero.

Example: *"continuous" functions on* $\mathbb{Q}$ *need not have* `IVP`.

Define $f : \mathbb{Q} \to \{0, 1\}$ by

$$f(x) := \begin{cases} 1 & x^2 < 2; \\ 0 & \text{else,} \end{cases}$$

then $f : \mathbb{Q} \to \mathbb{R}$ is continuous in the sense that

$$x_n,\ x \in \mathbb{Q},\ x_n \to x \implies f(x_n) \to f(x).$$

However $f(-3, 3) = \{0, 1\}$ is not an interval in $\mathbb{Q}$.

Example: *nowhere continuous functions on* $[0, 1]$ *may have* `IVP`.

To define a suitable function, expand $x \in [0, 1]$ into `binary expansion`:

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n(x)}{2^n} \quad \text{where } \epsilon_n = 0, 1 \ \&\ \epsilon_n \nrightarrow 1.$$

Define $F : [0, 1] \to [0, 1]$, by

$$F(x) := \varlimsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \epsilon_n(x).$$

If $x,\ x' \in [0, 1]$ & $\exists\ N \ge 1$ so that $\epsilon_n(x) = \epsilon_n(x')\ \forall\ n \ge N$, then $F(x) = F(x')$. This shows that for any nonempty interval $J \subset [0, 1]$, $F(J) = F([0, 1])$.

To show `IVP` for F, it suffices to show that $F([0, 1])$ is an interval. We claim that $F([0, 1]) = [0, 1]$ which also shows that F is nowhere continuous.

If $y = \frac{p}{q} \in (0,1) \cap \mathbb{Q}$ and $x \in [0,1]$ has binary expansion

$$(\epsilon_n(x))_{n\ge 1} = (\overline{b(p,q)}) := (b(p,q), b(p,q), \dots)$$

where

$$b(p,q) := (\underbrace{0,\dots,0}_{q-p\text{-times}}, \underbrace{1,\dots,1}_{p\text{-times}}) \in \{0,1\}^q$$

then

(i) $$\frac{1}{N}\sum_{k=1}^{N} \epsilon_k(x) \le y \ \forall\ N \ge 1;$$

(ii) $$\frac{1}{N}\sum_{k=1}^{N} \epsilon_k(x) \xrightarrow[N\to\infty]{} y = F(x).$$

If $y \in (0,1) \setminus \mathbb{Q}$, choose $y_n = \frac{p_n}{q_n} \uparrow y$ and set

$$(\epsilon_n(x))_{n\ge 1} = (\underbrace{b(p_1,q_1),\dots,b(p_1,q_1)}_{N_1\text{-times}}, \dots, \underbrace{b(p_k,q_k),\dots,b(p_k,q_k)}_{N_k\text{-times}}, \dots)$$

where $N_k \uparrow$ will be chosen to be very large.

Note that whatever the choice of N_k, if $M_k := \sum_{j=1}^{k} N_j$, then by (i) above

$$\frac{1}{N}\sum_{n=1}^{N} \epsilon_n(x) \le y_k \ \forall\ 1 \le N \le M_k$$

whence $(y_k \uparrow\ y)$

$$\frac{1}{N}\sum_{n=1}^{N} \epsilon_n(x) \le y \ \forall\ N \ge 1$$

whence $F(x) \le y$.

To ensure $F(x) = y$, choose N_k recursively so large so that (ii) ensures

$$\frac{1}{M_k}\sum_{n=1}^{M_k} \epsilon_n(x) > y_k - \frac{1}{k}.$$

It follows from $y_k \uparrow y$ that

$$\frac{1}{M_k}\sum_{n=1}^{M_k} \epsilon_n(x) \xrightarrow[k\to\infty]{} y$$

whence $F(x) \ge y$. ☑

Monotone functions

Proposition (one sided limits)

Suppose that $f:(a,b)\to\mathbb{R}$ is monotone and that $c\in(a,b)$, then

$$\exists \lim_{x\to c\pm} f(x) =: f(c\pm).$$

Proof

It suffices to prove that if $f:(a,b)\to\mathbb{R}$ is non-decreasing, then

$$f(x) \underset{x\to c-}{\longrightarrow} \mathtt{LUB}\{f(x):\ x\in(a,c)\},\ \&\ f(x) \underset{x\to c+}{\longrightarrow} \inf\{f(x):\ x\in(c,b)\}.$$

We only prove that $f(x) \underset{x\to c-}{\longrightarrow} L = \mathtt{LUB}\{f(x):\ x\in(a,c)\}$ (the other proof being similar).

Let $\epsilon>0$, then $L-\epsilon$ is not an upper bound for $\{f(x):\ x\in(a,c)\}$ and so $\exists\ x_\epsilon\in(a,c)$ with $f(x_\epsilon)>L-\epsilon$. It follows from the non-decreasing property that $\forall\ y\in(x_\epsilon,c)$:

$$L-\epsilon<f(x_\epsilon)\le f(y)\le L$$

whence $|f(y)-L|<\epsilon$. ☐

Corollary (continuity of monotone functions)

Suppose that $f:(a,b)\to\mathbb{R}$ is monotone and that $c\in(a,b)$, then f is continuous at c iff $f(c-)=f(c+)$.

Next topics

MONOTONE FUNCTIONS CTD., MAXIMA FOR CONTINUOUS FUNCTIONS, UNIFORM CONTINUITY, DIFFERENTIABILITY.

Lecture #16 [18]

Proposition (`IVP of monotone functions`)

Suppose that $I \subset \mathbb{R}$ is a bounded interval and that $f : I \to \mathbb{R}$ is monotone, then f is continuous $\Leftrightarrow$ $f(I)$ is an interval.

Proof of $\Leftarrow$)

Suppose that f is non-decreasing, and that f is not continuous at $c \in I = (a,b)$, then $f(c-) < f(c+)$ and $(f(c-), f(c+)) \cap f(I) \subseteq \{f(c)\}$.

$\exists\ u, v \in f(I),\ u \le f(c-),\ v \ge f(c+)$. Since $[u,v] \subset f(I)$ fails, $f(I)$ is not an interval. $\square$

Proposition (`continuity points of monotone functions`)

Suppose that $I = (a,b) \subset \mathbb{R}$ is an open interval and that $f : I \to \mathbb{R}$ is monotone, then

$$I \setminus C_f = \{x \in I :\ f \ \texttt{discontinuous at}\ x\}$$

is at most countable.

Proof for $f : I \to \mathbb{R}$ monotone, non-decreasing : Firstly,

$$\{x \in I :\ f \ \texttt{discontinuous at}\ x\} = \{x \in I :\ f(x+) > f(x-)\}.$$

We show that this set is at most countable by expressing it as an at most countable union of finite sets.

Note that possibly $f(x) \xrightarrow[x \to a, x > a]{} -\infty$ and/or $f(x) \xrightarrow[x \to b, x < b]{} \infty$.[19]

Given $s,\ t \in I$ such that $s < t$ and $\epsilon > 0$ we claim that $J_{(s,t),\epsilon} := \{x \in (s,t) :\ f(x+) - f(x-) \ge \epsilon\}$ is finite.

To see this, let $N \ge 3,\ x_1 < x_2 < \cdots < x_N \in J_{(s,t),\epsilon}$, then

$$f(t) - f(s) \ge f(x_N+) - f(x_1-) = \sum_{k=2}^{N} f(x_k+) - f(x_{k-1}+) + f(x_1+) - f(x_1-).$$

For $2 \le k \le N,\ f(x_k+) - f(x_{k-1}+) \ge f(x_k+) - f(x_k-)$ whence

$$f(t) - f(s) \ge f(x_N+) - f(x_1-) \ge \sum_{k=1}^{N} f(x_k+) - f(x_k-) \ge N\epsilon$$

[18] 21/5/17

[19] This possibility was not considered for in class.

and $N \le \frac{f(t)-f(s)}{\epsilon}$. This shows that $\#J_{(s,t),\epsilon} \le \frac{f(t)-f(s)}{\epsilon}$. Next, choose $s_n, t_n \in I$ with $s_n > s_{n+1} \downarrow a$ and $t_n < t_{n+1} \uparrow b$, then

$$\{x \in I : \ f(x+) > f(x-)\} = \bigcup_{n=1}^{\infty} J_{(s_n,t_n),\frac{1}{n}}$$

which is at most countable being an at most countable union of finite sets. ☑

Example: Monotone $f:(0,1)\to\mathbb{R}$ with $C_f = (0,1)\smallsetminus\mathbb{Q}$.

Define $f:(0,1)\to\mathbb{R}$ by

$$f(x) := \sum_{q=1}^{\infty} \frac{\lfloor qx \rfloor}{2^q}$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : \ n \le x\}$. The series converges absolutely and for $0 < x < y < 1$,

$$\begin{aligned} f(y) - f(x) \underset{N\to\infty}{\longleftarrow} & \sum_{q=1}^{N} \frac{\lfloor qy \rfloor}{2^q} - \sum_{q=1}^{N} \frac{\lfloor qx \rfloor}{2^q} \\ &= \sum_{q=1}^{N} \frac{(\lfloor qy \rfloor - \lfloor qx \rfloor)}{2^q} \\ &\underset{N\to\infty}{\longrightarrow} \sum_{q=1}^{\infty} \frac{(\lfloor qy \rfloor - \lfloor qx \rfloor)}{2^q} \\ &\ge 0. \end{aligned}$$

To see that $f(y) > f(x)$, note that $\exists\ \frac{P}{Q} \in \mathbb{Q}_+$ so that $x < \frac{P}{Q} < y$ whence $\lfloor Qy \rfloor - \lfloor Qx \rfloor \ge 1$ and

$$f(y) - f(x) \ge \frac{1}{2^Q}.$$

This shows that f is strictly increasing.

Claim *This function f is continuous at irrational points of $(0,1)$ and discontinuous at rational points of $(0,1)$.*

Proof

¶ Discontinuity at $r = \frac{P}{Q} \in (0,1)$: here

$$\lfloor Qr \rfloor = P \ \texttt{ whereas for } x < r, \ \lfloor Qx \rfloor \le P - 1$$

whence

$$f(r) - f(x) \ge \frac{\lfloor Qr \rfloor - \lfloor Qx \rfloor}{2^Q} \ge \frac{1}{2^Q}$$

and $f(r) - f(r-) \ge \frac{1}{2^Q}$.

¶ Continuity of $f(x) := \sum_{q=1}^{\infty} \frac{\lfloor qx \rfloor}{2^q}$ at $Z \in (0,1) \smallsetminus \mathbb{Q}$:

Let $\|x\| := \min\{|x-n| : n \in \mathbb{Z}\}$ and note that

(i) $x \notin \mathbb{Q}$ iff $\|qx\| > 0 \ \forall\ q \in \mathbb{N}$;

(ii) if $x \notin \mathbb{Z}$ and $y \in \mathbb{R}$, $|y-x| < \|x\|$, then $\lfloor y \rfloor = \lfloor x \rfloor$.

`Proof: Draw a picture.`

Now fix $Z \in (0,1) \setminus \mathbb{Q}$ and let $\epsilon > 0$, then

- $\exists\ Q = Q_\epsilon \in \mathbb{N}$ such that $\sum_{q=Q+1}^{\infty} \frac{q}{2^q} < \epsilon$;
- $\exists\ \delta = \delta_Q > 0$ such that if $|y - Z| < \delta$, then $|qy - qZ| < \|qZ\|$ and consequently $\lfloor qy \rfloor = \lfloor qZ \rfloor \ \ \forall\ 1 \le q \le Q$.

Thus, for $|y - Z| < \delta$,

$$\begin{aligned} |f(y) - f(Z)| &= |\sum_{q=1}^{\infty} \frac{(\lfloor qy \rfloor - \lfloor qZ \rfloor)}{2^q}| \\ &\le \sum_{q=Q+1}^{\infty} \frac{|\lfloor qy \rfloor - \lfloor qZ \rfloor|}{2^q} \\ &\le \sum_{q=Q+1}^{\infty} \frac{q}{2^q} < \epsilon. \quad \boxtimes \end{aligned}$$

Continuous functions on closed, bounded intervals

Weierstrass' Theorem

Suppose that $A \subset \mathbb{C}$ is a closed, bounded set and that $f : A \to \mathbb{C}$ is continuous on A, then $f(A)$ is closed and bounded.

Proof

Proof of boundedness:

Suppose that $f(A)$ is not bounded, then no $n \in \mathbb{N}$ is an upper bound for the set $\{|f(x)| : x \in A\}$, and so $\forall\ n \in \mathbb{N}\ \exists\ x_n \in A$ such that $|f(x_n)| > n$. By the BW theorem, $\exists\ n_k \to \infty$ and $x \in \mathbb{C}$ such that $x_{n_k} \to x$. Since A is closed, we have $x \in A$ and since $f : A \to \mathbb{C}$ is continuous, we have $f(x_{n_k}) \to f(x)$. Thus, the contradiction:

$$\infty \ \leftarrow\ n_k < |f(x_{n_k})| \to |f(x)| < \infty.$$

Proof of closure:

Let $w_n \in f(A)$ and assume that $w_n \to w$. We show that $w \in f(A)$.

Indeed $\exists\ x_n \in A$ with $w_n = f(x_n)$. Again by the BW theorem, $\exists\ n_k \to \infty$ and $x \in \mathbb{C}$ such that $x_{n_k} \to x$. Since A is closed, we have $x \in A$ and since $f : A \to \mathbb{C}$ is continuous, we have $f(x_{n_k}) \to f(x)$. Thus

$$f(x) \ \leftarrow\ f(x_{n_k}) \to w$$

and $w = f(x) \in f(A)$. $\boxtimes$

Corollary

Suppose that $A \subset \mathbb{C}$ is closed and bounded. If $f : A \to \mathbb{R}$ is continuous on A, then $\exists\ \alpha,\ \omega \in A$ so that

$$f(\alpha) \ge f(x) \ge f(\omega)\ \forall\ x \in A.$$

Proof Since $f(A) \subset \mathbb{R}$ is closed and bounded, we have
$\mathtt{LUB}\, f(A) \in f(A)\ \&\ \mathtt{GLB}\, f(A) \in f(A)$. ☑

Uniform continuity (רציפות במידה שווה)

Definition
Suppose that $A \subset \mathbb{R}$ and that $f : A \to \mathbb{R}$. Say that f is *uniformly continuous on A* if

☺ $\forall \epsilon > 0,\ \exists\ \delta = \delta(\epsilon, A) > 0$ such that

$$x, y \in A,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Evidently. if $f : A \to \mathbb{R}$ is uniformly continuous on A, then f is continuous at every $x \in A$.

Proposition
If $f : A \to \mathbb{R}$ is `Lip` *on A, then f is uniformly continuous on A.*

Proof Suppose that $M > 0$ and that

$$|f(x) - f(y)| \le M|x - y|\ \ \forall\ x, y \in A,$$

then f satisfies ☺ with $\delta(\epsilon, A) = \frac{\epsilon}{M}$. □

Examples.
1) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Note that for $x,\ \eta > 0$:

$$f(x + \eta) - f(x) = (2x + \eta)\eta > 2x\eta,$$

thus $\forall\ \delta > 0,\ \exists\ x, y \in \mathbb{R},\ |x - y| < \delta$ such that $|f(x) - f(y)| > 1$ and f is **NOT** uniformly continuous on $\mathbb{R}$.

2) We claim that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$. To see this we prove first the inequality

$$|\sqrt{y} - \sqrt{x}| \le \sqrt{|x - y|}\ \ \forall\ x, y > 0$$

To see the inequality, let $0 < x < y = x + h$, then

$$|\sqrt{y} - \sqrt{x}| = \frac{y - x}{\sqrt{y} + \sqrt{x}} = \frac{h}{\sqrt{x + h} + \sqrt{x}} \le \sqrt{h} = \sqrt{|x - y|}.$$

Thus f satisfies ☺ with $\delta(\epsilon, [0, \infty)) = \epsilon^2$.

Theorem (Cantor)

Suppose that $A \subset \mathbb{R}$ is closed and bounded. If $f : A \to \mathbb{R}$ is continuous on A, then uniformly continuous on A.

Proof

Suppose that $f : A \to \mathbb{R}$ is not uniformly continuous on A, then (!!) $\exists\ \epsilon > 0$ such that $\forall\ n \geq 1,\ \exists\ x_n, y_n \in A$ such that $|x_n - y_n| < \frac{1}{n}$ and $f(x_n) - f(y_n)| \geq \epsilon$.

The sequence $\{x_n\}$ is bounded (being in the bounded set A) and so by the BW theorem $\exists\ n_k \to \infty$ and $z \in \mathbb{R}$ such that $x_{n_k} \to z$. Since A is closed, $z \in A$ and by continuity of f at z, $f(x_{n_k}) \to f(z)$.

However, $y_{n_k} \to z$ too, whence (again) by continuity of f at z: $f(y_{n_k}) \to f(z)$. Thus the contradiction

$$\epsilon \leq |f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(z)| + |f(y_{n_k}) - f(z)| \to 0.$$

$\square$

`Next topics`

Modulus of continuity, differentiability, power series, mean slopes, graph sketching.

Lecture #17 [20]

Modulus of continuity.

Proposition

The function $f : A \subset \mathbb{C} \to \mathbb{C}$ is uniformly continuous on A iff

$$\omega_{f,A}(t) := \mathtt{LUB}\ \{|f(x) - f(y)| : \ x, y \in A, \ |x - y| < t\} \xrightarrow[t \to 0+]{} 0.$$

Proof of $\Rightarrow$) Let $\epsilon > 0$. By uniform continuity, $\exists\ \delta > 0$ such that

$$x, y \in A, \ |x - y| < \delta \implies |f(x) - f(y)| \le \epsilon,$$

whence for $t < \delta$:

$$\omega_{f,A}(t) = \mathtt{LUB}\ \{|f(x) - f(y)| : \ x, y \in A, \ |x - y| < t\} \le \epsilon.$$

$\Leftarrow$)

Let $\epsilon > 0$. $\exists\ \delta > 0$ such that $\mathtt{LUB}\ \{|f(x) - f(y)| : \ x, y \in A, \ |x - y| < \delta\} < \epsilon$. It follows that

$$x, y \in A, \ |x - y| < \delta \implies |f(x) - f(y)| \le \omega_{f,A}(|x - y|) < \epsilon.$$

$\square$

The function $\omega_{f,A} : \mathbb{R}_+ \to \mathbb{R}_+$ is known as the *modulus of continuity of the function $f : A \to \mathbb{C}$.*

The function $f : A \to \mathbb{R}$ is $\mathtt{Lip}$ in case $\exists\ M > 0$ so that $\omega_{f,A}(t) \le Mt \quad (t > 0)$.

For Rudin's function $T = \mathfrak{T}_{\frac{3}{4},4}$ (as on p. 85), by (iii) of the exercise on p. 86, $\exists\ C > 0$ so that

$$\omega_{T,\mathbb{R}}(t) \le Ct^{1 - \frac{\log 3}{\log 4}}.$$

Exercise

For $N \in \mathbb{N}$, show that $\omega_{P_{\frac{1}{N}},[0,\infty)}(t) = t^{\frac{1}{N}}$ where $P_{\frac{1}{N}}(t) := t^{\frac{1}{N}}$.

Exercise

Let J be a bounded interval. Prove that $f : J \to \mathbb{R}$ is uniformly continuous $\iff$ f has a continuous extension to $\overline{J}$.

Exercise

[20] 25/5/2017

Which of the following functions $f:(0,1)\to\mathbb{R}$ are uniformly continuous on $(0,1)$?
(i) $x\mapsto(1+x)^{\frac{1}{x}}$; (ii) $x\mapsto\frac{1}{x}$; (iii) $x\mapsto x\log x$.

Exercise

☆ Suppose that $f:[0,1]\to[0,1]$ has the `IVP`. Suppose also that for each $t\in[0,1]$, the set $\{x\in[0,1]:\ f(x)=t\}$ is closed. Show that $f:[0,1]\to[0,1]$ is continuous.

`Hint`: Suppose that f is not continuous at $x\in(0,1]$, then $\exists\ 0\le a<b\le 1$ and $u_n,\ v_n\ \to x,\ u_n,\ v_n<x$ so that $f(u_n)<a\ \&\ f(v_n)>b\ \forall\ n\ge 1$.....

TANGENTS AND DIFFERENTIALS

Definition

The function $f:(a,b)\to\mathbb{R}$ is said to be *differentiable at* (גזירה ב־) $x\in(a,b)$ if $\exists\ f'(x)\in\mathbb{R}$ such that

$$\frac{f(x+h)-f(x)}{h}\xrightarrow[h\to0,\ h\neq0]{}f'(x).$$

The number $f'(x)\in\mathbb{R}$ is known as the *derivative* (נגזרת) of f at x. It is sometimes denoted $f'(x)=\frac{df}{dx}(x)$. If f' is also differentiable at x, its derivative, the *second derivative* of f at x is denoted by $f''(x)=f^{(2)}(x)$, etc. ($f^{(k)}(x)$ being the k^{th} derivative of f at x).

Proposition (differentiability ⇒ continuity)

Suppose that $f:(a,b)\to\mathbb{R}$. If f is differentiable at $c\in(a,b)$, then f is continuous there.

Proof By assumption,

$$\frac{f(c+h)-f(c)}{h}\xrightarrow[h\to0,\ h\neq0]{}f'(c)\in\mathbb{R}.$$

We'll show $f(x)\xrightarrow[x\to c,\ x\neq c]{\text{Heine}}f(c)$. To this end suppose that $h_n\to0,\ h_n\neq0$, then

$$f(c+h_n)-f(c)=h_n\cdot\frac{f(c+h_n)-f(c)}{h_n}\longrightarrow0.$$

□

- We'll see later that if $f:(a,b)\to\mathbb{R}$ is differentiable on (a,b), then f is `Lipschitz continuous` on (a,b) iff $|f'|$ is bounded on (a,b).

Examples. ¶1 $\frac{d1}{dx}\equiv0$,
¶2 $\frac{dx^n}{dx}=nx^{n-1}\quad(n\in\mathbb{N})$.

Proof

$$(x+h)^n - x^n = x^n + hnx^{n-1} + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^k,$$

whence

$$\begin{aligned}
\left|\frac{(x+h)^n - x^n}{h} - nx^{n-1}\right| &= \left|\sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^{k-1}\right| \\
&\le |h| \sum_{k=2}^{n} \binom{n}{k} |x^{n-k} h^{k-2}| \\
&\xrightarrow[h\to 0,\ h\ne 0]{} 0. \quad \square
\end{aligned}$$

¶4 $\frac{de^x}{dx} = e^x$.

Proof By (⚕) on p. 49,

$$x < e^x - 1 < xe^x \ \forall \ x \in \mathbb{R} \smallsetminus \{0\}$$

whence

$$\frac{e^h - 1}{h} \xrightarrow[h\to 0,\ h\ne 0]{} 1$$

and for $x \in \mathbb{R}$,

$$\frac{e^{x+h} - e^x}{h} = e^x \frac{e^h - 1}{h} \xrightarrow[h\to 0,\ h\ne 0]{} e^x.$$

¶5 $\frac{d \log x}{dx} = \frac{1}{x}$.

Proof By (‡) on p.48, for $x,\ y\ >\ 0$,

$$\tfrac{1}{y}(y-x)\ <\ \log(y) - \log(x)\ =\ \log(\tfrac{y}{x})\ <\ \tfrac{1}{x}(y-x).$$

It follows that

$$\frac{\log(x+h) - \log(x)}{h} \xrightarrow[h\to 0,\ h\ne 0]{} \frac{1}{x}.$$

¶6 `Rudin's function` $T(x) = \sum_{n=1}^{\infty} (\frac{3}{4})^n \langle 4^n x \rangle$ is nowhere differentiable by (🚂) on p. 85.

One sided derivatives. Let $f : [a,b) \to \mathbb{R}$. We say that f *is differentiable from the right at* $c \in [a,b)$ if
$\exists\ \lim_{y\to c+} \frac{f(y)-f(c)}{y-c} =: f'_+(c) \in \mathbb{R}$.

Similarly f *is differentiable from the left at* $c \in (a,b]$ if
$\exists\ \lim_{y\to c-} \frac{f(y)-f(c)}{y-c} =: f'_-(c) \in \mathbb{R}$.

Evidently if $f : (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$ iff f is differentiable from both the right and the left at c and $f'_-(c) = f'_+(c)$.

Example. Let $A(x) := |x| \quad (x \in \mathbb{R})$, then A is differentiable at each $x \neq 0$ with $A'(x) = 1$ for $x > 0$ & $A'(x) = -1$ for $x < 0$. A is onesidedly differentiable at 0 with $A'_+(0) = 1$ & $A'_-(0) = -1$.

Routine Theorem on arithmetical operations

Suppose that $u, v : (a,b) \to \mathbb{R}$ are differentiable at $c \in (a,b)$, then $u+v$ and uv are also differentiable at c with

(a)
$$(u+v)'(c) = u'(c) + v'(c),$$
$$(uv)'(c) = u'(c)v(c) + v'(c)u(c).$$

In case $v'(c) \neq 0$, $\frac{u}{v}$ is differentiable at c with

(b)
$$(\frac{u}{v})'(c) = \frac{u'(c)v(c) - u(c)v'(c)}{v(c)^2}.$$

Proof Using the relevant theorems on arithmetic operations with limits, let $h \neq 0$, then

$$\frac{(u+v)(c+h) - (u+v)(c)}{h} = \frac{u(c+h) - u(c)}{h} + \frac{v(c+h) - v(c)}{h} \xrightarrow[h\to 0,\ h\neq 0]{} u'(c) + v'(c);$$

$$\begin{aligned}
&\frac{(uv)(c+h) - (uv)(c)}{h} = \\
&= \frac{u(c+h)v(c+h) - u(c)v(c+h) + u(c)v(c+h) - u(c)v(c)}{h} \\
&= v(c+h)\frac{u(c+h) - u(c)}{h} + u(c)\frac{v(c+h) - v(c)}{h} \\
&\xrightarrow[h\to 0,\ h\neq 0]{} v(c)u'(c) + u(c)v'(c).
\end{aligned}$$

$$\begin{aligned}
&\frac{\frac{u}{v}(c+h) - \frac{u}{v}(c)}{h} = \\
&= \frac{u(c+h)v(c) - u(c)v(c+h)}{hv(c)v(c+h)} \\
&= \frac{u(c+h)v(c) - u(c)v(c) + u(c)v(c) - u(c)v(c+h)}{hv(c)v(c+h)} \\
&= \frac{v(c)}{v(c)v(c+h)}\frac{u(c+h) - u(c)}{h} - \frac{u(c)}{v(c)v(c+h)}\frac{v(c+h) - v(c)}{h} \\
&\xrightarrow[h\to 0,\ h\neq 0]{} \frac{u'(c)v(c) - u(c)v'(c)}{v(c)^2}.
\end{aligned}$$

□

Corollary 1 *Any rational function is differentiable on its domain of definition.*

Proof

The constant functions ($C(x) \equiv c$) and the identity function ($I(x) \equiv x$) are differentiable on $\mathbb{R}$ (as can be shown directly) with derivatives $C' \equiv 0$ & $I' \equiv 1$. Applying (a) successively, we see that any polynomial is differentiable on $\mathbb{R}$.

A rational function is a function of form $R = \frac{P}{Q}$ where P, Q are polynomials and is thus differentiable (by (b)) on its domain of definition: $\{x \in \mathbb{R} : \ Q(x) \neq 0\}$. ☐

Derivatives of compositions and inverses

Derivative of composition (chain rule)

Suppose that $I, J \subset \mathbb{R}$ are open intervals and that $f : I \to J, \ g : J \to \mathbb{R}$.

If f is differentiable at $a \in I$ and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Proof Suppose first that $f'(a) \neq 0$, then $\exists \ \eta > 0$ so that $f(a+h) \neq f(a) \ \forall \ 0 < |h| < \eta$. Thus if $h_n \to 0, \ h_n \neq 0$ and $|h_n| < \eta$, then setting $k_n := f(a+h_n) - f(a) \neq 0$ and noting that $k_n \to 0$ (by continuity of f at a), we have

$$\frac{g(f(a+h_n)) - g(f(a))}{h_n} = \frac{g(f(a)+k_n) - g(f(a))}{k_n} \frac{f(a+h_n) - f(a)}{h_n} \xrightarrow[n\to\infty]{} g'(f(a))f'(a)$$

and

$$\frac{g(f(a+h)) - g(f(a))}{h} \xrightarrow[h\to 0, \ h\neq 0]{} g'(f(a))f'(a).$$

Now suppose that $f'(a) = 0$. We show that

$$\frac{g(f(a+h)) - g(f(a))}{h} \xrightarrow[h\to 0, \ h\neq 0]{} 0.$$

To see this, let $h_n \to 0, \ h_n \neq 0$ and set $k_n := f(a+h_n) - f(a) \ \forall \ n \geq 1$,

$$K_0 := \{n \in \mathbb{N} : \ k_n = 0\}, \ K_1 := \{n \in \mathbb{N} : \ k_n \neq 0\}.$$

- If $\#K_0 = \infty$, then
$$\frac{g(f(a+h_n)) - g(f(a))}{h_n} = \frac{g(f(a)+k_n) - g(f(a))}{h_n} = 0 \ \forall \ n \in K_0$$

and $\frac{g(f(a+h_n))-g(f(a))}{h_n} \underset{n\to\infty,\ n\in K_0}{\longrightarrow} 0$.

- If $\#K_1 = \infty$, then
$$\frac{g(f(a+h_n))-g(f(a))}{h_n} = \frac{g(f(a)+k_n)-g(f(a))}{k_n}\frac{k_n}{h_n}\ \forall\ n \in K_1$$

whence
$$\frac{g(f(a+h_n))-g(f(a))}{h_n} \underset{n\to\infty,\ n\in K_1}{\longrightarrow} g'(f(a))f'(a) = 0.$$

This shows that $\frac{g(f(a+h))-g(f(a))}{h} \underset{h\to 0,\ h\neq 0}{\longrightarrow} 0$. ☑

Corollary

For $a \in \mathbb{R}$, $F_a(x) = x^a$ is differentiable on $\mathbb{R}_+$ with $F'(x) = ax^{a-1}$.

Proof We first write $F_a = \exp\circ A \circ \log(x)$ where $A(x) = ax$. By the chain rule,

$$F_a'(x) = \exp'(A\circ\log(x))\cdot A'(\log(x))\cdot\log'(x) = \exp(a\log(x))\cdot a\cdot\frac{1}{x} = ax^{a-1}.$$

□

Theorem (Derivative of inverse function)

Let $I = (a,b)$, let $f : I \to \mathbb{R}$ be continuous and strictly monotone and let $f^{-1} : f(I) \to I$ be the inverse function.

If $x \in (a,b)$ and f is differentiable at x with $f'(x) \neq 0$, then f^{-1} is differentiable at $f(x)$ with $f^{-1\prime}(f(x)) = \frac{1}{f'(x)}$.

¶ Note that the assumption $f'(x) \neq 0$ is necessary: $x \mapsto f(x) = x^3$ is a differentiable bijection $(-1,1) \to (-1,1)$ with $f'(0) = 0$ and $f^{-1} : (-1,1) \to (-1,1)$ is not differentiable at $0 = f(0)$.

Proof Set $y := f(x)$. We'll show that
$$\frac{f^{-1}(y+h) - f^{-1}(y)}{h} \underset{h\to 0,\ h\neq 0}{\longrightarrow} \frac{1}{f'(x)}.$$

Write $k := f^{-1}(y+h) - f^{-1}(y)$, then

- $k \to 0$ as $h \to 0$ by continuity of f^{-1} at y; and
- $k \neq 0$ iff $h \neq 0$ by bijectivity of f^{-1}.

Let $h_n \to 0,\ h_n \neq 0$, then $k_n \to 0,\ k_n \neq 0$ and
$$\frac{f^{-1}(y+h_n) - f^{-1}(y)}{h_n} = \frac{k_n}{f(x+k_n) - f(x)} \longrightarrow \frac{1}{f'(x)}.$$

□

Lecture #18[21]

Higher derivatives

Let $f:(a,b)\to\mathbb{R}$. We say that f *is twice differentiable* on (a,b) if
(i) f is differentiable on (a,b),
and
(ii) f' is differentiable on (a,b). We denote $f^{(2)}=f'':=(f')'$, the *second derivative* of f on (a,b).

More generally, for $n\ge 2$, say that f *is n-times differentiable* on (a,b) if
(i) f is $n-1$-times differentiable on (a,b),
and
(ii) $f^{(n-1)}$ is differentiable on (a,b). We denote $f^{(n)}:=(f^{(n-1)})'$, the n^{th} *derivative* of f on (a,b).

Say that f *is n-times continuously differentiable* on (a,b) (or f is C^n on (a,b)) if f is n-times differentiable on (a,b) and $f^{(n)}$ is continuous there.

Exercise

Define $a:\mathbb{R}\to\mathbb{R}$ by $a(x+n):=(-1)^n x(1-x)$ for $n\in\mathbb{Z}$ & $x\in[0,1]$. Show that a is continuously differentiable on $\mathbb{R}$ with $a'(x+n)=(-1)^n(1-2x)$ for $n\in\mathbb{Z}$ & $x\in[0,1]$.
`Hint` Take one-sided derivatives at integer points

Exercise: Differentiable with discontinuous derivative

Let $a:\mathbb{R}\to\mathbb{R}$ be as above and define $b:\mathbb{R}\to\mathbb{R}$ by

$$b(x):=\begin{cases} x^2a(\frac{1}{x}) & x\ne 0;\\ 0 & x=0.\end{cases}$$

Show that
(i) b is differentiable on $\mathbb{R}$ with

$$b'(x):=\begin{cases} 2xa(\frac{1}{x})-a'(\frac{1}{x}) & x\ne 0;\\ 0 & x=0.\end{cases}$$

(ii) $b'(x)$ is not continuous at $x=0$.

Exercise: Leibniz's "product derivative theorem"

Suppose that $u,v:(a,b)\to\mathbb{R}$ are n-times differentiable on (a,b).
1) Prove `Leibniz's theorem`: that uv is also n-times differentiable on (a,b) and

$$(uv)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}u^{(k)}v^{(n-k)}.$$

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2) Given that $f : (-1,1) \to \mathbb{R}$ is C^2 and $f^{(2)}(x) = xf(x)\ \forall\ x \in (-1,1)$ show using Leibniz's "product derivative theorem" (or otherwise) that f is infinitely differentiable (i.e. n-times differentiable $\forall\ n$) on $(-1,1)$ and find $f^{(n)}(0)$ in terms of $a = f(0),\ b = f'(0)$.

3) Suppose that $x,\ y : (a,b) \to \mathbb{R}$ are twice differentiable and $y(t) = f(x(t))$ where $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable. Show that $f^{(2)}(x(t)) = \frac{y^{(2)}(t)x'(t) - x^{(2)}(t)y'(t)}{x'(t)^3}$ whenever $x'(t) \neq 0$.

Differentiation of power series

Let $S(x) := \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Recall that for $\theta \in \mathbb{R}$, the radius of convergence of the power series $\sum_{n=1}^{\infty} n^{\theta} a_n x^{n-1}$ is also R (see p.67).

Theorem

Let $S(x) := \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$; then $S : (-R,R) \to \mathbb{R}$ is differentiable on $(-R,R)$ with derivative $T(x) := \sum_{n=1}^{\infty} n a_n x^{n-1}$.

Proof The power series $T(x)$ has the same radius of convergence as $S(x)$ and also as $\sum_{n=1}^{\infty} n^2 a_n x^n$. It suffices to show differentiability on $(-r,r)\ \forall\ 0 < r < R$.

Fix $r \in (0,R)$, then

$$\sum_{n\geq 2} n^2 |a_n| r^{n-2} =: M < \infty.$$

We'll show that for $x \in (-r,r),\ h \neq 0$ so that $x + h \in (-r,r)$, we have

$$\left|\frac{S(x+h) - S(x)}{h} - T(x)\right| \leq M|h|. \tag{🚲}$$

To this end, let $x, y \in [-r,r]$, then for $n \geq 1$

$$y^n - x^n = (y - x)\sum_{k=0}^{n-1} y^k x^{n-k-1} \implies$$

$$|y^n - x^n| \leq |y - x| \cdot n r^{n-1}. \tag{☕}$$

It follows that for $n \geq 2$,

$$\begin{aligned}\frac{y^n - x^n}{y - x} - nx^{n-1} &= \sum_{k=0}^{n-1} y^k x^{n-k-1} - nx^{n-1} \\ &= \sum_{k=0}^{n-1} (y^k x^{n-k-1} - x^{n-1}) \\ &= \sum_{k=1}^{n-1} (y^k x^{n-k-1} - x^{n-1}) \\ &= \sum_{k=1}^{n-1} x^{n-k-1}(y^k - x^k)\end{aligned}$$

whence for $n \geq 2$,

(⚽)
$$\begin{aligned}\left|\frac{y^n - x^n}{y - x} - nx^{n-1}\right| &\leq \sum_{k=1}^{n-1} r^{n-k-1}|y^k - x^k| \\ &\leq |y - x| \sum_{k=1}^{n-1} r^{n-k-1} k r^{k-1} \quad \text{by (☕)} \\ &\leq |y - x| n^2 r^{n-2}.\end{aligned}$$

Now fix $x \in (-r, r)$ and let $h \in \mathbb{R}$ be so small that $y := x + h \in (-r, r)$, then $T(x)$ converges absolutely and

$$\begin{aligned}\left|\frac{S(x+h) - S(x)}{h} - T(x)\right| &\leq \sum_{n=1}^{\infty} |a_n| \left|\frac{(x+h)^n - x^n}{h} - nx^{n-1}\right| \\ &= \sum_{n=2}^{\infty} |a_n| \left|\frac{(x+h)^n - x^n}{h} - nx^{n-1}\right| \\ &\leq |h| \sum_{n=2}^{\infty} n^2 |a_n| r^{n-2} \quad \text{by (⚽)} \\ &= M|h| \xrightarrow[h \to 0]{} 0. \quad ☑\end{aligned}$$

Corollary

Let $S(x) := \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$; then $S : (-R, R) \to \mathbb{R}$ is infinitely differentiable on $(-R, R)$ with derivatives

$$S^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}$$

and

$$a_n = \frac{S^{(n)}(0)}{n!}.$$

Slopes & graph sketching

Local extrema & stationarity. Suppose that $f : (a,b) \to \mathbb{R}$.

A point $c \in (a,b)$ is called

- a *local maximum* (נקודת מקסימום מקומי) [*minimum* (נקודת מינימום מקומי)] if $\exists\ \epsilon > 0$ such that $(c-\epsilon, c+\epsilon) \subset (a,b)$ and $f(x) \le f(c)$ $[f(x) \ge f(c)]$ $\forall\ x \in (c-\epsilon, c+\epsilon)$;
- a *strong local maximum* [*minimum*] if $\exists\ \epsilon > 0$ such that in addition, $f(x) < f(c)$ $[f(x) < f(c)]$ $\forall\ c \ne x \in (c-\epsilon, c+\epsilon)$); and
- an *extreme point* (נקודת קיצון) if c is either a local maximum or a local minimum.

Fermat's theorem

Suppose that $f : (a,b) \to \mathbb{R}$ is differentiable on (a,b). If $c \in (a,b)$ is an extreme point, then $f'(c) = 0$.

Proof

Suppose that $f(x) \le f(c)$ $\forall\ x \in (c-\epsilon, c+\epsilon)$, then

$$f'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \le 0,$$

and

$$f'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c-} \frac{f(c) - f(x)}{c - x} \ge 0. \quad \square$$

`Consequences of Fermat's theorem.`

Darboux's IVT for derivatives

Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b) and one-sidedly differentiable at a and b, then

$$\forall\ K \in I(f'_+(a), f'_-(b)),\ \exists\ c \in (a,b) \text{ such that } f'(c) = K.$$

Here $I(X,Y) := (\min\{X,Y\}, \max\{X,Y\})$.

Proof in case $f'_+(a) < f'_-(b)$

Fix $K \in (f'_+(a), f'_-(b))$ and define $F : [a,b] \to \mathbb{R}$ by $F(x) := f(x) - K(x-a)$– continuous on $[a,b]$ and differentiable on (a,b). By Weierstrass' theorem $\exists\ c \in [a,b]$ such that $F(x) \ge F(c)$ $\forall\ x \in [a,b]$ and we claim that $c \in (a,b)$.

To see that $c \ne a$ note that $F'_+(a) = f'_+(a) - K < 0$ whence $\exists\ x > a$ such that $F(x) < F(a)$ and $a \ne c$. Similarly, $c \ne b$ as $F'_-(b) = f'_-(b) - K > 0$ whence $\exists\ x < b$ such that $F(x) < F(b)$.

Thus, $c \in (a,b)$ and is a local minimum for F, whence $F'(c) = 0 \Rightarrow f'(c) = K$. $\square$

Remark. Darboux's IVT is not a consequence of Cauchy's IVT as there are derivatives which are not continuous. Actually, the reverse is true. Every continuous function is a derivative (of its "integral", see calculus 2A) and so Cauchy's IVT is a consequence of Darboux's.

Rolle's theorem

Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then $\exists\ c \in (a,b)$ with $f'(c) = 0$.

Proof

Let $M := \mathtt{LUB}_{x\in[a,b]} f(x),\ m := \inf_{x\in[a,b]} f(x)$. In case $M = m,\ f \equiv f(a)$ on $[a,b]$ and $f' \equiv 0$ on (a,b).

To treat the remaining cases, we consider only the case $M > f(a)$ (the others being analogous). By Weierstrass' theorem, $\exists\ c \in [a,b]$ such that $f(c) = M$. Since $M > f(a) = f(b)$, we have $c \in (a,b)$ whence c is a local maximum for f. By Fermat's theorem, $f'(c) = 0$. $\square$

Exercise

Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Let $M := \mathtt{LUB}_{x\in[a,b]} f(x),\ m := \mathtt{GLB}_{x\in[a,b]} f(x)$. Show that either $\{f(a), f(b)\} = \{m, M\}$ or $\exists\ c \in (a,b)$ with $f'(c) = 0$.

Lagrange's mean slope theorem MST (משפט שיפוע ממוצע)

Suppose $I = [a,b] \subset \mathbb{R}$ is an interval, and that $f : I \to \mathbb{R}$ is continuous on I and differentiable on (a,b), then $\exists\ c \in (a,b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof

Define $g(x) := f(x) - D(x-a)$ with $D := \frac{f(b)-f(a)}{b-a}$, then $g(a) = g(b)$ and by Rolle's theorem, $\exists\ c \in (a,b)$ with $g'(c) = 0$, whence $f'(c) = D = \frac{f(b)-f(a)}{b-a}$. $\square$

Corollary (bounded derivative vs. Lip) *Let $f : (a,b) \to \mathbb{R}$ be differentiable, then f is Lipschitz continuous on (a,b) iff $\sup_{x\in(a,b)} |f'(x)| < \infty$.*

Proof of $\Rightarrow$

Suppose $|f(x) - f(y)| \le M|x-y|\ \forall\ x, y \in (a,b)$, then

$$|f'(x)| \xleftarrow[h\to 0,\ h\ne 0]{} \frac{|f(x+h) - f(x)|}{|h|} \le M. \quad ☑$$

Proof of $\Leftarrow$

Suppose $\sup_{x\in(a,b)} |f'(x)| =: M < \infty$. Fix $a < x < y < b$, then by `MST` $\exists\ z \in (x,y)$ so that $f(y) - f(x) = f'(z)(y-z)$, whence

$$|f(y) - f(x)| = |f'(z)| \cdot |y-z| \le M|y-z|. \quad ☑$$

Remark. It is easy to see that Lipschitz in $(a,b) \not\Rightarrow$ differentiability there. However it follows from an (advanced) theorem of `Henri Lebesgue` that if $f : (a,b) \to \mathbb{R}$ is Lipschitz continuous, then f is differentiable at some point of (a,b).

Graph-sketching.

Consider a differentiable $f : (a,b) \to \mathbb{R}$.

(1) $\qquad f' \ge 0$ on $(a,b) \iff f$ is non-decreasing on (a,b).

Proof of $\Rightarrow$ Suppose that $a < x < y < b$. By Lagrange's MST $\exists\ z \in (x,y)$ with $f(y) - f(x) = (y-x)f'(z) \ge 0$ ☐

(2) $\qquad f' > 0$ on $(a,b) \implies f$ is strictly increasing on (a,b).

Proof Similar to 1).

(3) $\qquad f$ is strictly increasing on $(a,b) \not\Rightarrow f' > 0$ on (a,b).

Proof If $f : (-1,1) \to \mathbb{R}$ is defined by $f(x) := x^3$ then f is strictly increasing on $(-1,1)$, but $f'(0) = 0$. ☐

Proposition *Let $f : (a,b) \to \mathbb{R}$ be differentiable.*
f is strictly increasing on $(a,b) \iff$
(i) $f' \ge 0$ on (a,b), and (ii) $\nexists$ a nontrivial subinterval $J \subset (a,b)$ with $f' \equiv 0$ on J.

Proof
$\Rightarrow$) Suppose that f is strictly increasing on (a,b), then evidently, $f' \ge 0$. If $f' \equiv 0$ on $(x,y) \subset (a,b)$, then by MST, $\exists\ z \in (x,y)$ so that $f(y) - f(x) = (y-x)f'(z) = 0$ whence $f|(x,y) \equiv f(x)$ contradicting the strict increase of f on (a,b).
$\Leftarrow$) Suppose (i) and (ii). As before, f is non-decreasing on (a,b). If f does not increase strictly, $\exists\ (x,y) \subset (a,b)$ with $f(x) = f(y)$ whence $f|(x,y) \equiv f(x)$, $f'|(x,y) \equiv 0$ contradicting (ii). ☐

Proposition: (`condition for strict local maximum`)*:*
Suppose that f is C^2 on (a,b). If $c \in (a,b)$ satisfies $f'(c) = 0$, $f''(c) < 0$, then c is a strict local maximum for f.

Proof By continuity of f'', $\exists\ \epsilon > 0$ such that $f'' < 0$ on $(c-\epsilon, c+\epsilon) \subset (a,b)$ whence f' strictly decreases on $(c-\epsilon, c+\epsilon)$.

It follows that $f' > f'(c) = 0$ on $(c-\epsilon, c)$ and $f' < f'(c) = 0$ on $(c, c+\epsilon)$.

Thus f strictly increases on $(c-\epsilon, c)$. and strictly decreases on $(c, c+\epsilon)$. Consequently,

$$f(x) < f(c)\ \ \forall\ x \in\ (c-\epsilon, c) \cup (c, c+\epsilon). \quad \boxtimes$$

Exercise: `Prove Bernoulli's inequalities`

:

(i) For $x > 0,\ \alpha > 0,$,

$$x^\alpha - \alpha x \begin{cases} \le 1-\alpha & 0 < \alpha < 1; \\ = 0 & \alpha = 1; \\ \ge 1-\alpha & \text{else.} \end{cases}$$

`Hint:` Consider the regions of positivity-negativity of the derivative of the function $f(x) = x^\alpha - \alpha x + \alpha - 1 \quad (x > 0)$.

(ii) Using (i) or otherwise, prove that

$$(1+x)^\beta \ge 1 + \beta x\ \ \forall\ x > -1\ \&\ \beta > 1.$$

When is there equality?

(iii) Show that for $x > 0,\ 0 < \alpha < 1,\ (1+x)^\alpha \le 1 + x^\alpha$.

(iv) Show that for $0 < \alpha < 1,\ \omega_{P_\alpha,[0,\infty)}(t) = t^\alpha$ where $P_\alpha(x) := x^\alpha$.

Next topics

TRIGONOMETRIC FUNCTIONS, POLAR COORDINATES IN $\mathbb{C}$, FUNDAMENTAL THEOREM OF ALGEBRA, EULER-MACLAURIN-TAYLOR EXPANSIONS I, REMOVAL OF SINGULARITIES

Lecture #19 [22]

Differential equations & antiderivatives

For differentiable functions $F : (a, b) \to \mathbb{R}$:

¶1 $F' \equiv 0$ on $(a, b) \Rightarrow F$ constant on (a, b).

Proof Suppose that $a < x < y < b$, then by the MST, $\exists\ z \in (x, y)$ such that $F(y) - F(x) = (y - x)F'(z) = 0$. □

¶2 $F' = x^k$ on (a, b) $(k \in \mathbb{Z}_*)$ $\Rightarrow$ $F(x) = \frac{x^{k+1}}{k+1} + c$ on (a, b) for some $c \in \mathbb{R}$.

Proof Define $G : (a, b) \to \mathbb{R}$ by $G(x) := F(x) - \frac{x^{k+1}}{k+1}$, then $G' \equiv 0$ and by ¶1, G is constant. □

¶3 $F : (a, b) \to \mathbb{R},\ F' \equiv F \Rightarrow F(x) \equiv Ke^x$ on (a, b) for some $K \in \mathbb{R}$.

Proof We show first that if $F > 0$ on $(c, d) \subset (a, b)$, then $F(x) \equiv Ke^x$ on (c, d) for some $K \in \mathbb{R}_+$.

To see this,

$$F' \equiv F \Rightarrow (\ln F)' = \frac{F'}{F} = 1 \overset{¶2}{\Rightarrow} \ln F(x) \equiv x + k \text{ (some } k \in \mathbb{R})$$

and $F(x) = e^k e^x =: Ke^x$.

To finish, we now claim that either $F \equiv 0$, $F < 0$, or $F > 0$ in (a, b).

If not, then (maybe passing to $-F$) we can find a subinterval $(c, d) \subset (a, b)$ where $F > 0$ on (c, d) but $\inf_{(c,d)} F = 0$. By the first part, $F(x) \equiv Ke^x$ on (c, d) where $K \in \mathbb{R}_+$ and $\inf_{(c,d)} F = Ke^c > 0$. Contradiction. ⊠

¶4 Let $S(x) := \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$; then the power series $T(x) := \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ also has radius of convergence $R > 0$. Moreover by the power series differentiation theorem, $T : (-R, R) \to \mathbb{R}$ is differentiable on $(-R, R)$ with $T' = S$.

Differentiation of series

Proposition *Suppose that $I \subset \mathbb{R}$ is an open interval and that $f_n : I \to \mathbb{R}$ $(n \geq 1)$ are differentiable on I and satisfy*

(i) $\sum_{n=1}^{\infty} v_n < \infty$ *and* (ii) $\sum_{n=1}^{\infty} w_n$ *where*

$$v_n := \sup_{x \in I} |f_n(x)| \ \&\ w_n := \sup_{x \in I} |f_n'(x)| < \infty,$$

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then both $S(x) = \sum_{n=1}^{\infty} f_n(x)$ *and* $T(x) = \sum_{n=1}^{\infty} f_n'(x)$ *converge absolutely* $\forall\ x \in I$ *and*

$S : I \to \mathbb{R}$ *is differentiable with* $S' = T$.

Proof Suppose that $x,\ x+h \in I$ where $h \neq 0$, then $x+\theta h \in I\ \forall\ \theta \in (0,1)$ and by the `MST`,

$$\forall\ n \geq 1,\ \exists\ \theta_n \in (0,1) \text{ so that } \tfrac{f_n(x+h)-f_n(x)}{h} = f_n'(x+\theta_n h).$$

It follows that

$$\left|\frac{f_n(x+h)-f_n(x)}{h} - f_n'(x)\right| = |f_n'(x+\theta_n h) - f_n'(x)| \leq 2w_n.$$

Now let $h_k \neq 0,\ x + h_k \in I\ \forall\ k \geq 1\ h_k \to 0,\ x$, then $\forall\ n \geq 1$,

$$a_n(k) := |\tfrac{f_n(x+h_k)-f_n(x)}{h_k} - f_n'(x)| \xrightarrow[k\to\infty]{} 0$$

and $|a_n(k)| \leq 2w_n\ \forall\ k, n \geq 1$.

It follows that

$$\left|\frac{S(x+h_k)-S(x)}{h_k} - T(x)\right| \leq \sum_{n=1}^{\infty}\left|\frac{f_n(x+h_k)-f_n(x)}{h_k} - f_n'(x)\right|$$

$$= \sum_{n=1}^{\infty} a_n(k) \xrightarrow[k\to\infty]{} 0 \text{ by the dominated convergence theorem for series. } \boxtimes$$

Exercise

Let $\|x\| := \min\{|x-n| :\ n \in \mathbb{Z}\}$ and define

$$\mathfrak{t}(x) := \sum_{n=0}^{\infty} \frac{\|2^n x\|}{4^n}.$$

(0) Show that $\mathfrak{t}(x+1) = \mathfrak{t}(x)$ and that $\mathfrak{T}_{\frac{1}{4},2}(x) = 2\mathfrak{t}(\frac{x}{2})\ \ \forall\ 0 < x < 2$.

In this exercise, you show that

☺ $\mathfrak{t} : (0,1) \to \mathbb{R}$ is differentiable with $\mathfrak{t}'(x) = 2 - 4x$

whence $\mathfrak{t}(x) = 2x(1-x)\ \forall\ x \in (0,1)$.

(i) Let $J(x) := \|x\|\ \ (x \in \mathbb{R})$. Show that J is onesidedly differentiable on $\mathbb{R}$ with $J_+'(x) = 1 - 21_{[\frac{1}{2},1)+\mathbb{Z}}(x)$ and $J_+'(x) = 1 - 21_{(\frac{1}{2},1]+\mathbb{Z}}(x)$.

Here $1_A(x) = 1$ if $x \in A$ & $1_A(x) = 0$ if $x \notin A$ and for $B \subset [0,1]$,

$$B + \mathbb{Z} := \{x + n :\ x \in\ B,\ n \in \mathbb{Z}\} = \bigcup_{n\in\mathbb{Z}} (B+n).$$

(ii) For $n \geq 0$, set $J_n(x) := \frac{J(2^n x)}{4^n}$, then J_n is differentiable both from the right and from the left with

$$J_{n+}'(x) = \frac{1 - 21_{[\frac{1}{2},1)+\mathbb{Z}}(2^n x)}{2^n} \ \ \&\ \ J_{n-}'(x) = \frac{1 - 21_{(\frac{1}{2},1]+\mathbb{Z}}(2^n x)}{2^n}.$$

(iii) Show (using binary expansion theory or otherwise) that for $x \in (0,1)$, the following series converge absolutely and

$$\sum_{n=0}^{\infty} J_{n+}'(x) = \sum_{n=0}^{\infty} J_{n-}'(x) = 2 - 4x.$$

(iv) Show that for $x \in (0,1)$, $h > 0$ so that $x \pm h \in (0,1)$, the series

$$\mathfrak{r}_+(x,h) := \sum_{n=0}^{\infty} \frac{J_n(x+h) - J(x)}{h} \ \& \ \mathfrak{r}_+(x,h) := \sum_{n=0}^{\infty} \frac{J_n(x-h) - J(x)}{h}$$

converge absolutely and that for $\eta = \pm$,

$$\mathfrak{r}_\eta(x,h) \xrightarrow[h\to 0,\ h>0]{} 2 - 4x.$$

(vi) Prove ☺.

Trigonometric functions

The geometric definitions of SIN & COS.

The classical geometric `SIN` & `COS` functions are defined as a function of an `angle`. An `angle` is defined in geometry as an equivalence class of `corners` (ordered pairs of half lines emanating from the same point).

Representative corners are indexed by the unit circle

$$\mathbb{S}^1 := \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 = 1\} \equiv \{z \in \mathbb{C} : \ |z| = 1\}$$

by

$$\angle P = (L_1, L_P) \quad P \in \mathbb{S}^1$$

where for $P = (x,y) \in \mathbb{S}^1$,

$$L_P := \{(xt, yt) : \ t \ge 0\} \ \& \ L_1 := \{(t,0) : \ t \ge 0\}.$$

A classical geometry result is that any corner is `equivalent` (i.e. can be `geometrically copied` on-) to a corner of this form.

Here, the geometric definitions of `SIN` & `COS` are

$$\mathtt{SIN}\angle P = y \ \& \ \mathtt{COS}\angle P = x \quad (\angle P = (x,y) \in \mathbb{S}^1).$$

Thus, the `representation` of angles as real numbers (via `polar coordinates` below) requires functions sin, $\cos : \mathbb{R} \to \mathbb{R}$ with certain properties, which we now proceed to define and construct analytically.

Theorem sin `by differential equation`

There is a unique C^2 function $\sin : \mathbb{R} \to \mathbb{R}$ *so that*

$$\sin'' \equiv -\sin, \quad \sin(0) = 0 \quad \& \quad \sin'(0) = 1.$$

Proof of existence The power series

$$\sin(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

which converges absolutely on $\mathbb{R}$, is infinitely differentiable there with

$$\begin{aligned}\sin''(x) &= \sum_{n=1}^{\infty}(2n+1)2n \cdot \frac{(-1)^n x^{2n-1}}{(2n+1)!}\\ &= \sum_{n=1}^{\infty} \cdot \frac{(-1)^n x^{2n-1}}{(2n-1)!}\\ &= -\sin(x)\end{aligned}$$

and satisfies $\sin(0) = 0,\ \sin'(0) = 1.$ ☑

Proof of uniqueness This follows from the

Uniqueness Lemma

If $F : \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathbb{R}$ with

$$F'' = -F,\ F(0) = 0 \ \&\ F'(0) = 0,$$

then $F \equiv 0$ on $\mathbb{R}$.

Proof Define $S : \mathbb{R} \to \mathbb{R}$ by $S := F^2 + F'^{\,2}$, then

$$S' = 2FF' + 2F'F'' = 2FF' - 2F'F \equiv 0.$$

Thus S is constant and $S \equiv S(0) = 0$, whence $F \equiv 0$. ☑

Historical note.

The expression of trigonometric functions as power series goes back (at least) to `Madhava of Sangamagrama` (~ 1350 – 1425).

Exercise: solutions of differential equations by power series

Show using power series (`or otherwise`) that

(i) $\forall\ n \geq 1,\ a_0, a_1, \dots, a_{n-1} \in \mathbb{R},\ \ \exists\ f : \mathbb{R} \to \mathbb{R}\ \ C^n$ so that $f^{(n)} = f$ and $f^{(k)}(0) = a_k$ for $0 \leq k \leq n-1$;

(ii)* `Airy's equation` $\exists\ f : \mathbb{R} \to \mathbb{R}\ \ C^2,\ f \not\equiv 0$ so that $f''(x) = xf(x)$.

Definition of $\cos$**.**

$$\cos(x) := \sin'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Theorem: `Trigonometric uniqueness`

If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathbb{R}$ and $f'' = -f$, then

$$f \equiv f'(0)\sin + f(0)\cos.$$

Proof Follows from the uniqueness lemma.

Properties of sin & cos.

Proposition (`addition formulæ`)

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \;\; \&$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \quad \forall\ x,y \in \mathbb{R};$$

Proof

For fixed y, define $f, g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \sin(x+y) - (\sin(x)\cos(y) + \cos(x)\sin(y)) \quad \&$$
$$g(x) := \cos(x+y) - (\cos(x)\cos(y) - \sin(x)\sin(y));$$

then $f' = g,\ g' = -f$ whence $(f^2+g^2)' \equiv 0$ and $f^2+g^2 \equiv f(0)^2+g(0)^2 = 0$, whence $f \equiv g \equiv 0$ and the addition formulæ hold. ☑

Proposition

$$\sin(-x) = -\sin(x) \;\; \& \;\; \cos(-x) = \cos(x).$$

Proof By inspection of the power series. ☑

Corollary.

$$\sin^2 + \cos^2 \equiv 1 \text{ on } \quad \mathbb{R}.$$

Proof

$$1 = \cos(x-x) = \cos x \cos x - \sin(-x)\sin x = \sin^2 x + \cos^2 x. \quad ☑$$

`Definition of` π.

$$\pi := 2\inf\{Z > 0 : \ \cos(Z) = 0\}.$$

Proposition.

$$\pi \in (0,8) \;\; \& \;\; \cos(\tfrac{\pi}{2}) = 0.$$

Proof *of* $\pi > 0$: By continuity of cos & $\cos(0) = 1$, $\exists\ \epsilon > 0$ so that $\cos(x) > \frac{1}{2}\ \forall\ |x| < \epsilon$, whence $\pi > 2\epsilon$. ☑

Proof *of* $\pi < 8$: By the MST for sin, $\exists\ \xi \in (0,2)$ such that

$$\sin 2 = \sin 2 - \sin 0 = 2\sin'(\xi) = 2\cos\xi$$

whence

$$|\cos(\xi)| = \frac{|\sin(2)|}{2} \le \frac{1}{2}.$$

It follows that

$$\cos(2\xi) = 2\cos^2(\xi) - 1 \le 0.$$

By the IVT for cos, $\exists\ \zeta \in (0,\xi]$ such that $\cos(\zeta) = 0$. Thus $\pi \le 2\zeta \le 2\xi < 8$.

By continuity, $\cos(\frac{\pi}{2}) = 0$. ☑

Properties.

¶0 $\cos > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof $\cos > 0$ on $(0, \frac{\pi}{2})$ by definition and hence on $(-\frac{\pi}{2}, \frac{\pi}{2})$ by evenness.

¶1 $\sin > 0$ on $(0, \frac{\pi}{2})$.

Proof Fix $x \in (0, \frac{\pi}{2})$. By MST $\exists\ \xi \in (0,x)$ such that

$$\sin(x) = x\cos(\xi) > 0. \quad ☑$$

¶2 cos strictly decreases and sin strictly increases on $(0, \frac{\pi}{2})$.

Proof On $(0, \frac{\pi}{2})$,

$$\sin' = \cos > 0 \ \& \quad \cos' = -\sin < 0. \quad ☑$$

¶3 $\sin(\frac{\pi}{2}) = 1$, $\cos(\pi) = -1$, $\sin(\pi) = \sin(2\pi) = 0$, $\cos(2\pi) = 1$;

Proof that $\sin(\frac{\pi}{2}) = 1$: Since $\cos(\frac{\pi}{2}) = 0$ we have $\sin(\frac{\pi}{2}) = \pm 1$. Positivity follows from ¶1.

The other bits follow from the addition formulæ.

Proposition

$$|\sin x| \le |x| \quad \forall\ x \in \mathbb{R}.$$

Proof It suffices to prove the inequality for $x > 0$. To this end, define $f : [0,\infty) \to \mathbb{R}$ by $f(x) := \sin x - x$, then f is differentiable with

$$f'(x) = \cos x - 1.$$

Now, $\cos x \le 1\ \forall\ x > 0$ and $f' \le 0$ on $(0,\infty)$. Thus $f : [0,\infty) \to \mathbb{R}$ is monotonic, non-increasing, whence

$$f(x) \le f(0) = 0\ \forall\ x > 0$$

and the inequality follows. ☑

There is no subinterval of $(0,\infty)$ where $f' \equiv 0$ and so $|\sin x| < |x|\ \forall\ x \ne 0$.

Exercise: Trigonometric Periodicities

Using the addition formulæ, show that

(0) $\sin(n\pi) = 0$ & $\cos(n\pi) = (-1)^n\ \forall\ n \in \mathbb{Z}$;
(i) $\cos(x+2\pi) = \cos(x)$ & $\sin(x+2\pi) = \sin(x)\ \forall\ x \in \mathbb{R}$;
(ii) $\cos(x) = \sin(\frac{\pi}{2} \pm x)\ \forall\ x \in \mathbb{R}$;
(iii) $\sin > 0$ on $(0,\pi)$.
(iv) $\sin(x) = 0$ iff $x \in \pi\mathbb{Z}$ and $\cos(x) = 0$ iff $x \in \pi(\mathbb{Z} + \frac{1}{2})$;
(v) if $c \in \mathbb{R}$, then $\exists\ x \in \mathbb{R}$ such that $\sin(x+c) = \sin(x)$ and $\cos(x+c) = \cos(x)$ iff $c \in 2\pi\mathbb{Z}$.

Exercise: `a discontinuous derivative`

Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) := \begin{cases} x^2 \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(i) Show that f is differentiable on $\mathbb{R}$ with
$$f'(x) := \begin{cases} \sin\frac{1}{x} + 2x\cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

(ii) Show that f' is not continuous at 0.

Exercise: an unbounded derivative
Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) := \begin{cases} |x|^{\frac{3}{2}} \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$
Show that f is differentiable on $\mathbb{R}$ with f' unbounded at 0 and thus not `Lip` at 0.

Polar coordinates in $\mathbb{C}$

Trigonometrical & complex exponentials.

Define $\mathtt{cis} : \mathbb{R} \to \mathbb{C}$ by $\mathtt{cis}(t) := \cos t + i\sin t$, then

(כפל) $$\mathtt{cis}(\theta + t) = \mathtt{cis}(\theta)\mathtt{cis}(t).$$

Proof Apply the addition formulae.
Now define $\mathtt{Exp} : \mathbb{C} \to \mathbb{C}$ by
$$\mathtt{Exp}(x+iy) := e^x\mathtt{cis}(y).$$

Euler's formula.

$$\mathtt{Exp}(z+w) = \mathtt{Exp}(z)\mathtt{Exp}(w)\quad \forall\ z,w \in \mathbb{C}.$$

Proof Let $z = x+iy,\ w = a+ib$, then
$$\begin{aligned}\mathtt{Exp}(z+w) &= e^{x+a}\mathtt{cis}(y+b)\\ &= e^x e^a \mathtt{cis}(y)\mathtt{cis}(b)\\ &= \mathtt{Exp}(z)\mathtt{Exp}(w).\ \boxtimes\end{aligned}$$

Exercise:

Let $c \in \mathbb{R}$. Show that $\sin(x+c) = \sin(x)\ \forall\ x \in \mathbb{R}$ iff $c \in 2\pi\mathbb{Z}$.

Theorem (`Polar coordinates in` $\mathbb{C}$) *For each* $z \in \mathbb{C},\ |z| = 1$, *there is a unique* $t \in [0, 2\pi)$ *so that*

$$z = \mathtt{cis}(t).$$

Proof

Both sin & $\cos : [0, \frac{\pi}{2}) \to [0,1)$ are bijections and that $\sin^2 + \cos^2 \equiv 1$ whence for each $x, y \geq 0,\ x^2 + y^2 = 1,\ \exists\ !\ t \in [0, \frac{\pi}{2}]$ so that $(x,y) = (\cos t, \sin t)$; equivalently $x + iy = \mathtt{cis}(t)$.

The rest of the proof is based on extending this.

For $\eta, \kappa = \pm 1$ write

$$\mathbb{S}_{\eta,\kappa} := \{z = x + iy \in \mathbb{S} :\ \eta x \geq 0\ \&\ \kappa y \geq 0\},$$

then

$$\mathbb{S} = \bigcup_{\eta,\kappa=\pm 1} \overline{\mathbb{S}_{\eta,\kappa}}.$$

The above shows that $\mathtt{cis} : [0, \frac{\pi}{2}] \to \mathbb{S}_{+,+}$ is a bijection.

Next, using again by (כפל), for $t \in [0, \frac{\pi}{2}]$ & $K = 1, 2, 3$, we see that

$$\mathtt{cis}(t + \frac{K\pi}{2}) = i^K \mathtt{cis}(t)$$

and consequently that

$$\mathtt{cis} : [\frac{K\pi}{2}, \frac{(K+1)\pi}{2}] \to i^K \mathbb{S}_{+,+}$$

is a bijection.

Now

$$i\mathbb{S}_{+,+} = \mathbb{S}_{-,+},\ i^2\mathbb{S}_{+,+} = \mathbb{S}_{-,-}\ \&\ i^3\mathbb{S}_{+,+} = \mathbb{S}_{+,-}$$

whence (checking that there's no problem with the overlaps)

$$\mathtt{cis} : [0, 2\pi] \to \bigcup_{K=0}^{3} i^K \mathbb{S}_{+,+} = \mathbb{S}$$

is a bijection. ☑

Fundamental theorem of trigonometry. Fix $0 \le t < 2\pi$,

$$\mathtt{SIN}(\angle \mathtt{cis}(t)) = \sin t \ \&\ \mathtt{COS}(\angle \mathtt{cis}(t)) = \cos t.$$

Algebra of $\mathbb{C}$

Proposition (`complex roots`) *For each $z \in \mathbb{C}$ & $n \ge 1$, $\exists\ w \in \mathbb{C}$ so that $w^n = z$.*

Proof WLOG $z \ne 0$ and so $z = r\mathtt{cis}(t)$ for some $r > 0$ & $t \in \mathbb{R}$. Define

$$w := r^{\frac{1}{n}} \mathtt{cis}(\tfrac{t}{n}),$$

then $w^n = z$. ☑

In fact it is not hard to show that $\forall\ z \in \mathbb{C} \setminus \{0\},\ n \ge 1 :\ \#\{w \in \mathbb{C} : w^n = z\} = n$.

The above proposition is a special case of the

`Next topics`

FUNDAMENTAL THEOREM OF ALGEBRA, POWER SERIES EXPANSIONS OF SOME ELEMENTARY FUNCTIONS, REMOVAL OF SINGULARITIES, RELATIVE `MST`, L'HOSPITAL'S RULES, LAGRANGE ERROR THEOREMS

Lecture #20 [23]

Fundamental theorem of algebra. (FTA) *Let $F:\mathbb{C}\to\mathbb{C}$ be a polynomial, then $\exists\ z\in\mathbb{C}$ so that $F(z)=0$.*

Proof The proof is a sequence of lemmas. Let $F(z):=\sum_{k=0}^{n}a_kz^k$ be a polynomial of degree $n\ge 2$ (i.e. where $a_i\in\mathbb{C},\ a_n\ne 0$).

Lemma 1

(min) $\exists\ \omega\in\mathbb{C}$ *such that* $|F(z)|\ge|F(\omega)|\ \forall\ z\in\mathbb{C}$.

Proof

Since

$$\left|\frac{F(z)}{z^n}\right|\ge|a_n|-\sum_{k=1}^{n-1}\frac{|a_{n-k}|}{|z|^k}\xrightarrow[|z|\to\infty]{}|a_n|>0$$

we have $|F(z)|\xrightarrow[|z|\to\infty]{}\infty$.

Fix $R>0$ such that $|F(z)|>|F(0)|\ \forall\ |z|\ge R$.

It follows that

$$m:=\mathtt{GLB}\{|F(z)|:\ z\in\mathbb{C}\}=\mathtt{GLB}\{|F(z)|:\ z\in N(0,R)\}.$$

Thus $\exists\ x_n\in N(0,R)$ so that $|F(x_n)|\xrightarrow[n\to\infty]{}m$.

The sequence $(x_n:\ n\ge 1)$ is bounded and by the BW theorem $\exists\ n_k\to\infty\ \&\ \omega\in\mathbb{C}$ so that $x_{n_k}\xrightarrow[k\to\infty]{}\omega$.

By continuity of $F:\mathbb{C}\to\mathbb{C}$ (it's Lip on bounded sets)

$$m\xleftarrow[k\to\infty]{}|F(x_{n_k})|\xrightarrow[k\to\infty]{}|F(\omega)|.\quad ☑$$

The proof of the Fundamental theorem of algebra is completed by showing that

$$F(\omega)=0.$$

- WLOG $\omega=0$.

Proof Let $G(z):=F(z+\omega)$, then G is a polynomial and

$$|G(z)|\ge|G(0)|\ \forall\ z\in\mathbb{C}.$$

Lemma 2

(no-pos-min) $|F(0)|>0\ \Rightarrow\ \exists\ \beta\in\mathbb{C}$ *s.t.* $|F(\beta)|<|F(0)|$.

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This establishes a contradiction if, as above

$$\texttt{GLB}\,\{|F(z)|:\ z\in\mathbb{C}\} = |F(0)| > 0.$$

Proof :

This is in stages.

¶1 If $F(z) = 1 - z^m + \sum_{k=m+1}^{n} a_k z^k$ with $1 \le m \le n$, then $|F(t)| < 1 = |F(0)|\ \forall\ t > 0$ sufficiently small.

Proof If $m = n$ then $F(z) = 1 - z^m$ and $F(\beta) < 1 = F(0)\ \forall\ \beta > 0$.

Otherwise, write

$$F(z) = 1 - z^m + \mathcal{E}(z) \texttt{ and set } M := \underset{|z|\le 1}{\texttt{LUB}}\ \frac{|\mathcal{E}(z)|}{|z|^{m+1}} < \infty.$$

For $0 < t < \frac{1}{2M} \wedge 1$,

$$\begin{aligned}|F(t)| &\le 1 - t^m + |\mathcal{E}(t)| \le 1 - t^m + Mt^{m+1}\\ &= 1 - t^m(1 - Mt) \le 1 - \frac{t^m}{2} < 1 = F(0).\ ☑¶1\end{aligned}$$

¶2 If $F(z) = 1 + \sum_{k=1}^{n} a_k z^k$, then $\exists\ u \in \mathbb{C}$ so that $|F(tu)| < 1 = |F(0)|\ \forall\ t > 0$ sufficiently small.

Let $m := \min\{k \ge 1:\ a_k \ne 0\} \le n$. By the `complex roots` existence proposition, $\exists\ u \in \mathbb{C} \setminus \{0\}$ such that $u^m = \frac{-1}{a_m}$. Let

$$Q(z) := F(uz) = 1 - z^m + \sum_{k=m+1}^{n} a_k u^k z^k$$

By ¶1, for $t > 0$ small

$$|F(ut)| = |Q(t)| < 1 = F(0).\ ☑¶2$$

Proof of Lemma 2 Set $Q(z) := \frac{1}{w}F(z)$ where $w := F(0) \ne 0$, then $Q(0) = 1$ and by ¶2, $\exists\ v \in \mathbb{C}$ such that $|Q(v)| < 1$. But then

$$|F(v)| = |w||Q(v)| < |w| = |F(0)|.\ ☑$$

More trigonometric functions.

Define $\tan x : \mathbb{R} \setminus \frac{1}{2}\mathbb{Z} \to \mathbb{R}$ by

$$\tan x := \frac{\sin x}{\cos x}.$$

Evidently (!) $\tan(x+\pi) = \tan(x)$ and $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is differentiable with

$$\tan'(x) = \frac{1}{\cos(x)^2}.$$

Proposition *For* $0 < x < \frac{\pi}{2}$,

(i)
$$\frac{\tan x}{x} > 1;$$

(ii)
$$\frac{\sin x}{x} > \frac{2}{\pi}.$$

Proof of (i) Set $f(x) := \frac{\tan x}{x}$, then $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is differentiable with

$$f'(x) = \frac{x\tan'(x) - \tan(x)}{x^2} = \frac{x - \sin(x)\cos(x)}{x^2\cos(x)^2} = \frac{2x - \sin(2x)}{2x^2\cos(x)^2} > 0$$

because $\sin(2x) < 2x$. Thus for $0 < x < \frac{\pi}{2}$, $1 = f(0+) < f(x)$. ☑

Proof of (ii) Set $g(x) := \frac{\sin x}{x}$, then $g : (0, \frac{\pi}{2}) \to \mathbb{R}$ is differentiable with

$$g'(x) = \frac{x\sin'(x) - \sin(x)}{x^2} = \frac{x\cos(x) - \sin(x)}{x^2} = \frac{(x - \tan(x))\cos(x)}{x^2} < 0$$

by (i). Thus for $0 < x < \frac{\pi}{2}$,

$$\frac{\sin(x)}{x} = g(x) > g(\tfrac{\pi}{2}) = \frac{2}{\pi}. \quad ☑$$

Inverse trigonometric functions

- $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is a strictly increasing bijection.

To see this: $\sin' = \cos > 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so by `MST`, $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is strictly increasing and hence 1-1.

The surjective property of $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ follows from $\sin(\pm\frac{\pi}{2}) = \pm 1$ and the `IVT`.

The inverse function is $\arcsin = \sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$.

- $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a strictly increasing bijection.

To see this: $\tan' = 1 + \tan^2 > 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so by `MST`, $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is strictly increasing and thus 1-1.

Surjectivity follows from $\tan(\pm\frac{\pi}{2}) = \pm\infty$ and the `IVT`.

The inverse function is $\arctan = \tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.

- $\cos : [0, \pi] \to [-1, 1]$ is a strictly decreasing bijection.

Proof: Exercise.

The inverse function is $\arccos = \cos^{-1} : [-1, 1] \to [0, \pi]$.

Derivative of arctan

$\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ *is differentiable on* $(-1,1)$ *with*

$$(\arctan x)' = \frac{1}{1+x^2}.$$

Proof For $x \in \mathbb{R}$,

$$(\arctan)'(x) = \tan^{-1\prime}(x) = \frac{1}{\tan'(\tan^{-1}(x))} = \cos^2 \tan^{-1}(x) \overset{!}{=} \frac{1}{1+x^2}. \quad ☑$$

Power series for $\arctan$**.**

For $|x| < 1$ the power series $S(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ converges. Moreover,

$$S'(x) = \sum_{k=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} = (\arctan x)'.$$

Moreover, $S(0) = \arctan(0) = 0$. Therefore

$$S(x) = \arctan(x) \ \forall \ |x| < 1.$$

This series is known as the `Madhava-Gregory` series.

Derivative of $\arcsin$

$\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ *is differentiable on* $(-1,1)$ *with*

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

Proof For $x \in [-1,1]$,

$$(\arcsin)'(x) = \sin^{-1\prime}(x) = \frac{1}{\sin'(\sin^{-1}(x))} = \frac{1}{\cos \sin^{-1}(x)} \overset{!}{=} \frac{1}{\sqrt{1-x^2}}. \quad ☑$$

Power series for arcsin depends on the

General binomial theorem. For $a \in \mathbb{R}$,

$$(1+x)^a = 1 + \sum_{n=1}^{\infty} \frac{a(a-1)\dots(a-n+1)}{n!} x^n \quad \forall \ |x| < 1.$$

Proof for $a \in \mathbb{R} \setminus \mathbb{Z}_*$

Define $f : (-1, \infty) \to \mathbb{R}_+$ by $f(x) := (1+x)^a$, then

$$f'(x) = a(1+x)^{a-1} = \frac{a}{1+x} f(x).$$

This gives a differential equation for f:

$$(1+x) f'(x) = a f(x).$$

Try to mimic this with a power series:

- If $G(x) = \sum_{n\ge0} A_n x^n$ for $|x| < r$ and $(1+x)G'(x) = aG(x)$, then

$$\begin{aligned}(1+x)G'(x) &= \sum_{n\ge1}(nA_nx^{n-1} + nA_nx^n)\\ &= \sum_{n\ge0}\Big((n+1)A_{n+1} + nA_n\Big)x^n\\ &= a\sum_{n\ge0}A_nx^n = aG(x)\end{aligned}$$

whence for $n \ge 0$, $A_{n+1} = \frac{a-n}{n+1}A_n$ and setting $A_0 := 1$, we obtain

$$A_n = \frac{a(a-1)\dots(a-n+1)}{n!} \quad (n \ge 1).$$

Evidently

$$\frac{|A_{n+1}|}{|A_n|} = \frac{|a-n|}{n} \to 1$$

whence by D'Alembert's ratio theorem, the radius of convergence of $\sum_{n=0}^\infty A_nx^n$ is 1.

Accordingly, we define $G : (-1,1) \to \mathbb{R}$ by $G(x) := \sum_{n=0}^\infty A_nx^n$. This G is differentiable on $(-1,1)$ and as above

- $(1+x)G'(x) = aG(x)$.
- To finish, we claim $G \equiv f$.

Proof Since $f(x) \ne 0 \ \forall\ x \in (-1,1)$ we may define $g := \frac{G}{f} : (-1,1) \to \mathbb{R}$, which is differentiable on $(-1,1)$ with

$$g' = \frac{fG' - f'G}{f^2} = \frac{a}{1+x}\frac{fG - fG}{f^2} \equiv 0$$

whence

$$\frac{G}{f} \equiv g \equiv g(0) \equiv \frac{G(0)}{f(0)} = 1.$$

□

Arcsine series.

$$\arcsin(x) = \sum_{n=0}^\infty \binom{2n}{n}\frac{x^{2n+1}}{(2n+1)4^n} \quad \forall\ |x| < 1.$$

Proof Recall that $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$ for $|x| < 1$. We claim first that

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^\infty \binom{2n}{n}\frac{x^{2n}}{4^n} \ \forall\ |x| < 1. \tag{\Ironing}$$

Proof By the general binomial theorem,

$$\frac{1}{\sqrt{1-x^2}} = (1+(-x^2))^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} A_n(-x^2)^n \quad \forall\ |x| < 1$$

where

$$\begin{aligned} A_n &= \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\ldots(-\frac{1}{2}-n+1)}{n!} \\ &= \frac{(-1)^n}{n!}(\frac{1}{2})(\frac{1}{2}+1)\ldots(\frac{1}{2}+n-1) \\ &\frac{(-1)^n}{2^n n!}1.3.5.\ldots.(2n-1) \\ &= \frac{(-1)^n}{2^n n!}\frac{(2n)!}{2.4.\ldots.2n} \\ &= \frac{(-1)^n}{2^n n!}\frac{(2n)!}{2^n n!} = \binom{2n}{n}\frac{(-1)^n}{4^n}. \ \boxslash \quad (\ ). \end{aligned}$$

• The radius of convergence of this power series is $R = 1$ as is the radius of convergence of $S(x) := \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{2n+1}}{(2n+1)4^n}$. Thus S is differentiable on $(-1, 1)$ and

$$S'(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{2n}}{4^n} = \frac{1}{\sqrt{1-x^2}} \ \forall\ |x| < 1$$

whence

$$S(x) = \arcsin(x) - \arcsin(0) + S(0) = \arcsin(x) \ \forall\ |x| < 1. \boxslash$$

In particular,

$$\frac{\pi}{6} = \arcsin(\frac{1}{2}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{(2n+1)2^{4n+1}}.$$

`Next topics`

Removal of singularities, antiderivative test, Cauchy's relative `MST`, L'Hospital's rule, Lagrange's error theorem, convexity.

Lecture[24] #21

Logarithmic series.

$$\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\quad \forall\ x\in(-1,1).$$

Proof For $|x|<1$,

$$\tfrac{d}{dx}\log(1+x)=\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n.$$

Set

$$S(x):=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$

then $S'(x)=\frac{1}{1+x}$ whence

$$S(x)=\log(1+x)+C$$

where

$$C=S(0)-\log 1=0.\ \boxtimes$$

Exercise

Prove that $\log 2=\frac{2}{3}\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{1}{9^n}$.
`Hint:` Find a power series for $\log\frac{1+x}{1-x}$.

Exercise

$$\frac{\pi}{6}=\tan^{-1}\frac{1}{\sqrt{3}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\Big(\frac{1}{\sqrt{3}}\Big)^{2n+1}=\frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{1}{3^n}.$$

Removal of singularities (השלמת הגדרה)

Proposition: `one-sided singularity` **(השלמת הגדרה לקצה קטע)**

Suppose that $f:(a,b)\to\mathbb{R}$ is differentiable on (a,b) and suppose that $f'(x)\xrightarrow[x\to a+]{} L\in\mathbb{R}$, then

(i)
$$\exists\ \lim_{x\to a+}f(x)=:M\in\mathbb{R}$$

(ii)
$$\frac{f(x)-M}{x-a}\xrightarrow[x\to a+]{}L.$$

i.e. if we set $f(a):=M$, then f is right-differentiable at a with $f'_+(a)=L$.

Proof

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To prove (i), we establish the Cauchy condition for $\exists \lim_{x\to a+} f(x) \in \mathbb{R}$.

Given $\epsilon \in (0,1)$, we'll find $\delta = \delta(\epsilon)$ such that $|f(x) - f(y)| < \epsilon, \ \forall \ x, y \in (a, a+\delta)$.

To do this, fix $K > |L|$ and let $\eta = \eta(\epsilon) > 0$ be such that $|f'(x)| < K \ \forall \ x \in (a, a+\eta)$, and let $\delta = \min\{\eta, \frac{\epsilon}{K}\}$. Given $a < x < y < a+\delta$, we have by Lagrange's MST that $\exists \ z \in (x,y)$ such that $f(y) - f(x) = (y-x)f'(z)$. Since $\delta \le \eta$, we have $|f'(z)| < K$, and since $\delta \le \frac{\epsilon}{K}$, $|f(y) - f(x)| = |y-x||f'(z)| < \delta K < \epsilon$ establishing (a).

To see (ii), let $\epsilon > 0$ and fix $\delta > 0$ such that $|f'(x) - L| < \epsilon \ \forall \ x \in (a, a+\delta)$. We claim that

$$|\frac{f(x) - M}{x-a} - L| < \epsilon \ \forall \ x \in (a, a+\delta).$$

To see this fix $x \in (a, a+\delta)$. By Lagrange's MST, $\exists \ z \in (a,x)$ such that $f(x) - M = (y-a)f'(z)$. It follows that

$$|\frac{f(x) - M}{x-a} - L| = |f'(z) - L| < \epsilon.$$

□

Symmetrically and analogously, if $f'(x) \underset{x\to b-}{\longrightarrow} L \in \mathbb{R}$, then

(iii) $$\exists \lim_{x\to b-} f(x) =: M \in \mathbb{R}$$

(iv) $$\frac{f(x) - M}{x-b} \underset{x\to b-}{\longrightarrow} L.$$

Proposition: `two-sided singularity`

Suppose that $f : (a,b) \smallsetminus \{c\} \to \mathbb{R}$ *is differentiable on* $x \in (a,b) \smallsetminus \{c\}$ *where* $c \in (a,b)$.

If

$$\exists \lim_{x\to c, \ x\ne c} f(x), \ \& \ \lim_{x\to c, \ x\ne c} f'(x) \in \mathbb{R},$$

then f *can be assigned a value at* c *to make it differentiable at* c.

Proof As above.

Example *Define* $f : (-1,1) \smallsetminus \{0\} \to \mathbb{R}$ *by* $f(x) := (1+x)^{\frac{1}{x}}$, *then if* $f(0) := e$, *then* f *is continuously differentiable on* $(-1,1)$.

Proof As shown before

$$f(x) \xrightarrow[x\to 0, \ x\ne 0]{} e.$$

Evidently f is a concatenation of continuously differentiable functions and therefore continuously differentiable on $(-1,1)\setminus\{0\}$. We complete the proof by showing that $f'(x) \xrightarrow[x\to 0,\ x\neq 0]{} -\frac{e}{2}$.

Using the logarithmic series

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} \quad \forall\ |x|<1,$$

we see that

$$|\log(1+x) - (x - \frac{x^2}{2})| \le \frac{2|x|^3}{3} \quad \forall\ |x| < \frac{1}{2}.$$

Thus for $x \neq 0$,

$$\begin{aligned}
f'(x) &= \frac{f(x)}{x}\left(\frac{1}{1+x} - \frac{\log(1+x)}{x}\right) \\
&= \frac{f(x)}{x}\left(\frac{1}{1+x} - (1 - \frac{x}{2})\right) + \frac{f(x)}{x^2}\left((x - \frac{x^2}{2}) - \log(1+x)\right) \\
&= \frac{f(x)(x-1)}{2(1+x)} + \frac{f(x)}{x^2}\left((x - \frac{x^2}{2}) - \log(1+x)\right) \\
&= I(x) + II(x).
\end{aligned}$$

Now

$$I(x) = f(x)\cdot\frac{x-1}{2(1+x)} \xrightarrow[x\to 0,\ x\neq 0]{} -\frac{e}{2}$$

and for $|x| < \frac{1}{2}$,

$$\begin{aligned}
|II(x)| &= \frac{f(x)}{x^2}|(x - \frac{x^2}{2}) - \log(1+x)| \\
&\le \frac{f(x)}{x^2}\cdot\frac{2|x|^3}{3} \\
&= \frac{2|x|}{3}f(x) \\
&\xrightarrow[x\to 0,\ x\neq 0]{} 0. \quad ☑
\end{aligned}$$

Rates of divergence of series

Suppose that $a_n > 0$. When does $\sum_{k=1}^n a_n \to \infty$ and if so, how fast?

Notations. For $A_n,\ B_n > 0$ write

- $A_n \sim B_n$ if $\frac{A_n}{B_n} \to 1$;
- $A_n \ll B_n$ if $\exists\ M > 0$ so that $A_n \le MB_n\ \forall\ n \ge 1$;
- $A_n \asymp B_n$ if $A_n \ll B_n\ \&\ B_n \ll A_n$.

Antiderivative test for series

Suppose that $f : [1,\infty) \to \mathbb{R}_+$ *is differentiable with* f' *positive, non-increasing, then*

$$(🚲)\qquad \left(\sum_{k=1}^{N} f'(k) - f(N+1)\right) \uparrow\ c\ \in\ \mathbb{R}\ \textit{as}\ \exists\ N \uparrow \infty.$$

It follows easily from (🚲) that

$$\text{(a)}\qquad \sum_{k=1}^{\infty} f'(k) < \infty \iff \underset{N\ge 1}{\mathtt{LUB}}\, f(N) < \infty\ \&$$

$$\text{(b)}\qquad \frac{1}{f(N+1)} \sum_{k=1}^{N} f'(k) \xrightarrow[N\to\infty]{} 1 \text{ if } \sum_{k=1}^{\infty} f'(k) = \infty.$$

Proof of (🚲) By MST, $\exists\ \theta_k \in (0,1)$ such that

$$f(N+1) = f(1) + \sum_{k=1}^{N} (f(k+1) - f(k)) = f(1) + \sum_{k=1}^{N} f'(k+\theta_k)$$

whence

$$J_n := \sum_{k=1}^{N} f'(k) - f(N+1) = -f(1) + \sum_{k=1}^{N} (f'(k) - f'(k+\theta_k)).$$

For each $k \ge 1,\ 0 \le f'(k) - f'(k+\theta_k) \le f'(k) - f'(k+1)$, whence $J_n \le J_{n+1}$ and

$$\sum_{k=1}^{N} (f'(k) - f'(k+\theta_k)) \le f'(1) - f'(N) \le f'(1).$$

Thus $\sum_{k=1}^{\infty} (f'(k) - f'(k+\theta_k)) < \infty$ and the $J_n \uparrow\ c < \infty$. ☑

Examples.

¶1 $f(x) = \log x\ f'(x) = \frac{1}{x}$. Here

$$\exists\ \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \log n =: \gamma \in \mathbb{R}$$

and consequently,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \xrightarrow[n\to\infty]{} 1.$$

In this situation, divergence can be obtained by condensation, but not $\sum_{k=1}^{n} \frac{1}{k} \sim \log n$. The constant γ appearing is aka `Euler's constant`. It is not known if $\gamma \in \mathbb{Q}$.

¶2 For $0 < a < 1$, set $f(x) := \frac{1}{1-a} \cdot x^{1-a}$, then $f'(x) = \frac{1}{x^a}$ and

$$\exists\ \lim_{n\to\infty} (\sum_{k=1}^{n} \frac{1}{k^a} - \frac{n^{1-a}}{1-a}) =: c_a \in \mathbb{R}$$

and consequently, $\sum_{k=1}^n \frac{1}{k^a} \sim \frac{n^{1-a}}{1-a}$.

The constants c_a are not as famous as γ but it is not known which $c_a \in \mathbb{Q}$.

Exercise

Show that if $f : (0, \infty) \to \mathbb{R}_+$ is differentiable with f' positive, non-decreasing, then

$$0 \le f(N+1) - \sum_{k=1}^{N} f'(k) \le f'(N+1) - f'(1) + f(1).$$

Exercise

For $a_n,\ b_n > 0$ write $a_n \sim b_n$ as $n \to \infty$ if $\frac{a_n}{b_n} \xrightarrow[n\to\infty]{} 1$.

(i) Suppose that $f : [1, \infty) \to \mathbb{R}_+$ is unbounded and differentiable with f' positive, non-increasing. Show that $\sum_{k=1}^N f'(k) \sim f(N+1)$ as $N \to \infty$.

(ii)★ Show that $\sum_{k=1}^n k^a (\log(k+2))^b \sim \frac{n^{a+1}(\log n)^b}{a+1}$ as $n \to \infty\ \forall\ a > -1,\ \ b \in \mathbb{R}$.

Exercises

(i) Show that for $\alpha > 0$, $\exists$

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \prod_{k=1}^{n} (1 + \frac{\alpha}{k}) \in \mathbb{R}_+.$$

(ii)★ Let $a_n > 0$. Show that

$$\overline{\lim_{n\to\infty}} (\frac{a_{n+1} + 1}{a_n})^n \ge e.$$

`Hint`: Assume otherwise and use $(1 + \frac{\alpha}{n})^n \to e^{\alpha n}$ to hit it with (i).

(iii) Find a sequence $a_n > 0$ so that

$$\overline{\lim_{n\to\infty}} (\frac{a_{n+1} + 1}{a_n})^n = e.$$

Relative MST's

Cauchy's MST

Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b).

If $g'(x) \ne 0\ \ \forall\ x \in (a, b)$, then (a) *$g(a) \ne g(b)$, and* (b) *$\exists\ c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof

By Rolle's theorem, $g(a) \ne g(b)$.

Define $F : [a,b] \to \mathbb{R}$ by

$$F(x) := f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(b)).$$

Evidently F is continuous on $[a,b]$ and differentiable on (a,b) with

$$F' = f' - \frac{f(b) - f(a)}{g(b) - g(a)} g'.$$

Moreover $F(a) = F(b) = f(b) - f(a)$. By Rolle's theorem, $\exists\ c \in (a,b)$ such that

$$0 = F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c).$$

$\square$

Note that Lagrange's MST is a special case of Cauchy's MST (with $g(x) = x$).

L'Hospital's rule (LHR)

Let $-\infty \le a < b \le \infty$ and suppose that $f, g : (a,b) \smallsetminus \{c\} \to \mathbb{R}$ are differentiable on $(a,b) \smallsetminus \{c\}$ where $c \in [a,b]$. Suppose also that

(i) $g(x),\ g'(x) \neq 0\ \forall\ x \in (a,b) \smallsetminus \{c\}$, *and either*

(ii) $f(x),\ g(x) \underset{x \to c,\ x \neq c}{\longrightarrow} 0$, *or*

(iii) $|g(x)| \underset{x \to c,\ x \neq c}{\longrightarrow} \infty$.

If $\exists\ \lim_{x \to c,\ x \neq c} \frac{f'(x)}{g'(x)} =: L \in \mathbb{R}$, *then*

$$\frac{f(x)}{g(x)} \underset{x \to c,\ x \neq c}{\longrightarrow} L.$$

Proof in case $c = a$

Remark. By (i) and Darboux's IVT, g' does not change sign on (a,b). Thus g is monotonic on (a,b).

Fix $a < x < b$, and define

$$m(x) := \mathtt{GLB}\left\{\frac{f'(t)}{g'(t)} :\ t \in (a,x)\right\}\ \&\ \ M(x) := \mathtt{LUB}\left\{\frac{f'(t)}{g'(t)} :\ t \in (a,x)\right\}.$$

By assumption, $m(x), M(x) \underset{x \to a+}{\longrightarrow} L$.

By Cauchy's MST, for each $y \in (a,x)$, $\exists\ z \in (y,x)$ such that

$$(\maltese) \qquad \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)} \in [m(x), M(x)].$$

¶Under assumption (ii), we have that

$$\frac{f(x)}{g(x)} \xleftarrow[y\to a+]{} \frac{f(x)-f(y)}{g(x)-g(y)} \in [m(x), M(x)]$$

whence

$$\frac{f(x)}{g(x)} \xrightarrow[x\to a+]{} L. \quad ☑$$

¶Now assume (iii) and let $\epsilon \in (0, \frac{1}{2})$.

- $\exists\ X_\epsilon > a$ so that $[m(x), M(x)] \subset (L-\epsilon, L+\epsilon)\ \forall\ a < x \le X_\epsilon$.
- $\exists\ Y_\epsilon \in (a, X_\epsilon)$ so that $|f(X_\epsilon)|,\ |g(X_\epsilon)| < \epsilon|g(y)|\ \forall\ y \in (a, Y_\epsilon)$. Now set for $a < y < Y_\epsilon$,

$$R(y) := \frac{f(y)}{g(y)},\ \mathcal{E}(y) := \frac{f(X_\epsilon)}{g(y)}\ \&\ \mathit{\Delta}(y) := \frac{g(X_\epsilon)}{g(y)}.$$

We have $|\mathcal{E}(y)|,\ |\mathit{\Delta}(y)| < \epsilon$ and, using (⚔) that

$$\frac{R(y)-\mathcal{E}(y)}{1-\mathit{\Delta}(y)} = \frac{f(X_\epsilon)-f(y)}{g(X_\epsilon)-g(y)} \in (L-\epsilon, L+\epsilon)$$

whence since $1-\mathit{\Delta}(y) > 0$,

$$(1-\mathit{\Delta}(y))(L-\epsilon) + \mathcal{E}(y) < R(y) < (1+\mathit{\Delta}(y))(L+\epsilon) + \mathcal{E}(y)$$

and

$$|R(y)-L| < (|L|+2+\epsilon)\epsilon. \quad ☑$$

`Next topics`

Lagrange's error theorem `aka LET(N)`; convexity; inflexion; Newton's method.

Lecture[25] #22

Example (`LHR` not perfect)**.**

📖 $$x^2 \sin\frac{1}{x},\ \sin x \xrightarrow[x\to 0,\ x\neq 0]{} 0 \ \ \&$$

$$\lim_{x\to 0,\ x\neq 0}\frac{x^2\sin\frac{1}{x}}{\sin x} = \lim_{x\to 0,\ x\neq 0}\frac{x}{\sin x}\cdot\lim_{x\to 0,\ x\neq 0} x\sin\frac{1}{x} = 0, \quad \textbf{BUT}$$

$$\lim_{x\to 0,\ x\neq 0}\frac{(x^2\sin\frac{1}{x})'}{(\sin x)'} = \lim_{x\to 0,\ x\neq 0}\frac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x}, \quad \textbf{AND}$$

(☢) $$\nexists \lim_{x\to 0,\ x\neq 0}\frac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x}.$$

So `LHR` is not perfect, but is it **wrong**? ☺

Corollary: Higher order `LHR`

Suppose that $f, g : [a,b) \to \mathbb{R}$ are n-times differentiable on $[a,b)$, that

$$f^{(k)}(a) = g^{(k)}(a) = 0 \ \forall\ 0 \le k \le n-1,\ \ g^{(n)}(a) \neq 0$$

and that $\frac{f^{(n)}(x)}{g^{(n)}(x)} \xrightarrow[x\to a+]{} \frac{f^{(n)}(a)}{g^{(n)}(a)}$, *then*

(i) $\exists\ \epsilon > 0$ *such that* $g(x) \neq 0$ *on* $(a, a+\epsilon)$ *and*

(ii) $$\frac{f(x)}{g(x)} \xrightarrow[x\to a+]{} \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

Proof First we prove (i), assuming WLOG that $g^{(n)}(a) > 0$. It follows that $\exists\ \epsilon > 0$ such that $g^{(n-1)} > 0$ on $(a, a+\epsilon)$ (since $\frac{g^{(n-1)}(x)}{x-a} \xrightarrow[x\to a+]{} g^{(n)}(a) > 0$), whence:

- $g^{(n-2)}$ increases strictly on $[a, a+\epsilon)$ whence $g^{(n-2)} > g^{(n-2)}(a) = 0$ on $(a, a+\epsilon)$ whence
- $g^{(n-3)}$ increases strictly on $[a, a+\epsilon)$ whence $g^{(n-2)} > 0$ on $(a, a+\epsilon)$ whence . . . . whence $g > 0$ on $(a, a+\epsilon)$.

By LHR, we have

$$\frac{f^{(n-1)}(x)}{g^{(n-1)}(x)} \xrightarrow[x\to a+]{} \frac{f^{(n)}(a)}{g^{(n)}(a)};$$

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whence again by LHR,

$$\frac{f^{(n-2)}(x)}{g^{(n-2)}(x)} \xrightarrow[x\to a+]{} \frac{f^{(n)}(a)}{g^{(n)}(a)};$$

whence again by LHR ......, whence again by LHR

$$\frac{f^{(n-k)}(x)}{g^{(n-k)}(x)} \xrightarrow[x\to a+]{} \frac{f^{(n)}(a)}{g^{(n)}(a)} \quad \forall\ 1 \le k \le n$$

and in particular (for $k = n$),

$$\frac{f(x)}{g(x)} \xrightarrow[x\to a+]{} \frac{f^{(n)}(a)}{g^{(n)}(a)}. \quad ☑$$

Approximation by polynomials

The idea is to approximate a C^N function $f : (-a, a) \to \mathbb{R}$ by a *Taylor polynomial*, ie one of form

$$P_{f,N}(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n.$$

• This can be done around any $c \in (-a, a)$. If $P(x)$ is a polynomial of degree N, $Q(h) := P(c + h)$ is also a polynomial in h of degree N and $Q^{(k)}(0) = P^{(k)}(c)$. Thus

$$P_{c,f,N}(x) = \sum_{n=0}^{N} \frac{1}{n!} P^{(n)}(c)(x - c)^n.$$

The *error* (aka *remainder*) is

$$R = R_{c,f,N} := f - P_{c,f,N}.$$

It satisfies $R^{(k)}(c) = 0\ \forall\ 0 \le k \le n - 1$ and $R^{(n)} \equiv f^{(n)}(c)$.

If $f : (a, b) \to \mathbb{R}$ is C^n on (a, b), then by the higher order `LHR`, for $c \in (a, b)$,

$$\frac{R_{c,f,n-1}(x)}{(x - c)^n} \xrightarrow[x\to c,\ x\ne c]{} \frac{f^{(n)}(c)}{n!}.$$

The following theorem improves this.

Lagrange's error theorem (`LET(`N`)`).

If $f : [a, b] \to \mathbb{R}$ is N-times differentiable, then $\exists\ \xi \in (a, b)$ such that

$$(⛺) \qquad f(b) - f(a) - \sum_{k=1}^{N-1} \frac{f^{(k)}(a)}{k!}(b - a)^k = \frac{f^{(N)}(\xi)}{N!}(b - a)^N.$$

Remark. `LET`(1) is the same as `MST`.

Proof of `LET`(N)

Fix $\lambda \in \mathbb{R}$ such that

$$f(b) - \sum_{k=0}^{N-1} \frac{(b-a)^k}{k!} f^{(k)}(a) - \frac{(b-a)^N}{N!}\lambda = 0.$$

We'll show that $\lambda = f^{(N)}(\xi)$ for some $\xi \in (a,b)$. To this end, define $H : [a,b] \to \mathbb{R}$ by

$$H(x) := f(b) - \sum_{k=0}^{N-1} \frac{(b-x)^k}{k!} f^{(k)}(x) - \frac{(b-x)^N}{N!}\lambda =: J(x) - \frac{(b-x)^N}{N!}\lambda.$$

- Clearly, $H : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable in (a,b) and $H(b) = 0$.
- From the choice of λ, $H(a) = 0$.
- By Rolle's theorem $\exists\ \xi \in (a,b)$ such that $H'(\xi) = 0$.

To use this we compute J' and then H':

$$\begin{aligned}
J'(x) &= -f'(x) - \sum_{k=1}^{N-1} \Big(\frac{(b-x)^k}{k!} f^{(k)}(x)\Big)' \\
&= -f'(x) - \sum_{k=1}^{N-1} \Big(-\frac{(b-x)^{k-1}}{(k-1)!} f^{(k)}(x) + \frac{(b-x)^k}{k!} f^{(k+1)}(x)\Big) \\
&= -f'(x) - \Big(\frac{(b-x)^N}{N!} f^{(N)}(x) - f'(x)\Big) \quad \texttt{(telescope!)} \\
&= \ -\frac{(b-x)^{N-1}}{(N-1)!} f^{(N)}(x).
\end{aligned}$$

Thus

$$\begin{aligned}
H'(x) &= J'(x) - \Big(\frac{(b-x)^N}{N!}\Big)'\lambda \\
&= -\frac{(b-x)^{N-1}}{(N-1)!} f^{(N)}(x) + \frac{(b-x)^{N-1}}{(N-1)!}\lambda \\
&= -\frac{(b-x)^{N-1}}{(N-1)!} \big(f^{(N)}(x) - \lambda\big)
\end{aligned}$$

and $H'(\xi) = 0,\ \xi \ne b \ \Rightarrow\ \lambda = f^{(N)}(\xi)$, whence

$$f(b) - \sum_{k=0}^{N-1} \frac{(b-a)^k}{k!} f^{(k)}(a) = \frac{(b-a)^N}{N!} f^{(N)}(\xi).$$ (⛺) ☑

Two sided `LET`

Suppose that $f : (a,b) \to \mathbb{R}$ is N-times differentiable. Let $c \in (a,b)$, *then* $\forall\ x \in (a,b) \smallsetminus \{c\},\ \exists\ \xi \in I(c,x)$ *such that*

$$f(x) = \sum_{k=0}^{N-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(N)}(\xi)}{N!}(x-c)^N.$$

Exercise

Show that $\exists$ at most one $f : \mathbb{R} \to \mathbb{R}\quad C^3$ so that $f^{(3)} = -f,\ f(0) = f'(0) = f''(0) = 1$.
`Hint`: LET.

Condition for a local maximum/minimum vs increasing/decreasing.

Suppose that $f : [a,b] \to \mathbb{R}$ is C^2. As before, if $c \in (a,b),\ f'(c) > 0$ then f increases strictly on some neighborhood of c and if $c \in (a,b),\ f'(c) = 0$ and $f''(c) < 0$, then c is a strict local maximum for f, ie $\exists\ \epsilon > 0$ such that $f(x) < f(c)\ \forall\ x \in (c-\epsilon, c+\epsilon) \in (a,b)$.

The following generalizes this:

Theorem

(i) *Suppose that* $f : [a,b] \to \mathbb{R}$ *is* C^{2N}.

If $c \in (a,b),\ f^{(k)}(c) = 0\ \forall\ 1 \le k \le 2N-1$ *and* $f^{(2N)}(c) > 0$, *then* c *is a strict local minimum for* f.

(ii) *Suppose that* $f : [a,b] \to \mathbb{R}$ *is* C^{2N+1}.

If $c \in (a,b),\ f^{(k)}(c) = 0\ \forall\ 1 \le k \le 2N$ *and* $f^{(2N+1)}(c) > 0$, *then* $\exists\ \epsilon > 0$ *such that* f *increases strictly on* $(c-\epsilon, c+\epsilon)$.

Proof

(i): By continuity of $f^{(2N)}$, $\exists\ \epsilon > 0$ such that $f^{(2N)} > 0$ on $(c-\epsilon, c+\epsilon) \subset (a,b)$. By the error theorem `LET`$(2N)$, $\forall\ x \in (c-\epsilon, c+\epsilon),\quad x \ne c,\ \exists\ \xi \in I(c,x)$ such that

$$\begin{aligned} f(x) &= f(c) + \sum_{k=1}^{2N-1} \frac{f^{(k)}(c)(x-c)^k}{k!} + \frac{f^{(2N)}(\xi)(x-c)^{2N}}{(2N)!} \\ &= f(c) + \frac{f^{(2N)}(\xi)(x-c)^{2N}}{(2N)!} \\ &> f(c). \end{aligned}$$

(ii) By (i), c is a strict local minimum for f' and $\exists\ \epsilon > 0$ so that

$$f'(x) > f'(c) = 0, \forall\ x \in (c-\epsilon, c+\epsilon),\ x \ne c.$$

Thus, f increases strictly on $(c-\epsilon, c+\epsilon)$. $\square$

Convexity (קמירות)

A function $f : (a, b) \to \mathbb{R}$ differentiable on (a, b) is said to be *convex* (קמורה) at $c \in (a, b)$ if $\exists\ \epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset (a, b)$ and

$$G_c(x) := (x - c)f'(c) + f(c) \le f(x) \quad \forall\ |x - c| < \epsilon,$$

and *strictly convex* (קמורה חזק) there if $\exists$ (a possibly smaller) $\epsilon > 0$ such that the inequality is strict except at $x = c$. The function G_c is the *tangent line* (קו מ▯וק) to the graph of f at $(c, f(c))$ aka a `supporting line` (קו תומך) (for f at c).

The function is *convex* on A if it is convex at each point of A.

The function is said to be *concave* (קעורה) at a point (or on a set) if the function $-f$ is convex at the point (or on the set). Note that the strict local maximum above is a point of strict local concavity (with zero slope).

Convexity proposition I

Suppose that $f : (a, b) \to \mathbb{R}$ is twice differentiable on (a, b).

If $f'' \ge 0$ $[\mathtt{f}'' > \mathtt{0}]$ on (a, b), then f is convex `[strictly convex]` *on (a, b).*

Proof Suppose that $f'' \ge 0$ on (a, b) and let $c \in (a, b)$. If $x \in (a, b)$, then by the error theorem $\mathtt{LET}(2)$, $\exists\ \xi \in I(c, x)$ such that

$$\begin{aligned} f(x) &= f(c) + (x - c)f'(c) + \frac{f''(\xi)}{2}(x - c)^2 \\ &= G_c(x) + \frac{f''(\xi)}{2}(x - c)^2 \ge G_c(x). \end{aligned}$$

A similar argument establishes strict global convexity in case $[\mathtt{f}'' > \mathtt{0}]$ on (a, b). ☑

Convexity proposition II

Suppose that $f : (a, b) \to \mathbb{R}$ is C^2 on (a, b). If f is convex on (a, b), then $f'' \ge 0$ on (a, b).

Proof Suppose otherwise, i.e. that f is convex on (a, b) but $\exists\ c \in (a, b)$ with $f''(c) < 0$. By continuity of f'', $\exists\ \epsilon > 0$ such that $f'' < 0$ on $(c - \epsilon, c + \epsilon) \subset (a, b)$. By the convexity proposition I, $-f$ is strictly convex on $(c - \epsilon, c + \epsilon)$. This contradicts convexity of f at c. ☐

`Next topics`

global convexity; Jensen's inequality, inflexion; Newton's method; C^∞ bump functions, power series expansions.

Lecture #23 [26]

Global convexity.

Call the differentiable function $f:(a,b)\to\mathbb{R}$ *globally convex* on (a,b) if $G_c\le f$ on (a,b) $\forall\ c\in(a,b)$ and *strictly globally convex* on (a,b) if $G_c<f$ on $(a,b)\smallsetminus\{c\}$ $\forall\ c\in(a,b)$.

Note that global convexity on (a,b) is *a priori* stronger than convexity which only requires $G_c\le f$ on a neighborhood of c $\forall\ c\in(a,b)$. However by Convexity propositions I & II, we have

Convexity proposition III

Suppose that $f:(a,b)\to\mathbb{R}$ is C^2 on (a,b), then f is globally convex on (a,b), iff $f''\ge 0$ on (a,b).

Jensen's Inequality *For $f:(a,b)\to\mathbb{R}$ globally convex on (a,b), $X_1,\dots,X_N\in(a,b)$, $p_1,\dots,p_N\ge 0$, $\sum_{k=1}^N p_k=1$:*

$$f\left(\sum_{k=1}^N p_kX_k\right)\le\sum_{k=1}^N p_kf(X_k).$$

Proof Using the "`mean notations`

$$\overline{X}:=\sum_{k=1}^N p_kX_k\ \&\ \overline{f(X)}:=\sum_{k=1}^N p_kf(X_k),$$

we must show that

$$f(\overline{X})\le\overline{f(X)}).$$

By global convexity, $G_{\overline{X}}\le f$ on (a,b), whence

$$G_{\overline{X}}(X_k)\le f(X_k)\quad\texttt{for}\ \ 1\le k\le N.$$

It follows that

$$\sum_{k=1}^N p_kG_{\overline{X}}(X_k)\le\sum_{k=1}^N p_kf(X_k)=\overline{f(X)}.$$

Evaluating the LHS,

$$\begin{aligned}\sum_{k=1}^N p_kG_{\overline{X}}(X_k)&=\sum_{k=1}^N p_k[f'(\overline{X})(X_k-\overline{X})+f(\overline{X})]\ \text{ by definition of } G_{\overline{X}}(X_k)\\&=f'(\overline{X})\sum_{k=1}^N p_k(X_k-\overline{X})+f(\overline{X})\\&=f(\overline{X}).\end{aligned}$$

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Thus

$$\overline{f(X)} = \sum_{k=1}^{N} p_k f(X_k) \geq \sum_{k=1}^{N} p_k G_{\overline{X}}(X_k) = f(\overline{X}). \quad \boxtimes$$

The following shows that for c^2 functions, Jensen's inequality characterizes convexity.

Convexity proposition IV

Suppose that $f : (a,b) \to \mathbb{R}$ is C^2 on (a,b), then f is convex on (a,b) $\iff$

(J) $\quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \;\forall\; x, y \in (a,b),\; 0 < t < 1.$

Proof

$\Rightarrow$) Suppose $f'' \geq 0$ on (a,b), then (as shown above) f is globally convex on (a,b) and (J) follows from Jensen's inequality.

$\Leftarrow$) Suppose (J). We first

claim ¶: if $a < x < w < y < b$, then $\frac{f(w)-f(x)}{w-x} \leq \frac{f(y)-f(w)}{y-w}$.

Proof of ¶: To see this, note that $w = tx + (1-t)y$ with $t = \frac{y-w}{y-x}$.

- By (J),

$$f(w) \leq tf(x) + (1-t)f(y) = \frac{y-w}{y-x} f(x) + \frac{w-x}{y-x} f(y)$$

whence

$$\begin{aligned}(y-w)f(w) + (w-x)f(w) &= (y-x)f(w)\\ &\leq (y-w)f(x) + (w-x)f(y),\end{aligned}$$

and

$$(y-w)(f(w) - f(x)) \leq (w-x)(f(y) - f(w))$$

and

$$\frac{f(w)-f(x)}{w-x} \leq \frac{f(y)-f(w)}{y-w}. \quad \boxtimes ¶.$$

Next, if $a < x < v < w < y < b$, then by ¶,

$$\frac{f(v)-f(x)}{v-x} \leq \frac{f(w)-f(v)}{w-v} \leq \frac{f(y)-f(w)}{y-w}.$$

It follows that for $a < x < y < b,\; 0 < \epsilon, \delta$ small enough:

$$\frac{f(x+\delta)-f(x)}{\delta} \leq \frac{f(y)-f(y-\epsilon)}{\epsilon}$$

whence $f'(x) \leq f'(y)$.

Thus $f'' \geq 0$ on (a,b). $\square$

Exercises

(i) Show that if $f : (a,b) \to \mathbb{R}$ is convex at $c \in (a,b)$ but not strictly convex there then $\exists\ A,\ B \in \mathbb{R},\ \eta > 0$ such that $f(x) = Ax + B\ \forall\ |x - c| < \eta$.

(ii) When is there equality in Jensen's inequality?

(iii) Suppose that $f : (a,b) \to \mathbb{R}$ is C^2. Show that f is strictly convex on (a,b) iff $f'' \ge 0$ on (a,b) and f'' does not vanish on any nontrivial subinterval of (a,b).

(iv) Prove that for $a > 0,\ t > 1$,

$$2t^a \quad \begin{cases} < (t-1)^a + (t+1)^a & a > 1, \\ > (t-1)^a + (t+1)^a & a < 1. \end{cases}$$

(v) Prove that for $n \ge 1,\ x_1, \dots, x_n \in (0,\pi)$ and $p_1, \dots, p_n > 0,\ \sum_{k=1}^n p_k = 1$,

$$\sin(\sum_{k=1}^n p_k x_k) \ge \prod_{k=1}^n (\sin x_k)^{p_k}.$$

`Hint`: $\log \sin : (0,\pi) \to \mathbb{R}$.

(vi) Prove that for $n \ge 1,\ x_1, \dots, x_n > 0$:

$$\left(\frac{1}{n}\sum_{k=1}^n \frac{1}{x_k}\right)^{-1} \le \left(\prod_{k=1}^n x_k\right)^{\frac{1}{n}} \le \frac{1}{n}\sum_{k=1}^n x_k.$$

Inflexion

Points of inflexion (נקודות פתול) Suppose that $f : (a,b) \to \mathbb{R}$ is C^2. The point $c \in (a,b)$ is a *point of inflexion* (נקודת פתול) of f if $f''(c) = 0$ and $\exists\ \epsilon > 0$ so that

either $f'' < 0$ on $(c - \epsilon, c)$ and $f'' > 0$ on $(c, c + \epsilon)$;

or $f'' > 0$ on $(c - \epsilon, c)$ and $f'' < 0$ on $(c, c + \epsilon)$. A point of inflexion indicates a change in the direction of (strict) convexity (i.e. convexity $\to$ concavity, or concavity $\to$ convexity).

Theorem (`condition for inflexion`)
Suppose that $f : [a,b] \to \mathbb{R}$ and that $c \in (a,b)$. Let $N \ge 1$.
(a) *If f is C^{2N} $(N \ge 1)$, $f^{(k)}(c) = 0\ \forall\ 2 \le k \le 2N-1$ and $f^{(2N)}(c) > 0$, then $\exists\ \epsilon > 0$ such that f is convex on $(c - \epsilon, c + \epsilon)$.*
(b) *If f is C^{2N+1}, $f^{(k)}(c) = 0$ $\forall\ 2 \le k \le 2N$ and $f^{(2N+1)}(c) \ne 0$, then c is a point of inflexion for f.*

Proof
(a) In case $N = 1$, (i) follows from the "Convexity proposition I" above. In case $N \ge 2$, by the `condition for a local minimum` (i), c is a strict local minimum for f'' and $\exists\ \epsilon > 0$ so that $f''(x) > f''(c) = 0\ \forall\ 0 < |x - c| < \epsilon$ whence (by Convexity I) f is convex on $(c - \epsilon, c + \epsilon)$.

(ii) in case $f^{(2N+1)}(c) > 0$. By the `condition for a local minimum` (ii), $\exists\ \epsilon > 0$ so that f'' increases strictly on $(c-\epsilon, c+\epsilon)$, thus

$$f'' < f''(c) = 0 \ \text{ on } \ (c-\epsilon, c), \ \& \ f'' > f''(c) = 0 \ \text{ on } \ (c, c+\epsilon),$$

whence f is concave on $(c-\epsilon, c)$; convex on $(c, c+\epsilon)$ and c is a point of inflexion for f. $\square$

The following complements `LET`.

Peano's theorem *Suppose that $f : (a,b) \to \mathbb{R}$ is $N-1$-times differentiable, and that $f(N-1)$ is differentiable at $c \in (a,b)$, then*

$$\frac{|R_{c,f,N}(x)|}{|x-c|^N} \xrightarrow[x\to c,\ x\neq c]{} 0.$$

As shown above, Peano's theorem follows from higher order `LHR` when f is C^N on (a,b).

Proof when $c = 0 \in (a,b)$

Write

$$g(x) := R_{0,f,N}(x) = f(x) - \sum_{k=0}^{N} \frac{f^{(k)}(0)x^k}{k!},$$

then $g : (a,b) \to \mathbb{R}$ is $N-1$-times differentiable with

$$g^{(k)}(0) = 0 \ \forall\ 1 \le k \le N-1$$

and $g^{(N-1)}$ is differentiable at $0 \in (a,b)$ with

$$g^{(N-1)\prime}(0) = 0.$$

To begin, for $x \neq 0$,

$$\frac{g^{(N-1)}(x)}{x} = \frac{g^{(N-1)}(x) - g^{(N-1)}(0)}{x} \xrightarrow[x\to 0,\ x\neq 0]{} g^{(N-1)\prime}(0) = 0.$$

Setting $h(x) := x^N$, we have that h is C^N with $h^{(k)}(0) = 0 \ \forall\ 0 \le k \le N-1$ & $h^{(N-1)}(x) = N!x$.

By higher order `LHR`,

$$\lim_{x\to 0,\ x\neq 0} \frac{g(x)}{h(x)} = \lim_{x\to 0,\ x\neq 0} \frac{g^{(N-1)}(x)}{N!x} = 0. \ ☑$$

Power series and uniqueness

Theorem (Euler-MacLaurin)

Suppose that $f : [a,b] \to \mathbb{R}$ is C^∞ in $[a,b]$ (i.e. $C^n \ \ \forall\ n \ge 1$), that $c \in (a,b)$ and that

$$\frac{1}{n!}\texttt{LUB}_{[a,b]}|f^{(n)}|(b-a)^n \xrightarrow[n\to\infty]{} 0$$

then the series $\sum_{k=0}^{\infty}\frac{f^{(k)}(c)(x-c)^k}{k!}$ *converges for* $x\in[a,b]$ *and*

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(c)(x-c)^k}{k!}\ \forall\ x\in[a,b].$$

Proof Fix $x\in[a,b]$. For each $n\ge 1$, by the error theorem $\mathtt{LET}(n)$, $\exists\ \xi_n\in I(c,x)$ such that $f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(c)(x-c)^k}{k!}+\frac{f^{(n)}(\xi_n)(x-c)^n}{n!}$ whence

$$\begin{aligned}|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(c)(x-c)^k}{k!}|&\le|\frac{f^{(n)}(\xi_n)(x-c)^n}{n!}|\\&\le\frac{1}{n!}\mathtt{LUB}_{[a,b]}|f^{(n)}|(b-a)^n\\&\xrightarrow[n\to\infty]{}0.\ \boxtimes\end{aligned}$$

Corollary (uniqueness of solutions of differential equations).
If $F:(-R,R)\to\mathbb{R}$ is C^κ, $(\kappa\in\mathbb{N})$ and $F^{(\kappa)}\equiv -F$ then $\forall\ x\in(-R,R)$:

$$F(x)=\sum_{n=0}^{\infty}(-1)^n x^{\kappa n}\left(\frac{F(0)}{(\kappa n)!}+\frac{F'(0)x}{(\kappa n+1)!}+\cdots+\frac{F^{(\kappa-1)}(0)x^{\kappa-1}}{(\kappa n+\kappa-1)!}\right).$$

Proof Evidently, F is C^∞ with $F^{(\kappa n+r)}\equiv(-1)^nF^{(r)}$. Thus for each $0<\rho<R$, $n=\kappa N_n+r_n$ where $0\le r_n<\kappa$, we have

$$\begin{aligned}\mathtt{LUB}_{[-r,r]}|F^{(n)}|&=\mathtt{LUB}_{[-r,r]}|F^{(\kappa N_n+r_n)}|=\mathtt{LUB}_{[-r,r]}|F^{(r_n)}|\\&=\max_{0\le r\le\kappa}\mathtt{LUB}_{[-r,r]}|F^{(r)}|=:M<\infty\end{aligned}$$

and the series expansion follows from the Euler-MacLaurin theorem. $\boxtimes$

Example 1: a C^∞ function with NO power series expansion.
Define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases}e^{-\frac{1}{x}} & x>0,\\ 0 & x\le 0.\end{cases}$$

Proposition. This f is C^∞ on $\mathbb{R}$ and $f^{(n)}(0)=0\ \forall\ n\ge 1$.

Proof Evidently f C^∞ on $\mathbb{R}\smallsetminus\{0\}$. Also f is left-differentiable of all orders at 0 and $f_-^{(k)}(0)=0\ \forall\ k\ge 0$. We must prove the

Claim. This f is right-differentiable of all orders at 0 and $f_+^{(k)}(0) = 0 \ \forall \ k \geq 0$.
Proof *(in steps):*

¶1 f is infinitely differentiable at each $x > 0$ and $f^{(n)}(x) = P_n(\frac{1}{x})e^{-\frac{1}{x}}$ where P_n is a polynomial of degree at most $2n$.
Proof: This is seen by induction:

$$(P_n(\tfrac{1}{x})e^{-\frac{1}{x}})' = -\tfrac{1}{x^2}P_n'(\tfrac{1}{x})e^{-\frac{1}{x}} + \tfrac{1}{x^2}P_n(\tfrac{1}{x})e^{-\frac{1}{x}} = P_{n+1}(\tfrac{1}{x})e^{-\frac{1}{x}}$$

where $P_{n+1}(z) = z^2(P_n(z) - P_n'(z))$. ☑

¶2 If P is a polynomial, then $P(\frac{1}{x})e^{-\frac{1}{x}} \underset{x\to 0+}{\longrightarrow} 0$.
Proof: It suffices to prove this for $P(z) = z^N$ where $N \geq 1$. For $x > 0$,

$$e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{1}{n!x^n} \geq \frac{1}{(N+1)!x^{N+1}},$$

whence

$$\frac{1}{|x|^N}e^{-\frac{1}{x}} \leq (N+1)!x \underset{x\to 0+}{\longrightarrow} 0. \quad \square$$

¶3 f is infinitely right-differentiable at 0 with $f_+^{(n)}(0) = 0 \ \forall \ n \geq 1$.
Proof: By induction. Assume that this is the case $\forall \ 1 \leq k \leq n$, then for $x \neq 0$,

$$\begin{aligned} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} &\overset{\text{assumption}}{=} \frac{f^{(n)}(x)}{x} \\ &\overset{¶1}{=} \frac{1}{x}P_n(\frac{1}{x})e^{-\frac{1}{x}} \\ &\underset{x\to 0+}{\overset{¶2}{\longrightarrow}} 0 = f_+^{(n+1)}(0). ☑¶3 \end{aligned}$$

Thus, f is C^∞ on $\mathbb{R}$, but there is no power series expansion about 0. Otherwise, $\exists \ \ \epsilon > 0$ such that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!} \equiv 0 \ \forall \ |x| < \epsilon. \quad ⊠$$

Exercise

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f is C^∞ on $\mathbb{R}$ and $f^{(n)}(0) = 0 \ \forall \ n \geq 1$.

`Next topics` Newton's method, C^∞ bump functions, power series continuity at endpoints of the convergence interval,product formula for sin, Wallis product, Stirling's formula.

Lecture #24 [27]

Newton's method

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is C^1. Newton's method is an iterative procedure to find $x\in f^{-1}\{0\}$.

The procedure

Given $u\in\mathbb{R}$, draw the tangent line L to the graph of f at $(u,f(u))$ and take v as the x-coordinate of the intersection of L with the x-axis, i.e. $\{(v,0)\}=L\cap(\mathbb{R}\times\{0\})$.

The equation of L is $\frac{y-f(u)}{x-u}=f'(u)$, whence v satisfies $\frac{-f(u)}{v-u}=f'(u)$, or

$$v=:Tu=u-\frac{f(u)}{f'(u)}.$$

Remark. Note that if $f(Z)=0$ & $f'(Z)\neq 0$, then $T(Z)=Z$. The map T is not defined when $f'(z)=0$, in particular when $f(Z)=f'(Z)=0$.

If f is C^r and $f(Z)=f'(Z)=f''(Z)=\cdots=f^{(r-1)}(Z)=0$ & $f^{(r)}(Z)\neq 0$, then by the higher order `LHR`,

for some $\epsilon>0$, $f'\neq 0$ on $N(Z,\epsilon)\smallsetminus\{Z\}$ and

$$\frac{f(u)}{f'(u)}\xrightarrow[u\to Z]{}\frac{f^{(r-1)}(Z)}{f^{(r)}(Z)}=0$$

and $T(u)\xrightarrow[u\to Z]{} Z$. So in this case we define $T(Z)=Z$.

The map T is called *Newton's transformation*. The idea is to study sequences $(T^n(u):\ n\geq 0)$ where $T^0(u):=u,\ T^{n+1}(u):=T(T^n(u))$.

Examples.

¶1 $f(x):=x^2,\ Tx=\frac{x}{2}$, then $T^n(u)=\frac{u}{2^n}\to 0\ \forall\ u\in\mathbb{R}$.

¶2 $f(x)=x^2-a\ \ (a>0)$ then $T(u)=u-\frac{u^2-a}{2u}=\frac{u}{2}+\frac{a}{2u}$. You'll see below that $T^n(u)\to\sqrt{a}\ \forall\ u>0$.

Raphson-Lagrange Convergence Theorem

Suppose that $r\geq 1$ and that

f is C^r and $Z\in\mathbb{R}$, $f(Z)=f'(Z)=\cdots=f^{(r-1)}(Z)=0$, $f^{(r)}(Z)\neq 0$, then $\exists\ \epsilon>0$ such that

(i) $f'(x)\neq 0\ \forall\ x\in N(Z,\epsilon):=(Z-\epsilon,Z+\epsilon)$;

(ii) $|Tx-Z|<(1-\frac{1}{2r})|x-Z|\ \forall\ x\in N(Z,\epsilon)\smallsetminus\{Z\}$;

[27] 18/6/2017

(iii) *If* $u \in N(Z,\epsilon)$, *then* $T^{n+1}(u) := T(T^n(u)) \in N(Z,\epsilon)\ \forall\ n \ge 1$ *and* $T^n(u) \xrightarrow[n\to\infty]{} Z$.

Proof Choose $\epsilon_0 > 0$ such that $f^{(r)}(x) \ne 0\ \forall\ x \in N(Z,\epsilon_0)$.

By Lagrange's error theorem (`LET`), $\forall\ u \in N(Z,\epsilon_0),\ \exists\ \alpha,\beta \in I(u,Z)$ such that

$$f(u) = \frac{f^{(r)}(\alpha)(u-Z)^r}{r!}, \quad f'(u) = \frac{f^{(r)}(\beta)(u-Z)^{r-1}}{(r-1)!}.$$

It follows that

(i) $f'(x) \ne 0\ \forall\ x \in N(Z,\epsilon_0) \smallsetminus \{Z\}$, and that

$$\frac{f(u)}{(u-Z)f'(u)} = \frac{f^{(r)}(\alpha)}{rf^{(r)}(\beta)} \xrightarrow[u\to Z,\ u\ne Z]{} \frac{1}{r}.$$

Let $0 < \epsilon < \epsilon_0$ satisfy:

$$|\frac{f(u)}{(u-Z)f'(u)} - \frac{1}{r}| < \frac{1}{2r} \quad \forall\ 0 < |u-Z| < \epsilon.$$

Multiplying by $(u-Z)$ we see that for $u \in N(Z,\epsilon)$: $|\frac{f(u)}{f'(u)} - \frac{(u-Z)}{r}| < \frac{|u-Z|}{2r}$, whence

- for $r = 1$,

$$|T(u) - Z| = |\frac{f(u)}{f'(u)} - (u-Z)| < \frac{|u-Z|}{2}; \quad ☑(ii)$$

- and for $r \ge 2$,

$$\begin{aligned} T(u) - Z &= \frac{f(u)}{f'(u)} - (u-Z) \\ &= (\frac{f(u)}{f'(u)} - \frac{(u-Z)}{r}) - (1-\frac{1}{r})(u-Z); \end{aligned}$$

whence

$$|T(u) - Z| \le (1-\frac{1}{r})|u-Z| + \frac{|u-Z|}{2r} \le (1-\frac{1}{2r})|u-Z|. \quad ☑(ii)$$

To establish (iii), let $u \in N(Z,\epsilon),\ u_0 \ne Z$, then by induction using (ii): for $r = 1$,

$T^{n+1}(u) := T(T^n(u)) \in N(Z,\epsilon)$ & $|T^n(u)-Z| \le (1-\frac{1}{2})^n|u-Z| < \epsilon\ \forall\ n \ge 1$,

whence $T^n(u) \xrightarrow[n\to\infty]{} Z$;

and for $r \ge 2$ using (ii) and (ii'),

$T^{n+1}(u) := T(T^n(u)) \in N(Z,\epsilon) \smallsetminus \{Z\}$ and $|T^n(u)-Z| \le (1-\frac{1}{2r})^n|u-Z| < \epsilon\ \forall\ n \ge 1$. ☑

Exercise

As above let $f(x) = x^2 - a$ & $Z = \sqrt{a}$. Show that

(o) Newton's transformation is $T(u) = \frac{u}{2} + \frac{a}{2u}$;

(i) for some $\epsilon > 0$,

$$T^n(u) \xrightarrow[n\to\infty]{} \sqrt{a} \ \forall \ u \in N(\sqrt{a}, \epsilon)$$

where $T^0(u) = u$ & $T^{n+1}(u) = T(T^n(u))$.

(ii) For $y > 0$ $T(y) < y \iff y^2 > a$ and $T(y) = y \iff y^2 = a$.

(iii) $T(y)^2 > a \ \forall \ y > 0, \ y^2 \neq a$.

(iv) For $y > 0, \ y^2 > a$, $T^n(y) \geq T^{n+1}(y) \geq \sqrt{a}$.

(v) $T^n(y) \xrightarrow[n\to\infty]{} \sqrt{a} \ \forall \ y > 0$.

Smooth bump functions

These are "`continuous indicator functions`".

Namely, a *bump function* is a continuous function $B : \mathbb{R} \to [0, 1]$ so that for some $a < \alpha < \beta < b$, one has

$$1_{(\alpha,\beta)} \leq B \leq 1_{[a,b]}$$

with B increasing on (a, α) & decreasing on (β, b).

Here 1_F is the *indicator function* of $F \subset \mathbb{R}$:

$$1_F(x) = \begin{cases} 0 & x \notin F; \\ 1 & x \in [\alpha, \beta]. \end{cases}$$

For example a `piecewise linear` bump function is defined by:

$$B(x) := \begin{cases} 0 & x \notin (a, b); \\ 1 & x \in [\alpha, \beta]; \\ \frac{x-a}{\alpha-a} & x \in [a, \alpha] \\ 1 - \frac{x-\beta}{b-\beta} & x \in [\beta, b]. \end{cases}$$

The question arises as "how smooth" can a bump function be?

C^∞ bump functions.

Define $S : \mathbb{R} \to [0, \infty)$ by

$$S(x) := \frac{A(x)}{A(x) + A(1-x)}$$

where

$$A(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0; \\ 0 & x \leq 0. \end{cases}$$

¶1 **Claim** S is C^∞, $0 \le S \le 1$ and

$$S(x) = \begin{cases} 0 & x \le 0; \\ 1 & x \ge 1. \end{cases}$$

Proof that S is C^∞

As shown above, A is C^∞. So is $J(x) := A(x) + A(1-x)$ which is also positive $\forall\ x \in \mathbb{R}$. Using the formula for the derivative of a quotient, we see that S is differentiable on $\mathbb{R}$ with

$$S'(x) = \frac{\alpha_1(x)}{J(x)^2}$$

where α_1 is C^∞. Repeating this (i.e. by induction), we see that $\forall\ n \ge 1$, S is n-times differentiable with

$$S^{(n)}(x) = \frac{\alpha_n(x)}{J(x)^{2^n}}$$

with α_n C^∞. ☑

For $0 < a < b$, define

$$B_{a,b}(x) := 1 - S(\tfrac{x^2-a^2}{b^2-a^2}).$$

¶2 **Claim** $B_{a,b} : \mathbb{R} \to [0,1]$ is a smooth bump function, `i.e.` $B_{a,b}$ is C^∞ and that

$$B_{a,b}(x) = \begin{cases} 0 & x \notin (-b,b); \\ 1 & x \in [-a,a]. \end{cases}$$

Proof that $B_{a,b}$ is C^∞ We have that $B_{a,b}(x) = B = C \circ D(x)$ where C is C^∞ and D is a degree 2 polynomial (with D'' constant).

We claim that in this situation, there are polynomials $P_{N,k}$ $(1 \le k \le N)$ so that

$$B^{(N)}(x) = \sum_{k=1}^{N} C^{(k)} \circ D(x) \cdot P_{N,k}(D'(x)).$$

To see this by induction, note that

$$P_{0,0} \equiv 1,\ P_{N+1,0}(x) = D'' P'_{N,0}(x),\ P_{N+1,N+1}(x) = x P_{N,N}(x)\ \&$$
$$P_{N+1,k}(x) = x P_{N,k-1}(x) + D'' P'_{N,k}(x)\ \ (1 \le k \le N).$$

Thus B is C^∞ on $\mathbb{R}$. ☑

Continuity of power series at endpoints of the convergence interval

Abel's continuity theorem

Let $a_n \in \mathbb{R}$ $(n \geq 0)$ *and suppose that* $\sum_{n=0}^{\infty} a_n x^n$ *converges* $\forall\ |x| < 1$ (i.e. $R \geq 1$). *If* $\sum_{n=0}^{\infty} a_n$ *converges, then*

$$\sum_{n=0}^{\infty} a_n x^n \underset{x \to 1-}{\longrightarrow} \sum_{n=0}^{\infty} a_n.$$

i.e. $f : (-1, 1] \to \mathbb{R}$ *defined by* $f(x) := \sum_{n=0}^{\infty} a_n x^n$ *is continuous on* $(-1, 1]$.

In particular

$$\log 2 \underset{x \to 1-}{\longleftarrow} \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \underset{x \to 1-}{\longrightarrow} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Proof Set $A_n := \sum_{k=0}^{n} a_k$ $(n \geq 0)$, $A_{-1} := 0$, and $A := \sum_{k=0}^{\infty} a_k$, then for $|x| < 1$, $n \geq 0$,

$$\begin{aligned}
\sum_{k=0}^{n} a_k x^k &= \sum_{k=0}^{n} (A_k - A_{k-1}) x^k \\
&= \sum_{k=0}^{n} A_k x^k - \sum_{k=0}^{n} A_{k-1} x^k \\
&= (1-x) \sum_{k=0}^{n-1} A_k x^k + A_n x^n
\end{aligned}$$

Now $A_n \to A \in \mathbb{R}$ by assumption, so for $|x| < 1$:

- $A_n x^n \to 0$ as $n \to \infty$ and $\sum_{k=0}^{\infty} A_k x^k$ converges; and

$$f(x) \underset{n \to \infty}{\longleftarrow} \sum_{k=0}^{n} a_k x^k \underset{n \to \infty}{\longrightarrow} (1-x) \sum_{k=0}^{\infty} A_k x^k.$$

Recall that also $(1-x) \sum_{k=0}^{\infty} x^k = 1$ $\forall\ |x| < 1$.

- Using these, we show that $f(x) \xrightarrow[x\to 1-]{} A$. For $|x|<1$, $N\in\mathbb{N}$,

$$\begin{aligned}|f(x)-A| &= |(1-x)\sum_{k=0}^{\infty}(A_k-A)x^k| \\ &\le (1-x)\sum_{k=0}^{\infty}|A_k-A||x|^k \\ &\le (1-x)\sum_{k=0}^{N}|A_k-A||x|^k+(1-x)\sum_{k=N+1}^{\infty}|A_k-A||x|^k \\ &\le (1-x)\sum_{k=0}^{N}|A_k-A|+\sup_{k>N}|A_k-A|. \quad \square\end{aligned}$$

To see that $f(x) \xrightarrow[x\to 1-]{} A$, let $\epsilon>0$.

- $\exists\ N$ such that $\sup_{k>N}|A_k-A|<\epsilon$; and
- $\exists\ \delta>0$ such that $(1-x)\sum_{k=0}^{N}|A_k-A|<\epsilon\ \forall\ 1-\delta<x<1$, whence

$$|f(x)-A|<\epsilon\ \forall\ 1-\epsilon<x<1. \quad ☑$$

Corollary

$$\arctan x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \quad \forall\ |x|\le 1.$$

Exercise: Convergence at the endpoints

⋆

Show that $\arcsin(x)=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{2n+1}}{(2n+1)4^n} \quad \forall\ |x|\le 1.$

`Hint`: $\binom{2n}{n}\frac{1}{4^n}=\prod_{k=1}^{n}(1-\frac{1}{2k})\asymp\frac{1}{\sqrt{n}}$.

Euler's sine product formula

$$\pi x\prod_{k=1}^{n}(1-\frac{x^2}{k^2}) \xrightarrow[n\to\infty]{} \sin\pi x.$$

The proof is in stages.[28]

¶1 *For* $|x|<1$,

(1) $$\exists\ \lim_{n\to\infty}\prod_{k=1}^{n}(1-\frac{x^2}{k^2}) =: F(x)>0 \quad \forall\ |x|<1;$$

(2) $F:(-1,1)\to\mathbb{R}$ *is* C^1.

[28]This proof is an example of the "`Herglotz trick`" to prove equality of functions. See Carathéodory, C. *Theory of functions of a complex variable*, Vol. 1, §259-262 & Aigner, M., Ziegler, GM. *Proofs from The Book*, Ch. 23 & Landau.

Proof of (1):

For $|x| < 1$,

$$\prod_{k=1}^{n}(1 - \frac{x^2}{k^2}) = \exp[-\sum_{k=1}^{n} a_k(x)]$$

where $a_k(x) = f(\frac{x^2}{k^2})$ with $f(t) := -\log(1-t) \in \mathbb{R}_+$, for $t \in (0,1)$.

By MST $\exists\ \theta_t \in (0,1)$ such that

$$0 < f(t) = f(0) + tf'(\theta_t t) = \frac{t}{1-\theta_t t} < \frac{t}{1-t}.$$

Thus, for $k \geq 2$,

$$a_k(x) = f(\frac{x^2}{k^2}) < \frac{x^2}{k^2}\frac{1}{1-\theta(k)\frac{x^2}{k^2}} \leq \frac{4x^2}{3k^2}$$

where $\theta(k) := \theta_{\frac{x^2}{k^2}}$.

Thus

$$\sum_{k\geq 2}\sup_{|x|\leq 1}|a_k(x)| < \infty. \tag{🦇}$$

In particular, $\exists\ \lim_{n\to\infty}\sum_{k=1}^{n} a_k(x) =: g(x) \in \mathbb{R}_+$ and

$$\prod_{k=1}^{n}(1 - \frac{x^2}{k^2}) = e^{-\sum_{k=1}^{n} a_k(x)} \underset{n\to\infty}{\longrightarrow} e^{-g(x)} =: F(x) \in (0,1). \boxtimes(1)$$

`Next topics`

Euler's product formula for sin `ctd.`, Wallis' product, Stirling's formula.

Lecture #25[29]

Proof of (2):

For $F:(-1,1)\to\mathbb{R}$ to be C^1, it suffices that $g:(-1,1)\to\mathbb{R}$ is C^1.

To see this we'll need (🦇) and

$$(🚗)\qquad \sum_{k\ge 2}\sup_{|x|\le 1}|a_k'(x)|<\infty.$$

Proof of (🚗)

$$|a_n'(x)|=\frac{2x}{n^2}\frac{1}{1-\frac{x^2}{n^2}}\le\frac{2}{n^2-1}\le\frac{8}{3n^2}\ \forall\ n\ge 2,\ |x|\le 1.\ \boxtimes$$

Proof that g is C^1

Write $A(x):=\sum_{k\ge 2}a_k(x)$ and $B(x):=\sum_{k\ge 2}a_k'(x)$.

- By the `continuity of series` theorem (on p. 83),

$A,B:(-1,1)\to\mathbb{R}$ are continuous.

- By the `differentiation of series` proposition (on p. 111),

$A:(-1,1)\to\mathbb{R}$ is differentiable with $A'=B$.

Thus $g=A+a_1$ is differentiable with $g'=A'+a_1'$ which is continuous.

☑

¶2

$$\exists\ \lim_{n\to\infty}\ x\cdot\prod_{k=1}^{n}(1-\frac{x^2}{k^2})\ =: f(x)\in\mathbb{R}\ \forall\ x\in\mathbb{R}$$

where $f:\mathbb{R}\to\mathbb{R}$ is C^1 and $f(x+1)=-f(x)$.

Proof

Set $f_n(x):=x\prod_{k=1}^{n}(1-\frac{x^2}{k^2})$, then by ¶1,

$$f_n(x)\xrightarrow[n\to\infty]{} xF(x)=:f(x)\ \forall\ |x|<1.$$

Since $f_n(x)=0\ \forall\ x\in\mathbb{Z},\ n>|x|$, it suffices to show that

$$\frac{f_n(x+1)}{f_n(x)}\xrightarrow[n\to\infty]{}-1\ \forall\ x\in\mathbb{R}\smallsetminus\mathbb{Z}.$$

[29] 22/6/2017

We develop first some alternate expressions for $f_n(x)$.

(🕮)
$$\begin{aligned} f_n(x) &:= x\prod_{k=1}^{n}(1-\frac{x^2}{k^2}) = x\prod_{k=1}^{n}\frac{k^2-x^2}{k^2} \\ &= x\prod_{k=1}^{n}\frac{x-k}{-k}\cdot\frac{x+k}{k} \\ &= x\prod_{1\le|k|\le n}\frac{x+k}{k} \\ &= \frac{(-1)^n}{n!^2}\prod_{k=-n}^{n}(x+k) \end{aligned}$$

Using (🕮),

$$\begin{aligned} \frac{f_n(x+1)}{f_n(x)} &= \prod_{k=-n}^{n}\frac{(k+x+1)}{(k+x)} \\ &= \frac{n+1+x}{-n+x} \\ &\underset{n\to\infty}{\longrightarrow} -1. \quad \boxtimes \end{aligned}$$

Next we show that f shares another property with $x \mapsto \sin\pi x$:

¶3

$$f(x)f(x+\tfrac{1}{2}) = \frac{1}{2}f(\tfrac{1}{2})f(2x).$$

To see this for $x \mapsto \sin\pi x =: s(x)$:

$$s(x)s(x+\tfrac{1}{2}) = \sin\pi x\cos\pi x = \frac{1}{2}\sin 2\pi x = \frac{1}{2}s(\tfrac{1}{2})s(2x).$$

Proof of ¶3: Using (🕮), we have

$$\begin{aligned}
f(x)f(x+\tfrac{1}{2}) &\underset{n\to\infty}{\longleftarrow} f_n(x)f_n(x+\tfrac{1}{2})\\
&= x(x+\tfrac{1}{2})\prod_{1\le|k|\le n}\frac{k+x}{k}\cdot\frac{k+x+\frac{1}{2}}{k}\\
&= x(x+\tfrac{1}{2})\prod_{1\le|k|\le n}\frac{2k+2x}{2k}\cdot\frac{2k+1+2x}{2k+1}\cdot\frac{2k+1}{2k}\\
&= x(x+\tfrac{1}{2})\prod_{-2n\le\nu\le 2n+1,\ \nu\ne 0,1}\frac{\nu+2x}{\nu}\cdot\prod_{1\le|k|\le n}\frac{k+\frac{1}{2}}{k}\\
&= \frac{x(x+\frac{1}{2})}{1+2x}\prod_{1\le|\nu|\le 2n}\frac{\nu+2x}{\nu}\cdot\frac{2n+1+x}{2n+1}\cdot\prod_{1\le|k|\le n}\frac{k+\frac{1}{2}}{k}\\
&= \frac{x}{2}\prod_{1\le|\nu|\le 2n}\frac{\nu+2x}{\nu}\cdot\frac{2n+1+x}{2n+1}\cdot\prod_{1\le|k|\le n}\frac{k+\frac{1}{2}}{k}\\
&= \frac{1}{2}\cdot f_{2n}(2x)f_n(\tfrac{1}{2})\cdot\frac{2x+n+1}{2n+1}\\
&\underset{n\to\infty}{\longrightarrow} \frac{1}{2}\cdot f(2x)f(\tfrac{1}{2}). \quad ☑
\end{aligned}$$

To finish the proof we show that

¶4

$$f(x)=\frac{\sin\pi x}{\pi}\ \ \forall\ x\in\mathbb{R}.$$

Proof Let

$$C(x):=\frac{\sin\pi x}{x} \quad = \quad \sum_{n=0}^{\infty}\frac{\pi^{2n+1}(-x^2)^n}{(2n+1)!} \quad ,$$

then C is infinitely differentiable on $\mathbb{R}$, $C(x)>0$ on $(-1,1)$ & $C(0)=\pi$..
Thus, $\Phi:\mathbb{R}\to\mathbb{R}$ defined by

$$\Phi(x):=\frac{F(x)}{C(x)}=\frac{f(x)}{\sin\pi x}$$

is positive and C^1 on $(-1,1)$.

Moreover, on $\mathbb{R}$,

$$\text{(a)}\ \ \Phi(x+1)=\Phi(x)\ \ \&\ \ \text{(b)}\ \ \Phi(\tfrac{1}{2})\Phi(2x)=\Phi(x)\Phi(x+\tfrac{1}{2}).$$

By (a), Φ is positive and C^1 on $\mathbb{R}$ and it follows that

$$H:=\log\Phi-\log\Phi(\tfrac{1}{2}):\mathbb{R}\to\mathbb{R}\quad \texttt{is } C^1 \texttt{ on } \mathbb{R}$$

with

$$\text{(c)}\quad H(x+1) = H(x) \quad \& \quad \text{(d)}\quad H(2x) = H(x) + H(x+\tfrac{1}{2}).$$

It suffices to show that H is constant for then so is $\varPhi$ and

$$\varPhi \equiv \varPhi(0) = \frac{1}{\pi}.$$

Proof that H is constant :

By (c), $H'(x+1) = H'(x)$. We claim next that $\forall\ n \geq 1$,

$$\text{(e)}_n \qquad H(x) = \sum_{k=0}^{2^n-1} H\Big(\frac{x+k}{2^n}\Big).$$

This proven by induction using (d) which is $(\mathrm{e})_1$. To show $(\mathrm{e})_n \implies (\mathrm{e})_{n+1}$, assume that

$$H(x) = \sum_{k=0}^{2^n-1} H\Big(\frac{x+k}{2^n}\Big).$$

By (d), for each k,

$$H\Big(\frac{x+k}{2^n}\Big) = H\Big(\frac{x+k}{2^{n+1}}\Big) + H\Big(\frac{x+k}{2^{n+1}} + \frac{1}{2}\Big) = H\Big(\frac{x+k}{2^{n+1}}\Big) + H\Big(\frac{x+k+2^n}{2^{n+1}}\Big)$$

and so

$$\begin{aligned} H(x) &\overset{(\mathrm{e})_n}{=} \sum_{k=0}^{2^n-1} H\Big(\frac{x+k}{2^n}\Big) \\ &= \sum_{k=0}^{2^n-1} \Big(H\Big(\frac{x+k}{2^{n+1}}\Big) + H\Big(\frac{x+k+2^n}{2^{n+1}}\Big)\Big) \\ &= \sum_{k=0}^{2^n-1} H\Big(\frac{x+k}{2^{n+1}}\Big) + \sum_{k=2^n}^{2^{n+1}-1} H\Big(\frac{x+k}{2^{n+1}}\Big) \\ &= \sum_{k=0}^{2^{n+1}-1} H\Big(\frac{x+k}{2^{n+1}}\Big).\quad \boxtimes\ (\mathrm{e})_{n+1} \end{aligned}$$

Differentiating (e), we obtain the `averaging property`

$$\text{(f)} \qquad H'(x) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} H'\Big(\frac{x+k}{2^n}\Big).$$

Since H' is continuous and periodic $\exists\ \xi \in [0,1]$ such that

$$H'(\xi) = M := \max\{H'(x) :\ x \in \mathbb{R}\}.$$

By maximality

$$H'\Big(\frac{\xi+k}{2^n}\Big) \leq M\ \ \forall\ n \geq 1,\ 0 \leq k \leq 2^n - 1$$

and by the averaging property (f) (!)

$$H'(\frac{\xi+k}{2^n}) = M \ \forall \ n \ge 1, \ 0 \le k \le 2^n - 1.$$

By continuity of H, for $x \in [0,1]$,

$$H'(x) \xleftarrow[n\to\infty]{} H'(\frac{\xi + \lfloor 2^n x\rfloor}{2^n}) = M.$$

By (c), $H' \equiv M$ and $H(x) = Mx + H(0) \ \forall \ x \in \mathbb{R}$.

Moreover

$$M = H(x+1) - H(x) = 0$$

whence H is constant as advertised. ☑

Corollary: Wallis' Product

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right),$$

Proof

$$\frac{\pi}{2}\prod_{k=1}^{n}(1 - \frac{1}{4k^2}) \xrightarrow[n\to\infty]{} \sin\frac{\pi}{2} = 1. \quad ☑$$

Basel problem.

The *Zeta function* $\zeta : (1,\infty) \to \mathbb{R}_+$ is defined by

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The "`Basel problem`" was[30] to give a "closed formula" for $\zeta(2)$.

Corollary: (Euler)

$$\zeta(2) = \frac{\pi^2}{6}.$$

Proof

By the sine product formula, for $|x| < 1$,

$$S(x) := \frac{\sin\pi\sqrt{x}}{\pi\sqrt{x}} = \exp[\sum_{n=1}^{\infty}\log(1 - \frac{x}{n^2})] =: e^{-g(x)}$$

where

$$g(x) = -\sum_{n=1}^{\infty}\log(1 - \frac{x}{n^2}).$$

[30] proposed 1644 by Mengoli, solved 1734 without proof by Euler, many proofs in 1800's

From the definition of sin,

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n+1)!} \cdot x^n,$$

whence

$$S^{(n)}(0) = \frac{(-1)^n n! \pi^{2n}}{(2n+1)!}.$$

In particular

$$S'(0) = -\frac{\pi^2}{6}.$$

By (repeated application of) the differentiation of series theorem, g is C^∞ on $(-1,1)$ with

$$g^{(k)}(x) = \sum_{n=1}^{\infty} \frac{(k-1)!}{(n^2-x)^k} \quad (k \geq 1)$$

and

$$g^{(k)}(0) = \zeta(2k) \quad (k \geq 1).$$

In particular $S : (-1,1) \to \mathbb{R}$ is C^1 with

$$S'(x) = -g'(x)S(x)$$

whence

$$S'(0) = -2g'(0)S(0) = -2\zeta(2).$$

On the other hand, using the power series for S as above,

$$\zeta(2) = -\frac{S''(0)}{2} = \frac{\pi^2}{6}. \quad ☑$$

Exercise

(i) $\zeta(4) = \frac{\pi^4}{90}$; (ii) $\zeta(6) = \frac{\pi^6}{945}$.

Stirling's formula

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \quad \text{as} \quad n \to \infty.$$

Here $a_n \sim b_n$ means $\frac{a_n}{b_n} \to 1$. Stirling's formula is important in combinatorics and probability theory.

Proof of Stirling's formula.

We show first that $\exists\ C > 0$ such that

$$n! \sim Cn^{n+\frac{1}{2}}e^{-n} \quad \text{as} \quad n \to \infty \tag{$\triangle$}$$

and then calculate C using Wallis' Product.

Too see ($\triangle$), it suffices to show that

$$\log(n!) = n\log n - n + \frac{1}{2}\log n + c_n$$

where $c_n \to c \in \mathbb{R}$ (and then $C = e^c$ in ($\triangle$)).

Accordingly, define

$$A_n := \log(n!) - n\log n,$$

then $\log(n!) = n\log n + A_n$ and

$$\begin{aligned} A_{n+1} - A_n &= \log(n+1) - \left[(n+1)\log(n+1) - n\log n\right] \\ &= -n\log(1+\frac{1}{n}) \\ &= \log\left((1+\frac{1}{n})^{-n}\right) \xrightarrow[n\to\infty]{} -1 \end{aligned}$$

and

$$\frac{A_n}{n} = \frac{1}{n}\sum_{k=1}^{n-1}(A_{k+1} - A_k) + \frac{A_1}{n} \xrightarrow[n\to\infty]{} -1.$$

Thus,

$$\log(n!) = n\log n - n + B_n$$

where $\frac{B_n}{n} \to 0$. To continue,

$$B_n = \log(n!) - n\log n + n, \text{ whence } B_{n+1} - B_n = 1 - n\log(1+\frac{1}{n}).$$

By the error theorem `LET`(3), $\exists\ \theta_n \in (0, \frac{1}{n}) \quad (n \geq 1)$ so that

$$\log(1+\frac{1}{n}) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3(1+\theta_n)^3},$$

whence

$$B_{n+1} - B_n = 1 - n(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3(1+\theta_n)^3}) = \frac{1}{2n} - \gamma_n$$

where $\gamma_n = \frac{1}{3n^2(1+\theta_n)^3} \leq \frac{8}{3n^2}$.

It follows that $\sum_{n\geq 1}|\gamma_n| < \infty$ whence

$$B_n = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} + \varGamma_n$$

where

$$\Gamma_n = B_1 + \sum_{k=1}^{n} \gamma_k \to C = B_1 + \sum_{k=1}^{\infty} \gamma_k \in \mathbb{R}.$$

To finish the proof of (🧺), we recall that by the antiderivative test

$$\exists \lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \log n\Big) \in \mathbb{R}. \quad \square(🧺)$$

To finish the proof of Stirling's formula, we show that $C = \sqrt{2\pi}$.
Firstly, Wallis' partial products can be written in the form

$$\begin{aligned}\prod_{k=1}^{n}\left(1 - \frac{1}{4k^2}\right) &= \prod_{k=1}^{n} \frac{4k^2 - 1}{4k^2} \\ &= \frac{1}{4^n (n!)^2} \prod_{k=1}^{n} (2k+1)(2k-1) \\ &= \frac{2n+1}{4^n (n!)^2} \left(\prod_{k=1}^{n} (2k-1)\right)^2 \\ &= \frac{2n+1}{4^n (n!)^2} \left(\frac{(2n)!}{2^n n!}\right)^2 \\ &= \frac{2n+1}{4^{2n}} \binom{2n}{n}^2.\end{aligned}$$

By (△),

$$\binom{2n}{n} = \frac{(2n)!}{n!^2} \sim \frac{\sqrt{2}}{C\sqrt{n}} \cdot 4^n$$

whence using Wallis' product theorem:

$$\frac{2}{\pi} \underset{n\to\infty}{\longleftarrow} \frac{2n+1}{4^{2n}} \binom{2n}{n}^2 \sim \frac{2n+1}{4^{2n}} \left(\frac{\sqrt{2}}{C\sqrt{n}} \cdot 4^n\right)^2 \underset{n\to\infty}{\longrightarrow} \frac{4}{C^2}$$

and $C = \sqrt{2\pi}$. ☑

Demoivre's local, central limit theorem (`LCLT`. A fair coin is tossed n-times yielding "heads" s_n times and "tails" $n - s_n$ times. The probability distribution of s_n is "binomial", given by

$$\mathtt{Prob}(s_n = k) := \binom{n}{k} \frac{1}{2^n}.$$

Exercise

Prove Demoivre's `LCLT`: that $\forall\ t \in \mathbb{R}$,

$$\sqrt{\frac{\pi n}{2}} \cdot \mathtt{Prob}(s_{2n} = n + x_n) = \sqrt{\frac{\pi n}{2}} \cdot \frac{1}{4^n}\binom{2n}{n+x_n} \xrightarrow[n\to\infty]{} \epsilon^{-t^2}$$

$$\text{as } \ n \to \infty \ \&\ x_n \in \mathbb{Z},\ \frac{x_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} t.$$

end of coursenotes