INTRODUCTION TO DYNAMICAL SYSTEMS, COURSE NOTES, WINTER 2013.

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Basic Concepts

What is a dynamical system? In this course a "dynamical system" will (mainly) be a pair (X, T) where X is a set equipped with some structure (e.g. a topological space, a measure space or a differentiable manifold) and $T : X \to X$ is a map preserving the structure on X, i.e. T continuous if X is a topological space, T measurable and "non-singular" (preserving measure zero) if X is a measure space and T a differentiable map if X is a differentiable manifold.

Usually (but not always), we are interested in the "asymptotic behavior" of a dynamical system ($T^n x$ for large n).

`It's also possible to consider ``generalized" dynamical systems` $(X, \Gamma)$ `where` $X$ `is as above and` $\Gamma$ `is a semigroup under composition of maps preserving the structure on` X.

Stable behavior, attractors, contracting maps. Let (X, T) be a continuous map of a metric space.

- An *attractor* (for (X, T)) is a point $a \in X$ for which there is an open set $\varnothing \neq U \subset X$ with $T^n x \underset{n\to\infty}{\longrightarrow} a \ \forall \ x \in U$. The *domain of attraction* of the attractor a is the largest such open set. By continuity of T, an attractor a for T is necessarily a *fixed point* i.e. $Ta = a$.

The attractor is called *global* if its domain of attraction is X.

The map $T : X \to X$ is called a *contraction with respect to d* if $\exists \ \lambda = \lambda(T) = \lambda(T, d) < 1$ (the *contraction factor*) such that $d(Tx, Ty) \leq \lambda d(x, y)$.

`The metric is important here and we should say that` (X, T, d) `is a contraction under these conditions.`

1.1 Contraction mapping theorem

If $T : X \to X$ is a contraction of a complete metric space (X, d), then there is a global attractor for T.

Proof Let $\lambda \in (0,1)$ be the contraction factor and fix $x \in X$. It follows that for $n,\ k \ge 1$,

$$\begin{aligned} d(T^n x, T^{n+k}x) &\le \sum_{j=0}^{k-1} d(T^{n+j}x, T^{n+j+1}x) \\ &\le \sum_{j=0}^{k-1} \lambda^{n+j} d(x,Tx) \le \lambda^n \frac{d(x,Tx)}{1-\lambda}. \end{aligned}$$

Thus $(x, Tx, T^2x, \dots)$ is a Cauchy sequence in X and by completeness $\exists\ a(x) \in\ X$ so that $d(T^n x, a(x)) \xrightarrow[n\to\infty]{} 0$.

By continuity of T, $Ta(x) = a(x)$. To see that $a(x)$ does not depend on X:

$$d(a(x), a(y)) = d(T^n a(x), T^n a(y)) \le \lambda^n d(a(x), a(y)) \xrightarrow[n\to\infty]{} 0. \quad \boxtimes$$

Iterated function systems & hyperspace

Let (X,d) be a metric space. An *iterated function system (IFS)* on (X,d) is a finite collection of contractions $w_1, \dots w_N : X \to X$.

Associated to an iterated function system, there is an interesting contraction of the *hyperspace* $\mathcal{H}(X)$ space of nonempty compact subsets of X:

$$W(K) := \bigcup_{k=1}^{N} w_k(K).$$

The *Hausdorff metric* h on $\mathcal{H}(X)$ is defined by

$$h(K,K') := \max\{\overline{d}(K,K'), \overline{d}(K',K)\}$$

where

$$\overline{d}(A,B) := \max_{x\in A}(\min_{y\in B} d(x,y)).$$

Note that (!)

$$h(K,K') = \min\{\epsilon > 0 :\ K \subset B(K',\epsilon),\ \&\ K' \subset B(K,\epsilon)\}$$

Proposition 2

$(\mathcal{H}(X), h)$ is a metric space.

Proof

In case $A \smallsetminus B \ne \varnothing$ choose $a \in A \smallsetminus B$, then by compactness, $\min_{y\in B} d(a,y) > 0$, whence $d(A,B) > 0$. It follows that $h(A,B) = 0$ iff $A = B$.

To prove the triangle inequality, note first that

$$d(a,c) \le d(a,b) + d(b,c)\ \forall\ a \in A,\ b \in B,\ c \in C.$$

Fixing $a \in A,\ c \in C$ and minimizing over $b \in B$ we obtain that $\exists\ b_0 \in B$ such that

$$d(a,c) \le d(a,b_0) + d(b_0,c) = \min_{y \in B} d(a,y) + d(b_0,c),$$

whence fixing $a \in A$ and minimizing over $c \in C$:

$$\min_{x \in C} d(a,x) \le \min_{y \in B} d(a,y) + \min_{z \in C} d(b_0,z) \le \overline{d}(A,B) + \overline{d}(B,C).$$

□

Proposition 3 *If (X,d) is either compact, or $\mathbb{R}^d$ with the Euclidean distance, then $(\mathcal{H}(X),h)$ is complete.*

Proof sketch
Suppose that $A_n \in \mathcal{H}(X),\ \ (n \ge 1)$ is a h-Cauchy sequence and define

$$A := \{x \in X:\ \exists\ x_n \in A_n \text{ s.t. } x_n \to x\}.$$

¶1 $A \ne \varnothing$

Choose $n_i \uparrow$ such that $h(A_k,A_\ell) < \frac{1}{2^i}\ \forall\ k,\ell \ge n_i$. Fix $x_1 \in A_{n_1}$, then $\exists\ x_2 \in A_{n_2}$ with $d(x_1,x_2) < \frac{1}{2}$. Continuing, get $x_k \in A_{n_k}\ \ (k \ge 1)$ such that $d(x_k,x_{k+1}) < \frac{1}{2^k}$. Evidently (!) $\{x_k\}_k$ is a d-Cauchy sequence. Let $x_k \to a$. We need to show $\exists\ a_n \in A_n,\ a_n \to a$. To do this we show $\exists\ a_n \in A_n$, Cauchy s.t. $a_{n_i} = x_i$.

Indeed for $n_i < k \le n_{i+1}$, choose $a_k \in A_k$ such that
$d(a_k, x_{n_{i+1}}) = \min_{y \in A_k} d(y, x_{n_{i+1}}) \le h(A_k, A_{n_{i+1}}) < \frac{1}{2^i}$. □

¶2 A is closed.

Suppose $b_i \in A,\ b_i \to b$. $\exists\ n_i \uparrow$ such that $\forall\ i,\ \exists\ x_i \in A_{n_i}$ with $d(x_i,b_i) < \frac{1}{2^i}$. As before, $\exists\ a_n \in A_n$, Cauchy s.t. $a_{n_i} = x_i$. It follows that $a_n \to b$ whence $b \in A$. □

¶3 $\forall\ \epsilon > 0\ \exists\ N$ such that $A \subset B(A_n,\epsilon)\ \forall\ n \ge N$.

Fix $N \ge 1$ such that $h(A_k,A_\ell) < \epsilon\ \forall\ k,\ell \ge N$, then $A_k \subset B(A_\ell,\epsilon)\ \forall\ k,\ell \ge N$. Fix $a \in A$ and let $a_n \in A_n,\ a_n \to A$. Since $B(A_\ell,\epsilon)$ is closed,

$$a\ \leftarrow a_k \in B(A_\ell,\epsilon)\ \forall\ k,\ell \ge N.$$

□

¶4 A is compact.

Closed by ¶2 and precompact by ¶3.

¶5 $\forall\ \epsilon > 0\ \exists\ N$ such that $A_n \subset B(A,\epsilon)\ \forall\ n \ge N$.

Fix $N \ge 1$ such that $h(A_k,A_\ell) < \epsilon/2\ \forall\ k,\ell \ge N$, then $A_k \subset B(A_\ell,\epsilon)\ \forall\ k,\ell \ge N$. We show that $A_k \subset B(A,\epsilon)\ \forall\ k \ge N$.

Fix $y \in A_k$. $\exists\ k \le N_i \uparrow$ such that $A_m \subset B(A_n, \frac{\epsilon}{2^j})\ \forall m,n \ge N_j$. $\exists\ x_j \in A_{n_j}$ such that $d(y,x_1) < \frac{\epsilon}{2},\ d(x_j,x_{j+1}) < \frac{\epsilon}{2^{j+1}}$. It follows that

$x_j \to z \in X$. As before, $z \in A$. Also $d(y, x_j) \le \epsilon\ \forall\ j$ whence $d(y, z) \le \epsilon$. It follows that $y \in B(A, \epsilon)$. □ □

Exercises

Prove that
1) if (X, d) is compact, then so is $(\mathcal{H}(X), h)$.
2) $(\mathcal{H}(\mathbb{R}^d), h)$ is pathwise connected.

Proposition 4

$$h(W(A), W(B)) \le \max_{1\le k\le n} \lambda(w_k) h(A, B).$$

Proof Note that

$$h(K, K') = \min\{\epsilon > 0 : \ K \subset B(K', \epsilon),\ \&\ K' \subset B(K, \epsilon)\}$$

Thus

$$h(W(A), W(B)) \le \max_{1\le k\le n} h(w_k(A), w_k(B)).$$

Now for $a \in A,\ b \in B$,

$$d(w_k(a), w_k(b)) \le \lambda(w_k) d(a, b)$$

whence

$$\min_{y\in w_k(B)} d(w_k(a), y) \le \lambda(w_k) \min_{b\in B} d(a, b)$$

and

$$\overline{d}(w_k(A), w_k(B)) \le \lambda(w_k)\overline{d}(A, B).$$

□

Corollary 5

Each IFS has a unique attractor.

Proof By propositions 3 and 4 and the contraction mapping theorem, $\exists\ K \in \mathcal{H}(X)$ such that $W^n(A) \to K\ \forall\ A \in \mathcal{H}(X)$. □

Exercise. Let

$$X = [0, 1],\ w_0(x) := \frac{x}{3}\ \&\ \ w_1(x) := \frac{x+2}{3};$$

then the attractor of the IFS (w_0, w_1) is the (middle third) Cantor set.

Hutchinson's formula *Let $W(K) = K$ and suppose that $w_i(K)$ $(1 \le i \le N)$ are disjoint, then the box dimension d of K coincides with its Hausdorff dimension, and satisfies*

$$\sum_{i=1}^{N} \lambda(w_i)^d = 1.$$

Proof Exercise, or see Barnsley's book *Fractals Everywhere.*

Picard's solution of initial value ODE

Let $d \geq 1$ & $U \subset \mathbb{R} \times \mathbb{R}^d$ be open and let $f : U \to \mathbb{R}^d$ be continuous. Given $(t_0, x_0) \in U$ and $\epsilon > 0$, we say that $x : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^d$, C^1 *solves the initial value problem* $\texttt{IVP}(t_0, x_0)$ if

$$x(t_0) = x_0, \quad (t, x(t)) \in U, \ \& \quad \frac{dx}{dt}(t) = f(t, x(t)) \quad \forall \ t \in (t_0 - \epsilon, t_0 + \epsilon).$$

Picard's Theorem *If f is Lipschitz continuous, then* $\forall (t_0, x_0) \in U$, $\exists$ *a unique solution of* $\texttt{IVP}(t_0, x_0)$.

Proof Fix $(t_0, x_0) \in U$. Suppose that V is open with $(t_0, x_0) \in V$ and $\overline{V} \subset U$. Let $\epsilon > 0$ with $B((t_0, x_0), \epsilon) \subset \overline{V}$. Let

$$X = X_{(t_0,x_0),\epsilon} := \{x : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^d : \ (t, x(t)) \in \overline{V} \ \ \forall \ t \in (t_0 - \epsilon, t_0 + \epsilon)\}$$

with the metric $d = d_{(t_0,x_0),\epsilon}$ defined by

$$d(x, y) := \sup_{t \in (t_0 - \epsilon, t_0 + \epsilon)} \|x(t) - y(t)\|_2.$$

It follows that (X, d) is a complete metric space.

Next, define $\Phi : X \to X$ by

$$\Phi(x)(t) := x_0 + \int_{t_0}^{t} f(x(s), s) ds$$

where $\int_u^v := -\int_v^u$.

We claim that for $\epsilon > 0$ small enough, Φ is a contraction.

To see this, for $x, y \in X_{(t_0,x_0),\epsilon}$,

$$\begin{aligned}
\|\Phi(x)(t) - \Phi(y)(t)\|_2 &= \| \int_{t_0}^{t} (f(x(s), s) - f(y(s), s)) ds \|_2 \\
&\leq \int_{t_0}^{t} \|(f(x(s), s) - f(y(s), s))\|_2 ds \\
&\leq \text{Lip(f)} \int_{t_0}^{t} \|x(s) - y(s)\|_2 ds \\
&\leq \text{Lip(f)} \cdot |t - t_0| \cdot d(x, y);
\end{aligned}$$

whence

$$d(\Phi(x), \Phi(y)) \leq \Lambda_\epsilon d(x, y)$$

where $\Lambda_\epsilon := \text{Lip(f)}\epsilon < 1$ for ϵ small.

For such small $\epsilon > 0$, there is a unique global attractor $x \in X$ satisfying $\Phi(x) = x$ or:

$$x(t) := x_0 + \int_{t_0}^{t} f(s, x(s))ds$$
$$\implies x(t_0) = x_0 \ \& \ \frac{dx}{dt}(t) = f(t, x(t)). \quad ☑$$

Newton's method

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is C^1. Newton's method is an iterative procedure to find $x \in f^{-1}\{0\}$.

The procedure

Given $u \in \mathbb{R}$, draw the tangent line L to the graph of f at $(u, f(u))$ and take v as the x-coordinate of the intersection of L with the x-axis, i.e. $\{(v, 0)\} = L \cap (\mathbb{R} \times \{0\})$.

The equation of L is $\frac{y-f(u)}{x-u} = f'(u)$, whence v satisfies $\frac{-f(u)}{v-u} = f'(u)$, or

$$v =: T_f u = u - \tfrac{f(u)}{f'(u)}.$$

Almost any $T = T_f$ with

$$f(x) = e^{\int \frac{1}{x - Tx} dx}.$$

Set $u_0 := u, \ u_{n+1} := T_f u_n$.

Theorem 1.3 (Raphson)

Suppose that $r \geq 1$ and that
f is C^r and $z \in \mathbb{R}$, $f(z) = 0$, $f'(z) = \cdots = f^{(r-1)}(z) = 0$, $f^{(r)}(z) \neq 0$,
then $\exists \ \epsilon > 0$ such that $u_n \to z$ as $n \to \infty$ whenever $|u_0 - z| < \epsilon$.

Proof

WLOG, $z = 0$ & $f^{(r)}(0) > 0$. Fix $\delta > 0$ such that $|e^{\pm 2\delta} - 1| \leq \frac{1}{2}$ and let $\epsilon > 0$ satisfy:

$$\tfrac{f(u)}{u^r} = \tfrac{f^{(r)}(0)}{r!} e^{\pm\delta}, \qquad \tfrac{f'(u)}{u^{r-1}} = \tfrac{f^{(r)}(0)}{(r-1)!} e^{\pm\delta} \ \forall \ |u| < \epsilon.$$

For $|u| < \epsilon$: $\quad \frac{f(u)}{f'(u)} = \frac{u}{r} e^{\pm 2\delta}$, whence

$$|T_f(u)| = |u - \tfrac{u}{r} e^{\pm 2\delta}| \leq (1 - \tfrac{1}{r})|u| + \tfrac{u}{r}|e^{\pm 2\delta} - 1| \leq (1 - \tfrac{1}{2r})|u| \leq |u| < \epsilon$$

and

$$|u_n| = |T_f^n(u)| < (1 - \tfrac{1}{2r})^n |u| \to 0.$$

□

Example.

If $f(x) := x^2 - a$, $(a > 0)$, then $T_f x = \frac{x}{2} + \frac{a}{2x}$ and that $T_f^n(x) \to \sqrt{a} \ \forall \ x > 0$ and $T_f^n(x) \to -\sqrt{a} \ \forall \ x < 0$.

Question for later.

How does $T_f^n(z)$ behave for $z \in \mathbb{C}$?

Example.

Let (X,T) be a dynamical system with a global attractor $a \in X$. Fix $d \geq 2$ and define $T_d : X_d := X \times \mathbb{Z}_d \to X_d$ by $T_d(x,i) := (Tx, i+1)$ where $Z_d := \mathbb{Z}/d\mathbb{Z}$. It follows that T_d^i has no attractor for $1 \leq i < d$ and that (a,i) is a global attractor for T_d^d.

Exercise 1.0

1) No contraction of a compact metric space is a homeomorphism.

Exercise 1.1.

Let $X_r := \{z \in \mathbb{C} : \ |z| \leq r\}$, $S_2 z := z^2$, then $S_2 : X_r \to X_r \ \forall \ r \leq 1$. Fix $d(x,y) := |x-y|$.

(i) Show that (X_r, S_2, d) is a contraction with respect to $d \ \forall \ r < \frac{1}{2}$ with $\lambda(S_2, d) = 2r$ but not for $\frac{1}{2} \leq r \leq 1$.

(ii) For which $r \in [\frac{1}{2}, 1]$ can you find a metric $\rho \sim d$ so that (X_r, S_2, ρ) is a contraction?

Exercise 1.2.

Let the continuous map of a Polish space (X,T) be *nowhere-expanding* in the sense that $d(Tx, Ty) < d(x,y) \ \forall \ x, \ y \in X$.

(i) Show that (X,T) has a global attractor if either

(a) X is compact; or

(b)☆ there is a complete metric d on X and $\psi : [0,\infty) \to \mathbb{R}$ continuous, strictly increasing satisfying $\psi(0) = 0$ and

$$d(Tx, Ty) \leq d(x,y) - \psi(d(x,y)) \ \forall \ x, \ y \in X.$$

(ii) Does every nowhere-expanding map (X,T) of a Polish space have a fixed point?

Some unstable dynamical properties

An "`unstable property`" should ensure that there are no attractors.

Minimality & transitivity.

A homeomorphism $T : X \to X$ of a metric space X is called:

- (*topologically*) *transitive* if some orbit is dense (i.e. $\exists \ x \in X$, $\overline{\{T^n x : \ n \in \mathbb{Z}\}} = X$);
- *minimal* if every orbit is dense (i.e. $\overline{\{T^n x : \ n \in \mathbb{Z}\}} = X \quad \forall \ x \in X$).

Note that T minimal $\implies$ T transitive $\implies$ no power of T has an attractor.

Rotations of $\mathbb{T}$. Recall that $\mathbb{T} := \mathbb{R}/\mathbb{Z} \cong [0,1)$. For $\alpha \in \mathbb{T}$ define $R_\alpha : \mathbb{T} \to \mathbb{T}$ by $R_\alpha x = x + \alpha \mod 1$.

Proposition

If $\alpha \notin \mathbb{Q}$, then R_α is minimal.

Proof Consider $\mathbb{T} \cong [0,1)$ equipped with the metric $d(x,y) := \min\{|x-y|,\ 1-|x-y|\}$, then R_α is an *isometry* in the sense that $d(R_\alpha x, R_\alpha y) = d(x,y)$.

• It suffices to show that $\overline{\{n\alpha \mod 1\}_{n\ge 1}} = \mathbb{T}$
(as then $\overline{\{R_\alpha^n x :\ n \in \mathbb{Z}\}} = x + \overline{\{n\alpha \mod 1\}_{n\ge 1}} = \mathbb{T}\ \forall\ x \in \mathbb{T}$).

• To this end, we claim that $\forall\ \epsilon > 0,\ \exists\ \ell \ge 1,\ d(\{\ell\alpha\}, 0) < \epsilon$. To see this, let $\mathfrak{p}$ be a finite partition of $\mathbb{T}$ into sets of diameter $< \epsilon$. Since $\{k\alpha\} \ne \{k'\alpha\}$ for $k \ne k'$, we have (using the pigeon-hole principle) that $\exists\ j \le k \le \#\mathfrak{p} + 1$ and $p \in \mathfrak{p}$ with $\{j\alpha\},\ \{k\alpha\} \in p$, whence $d(\{j\alpha\}, \{k\alpha\}) < \epsilon$. If $\ell = k - j$ then (since R_α is an isometry), $d(\{\ell\alpha\}, 0) < \epsilon$. This shows that $\forall\ x \in \mathbb{T},\ \exists\ n \in \mathbb{Z},\ d(x, \{n\ell\alpha\}) < \epsilon$. ☐

An example

Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$.
Let $S_2 z := z^2$ & $S_2(\infty) = \infty$, then $S_2 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.
Note that

$$S_2^n(x) \xrightarrow[n\to\infty]{} \begin{cases} 0 & |x| < 1; \\ \infty & |x| > 1. \end{cases}$$

Proposition 1.2

The dynamical system $(\mathbb{S}^1, S_2)$ is transitive where $\mathbb{S}^1 := \{x \in \mathbb{C} :\ |x| = 1\}$.

Proof

Let $\Omega := \{0,1\}^{\mathbb{N}}$ and define $\psi : \Omega \to \mathbb{S}^1$ by

$$\psi(\omega) := \exp\left[2\pi i \sum_{n=1}^{\infty} \frac{\omega_n}{2^n}\right].$$

Note that for $\omega = (\omega_1, \omega_2, \dots) \in \Omega$,

$$\begin{aligned} S_2(\psi(\omega)) &= \exp\left[2\pi i 2 \sum_{n=1}^{\infty} \frac{\omega_n}{2^n}\right] \\ &= \exp\left[2\pi i \left(\omega_1 + \sum_{n=1}^{\infty} \frac{\omega_{n+1}}{2^n}\right)\right] \\ &= \psi(\sigma\omega) \end{aligned}$$

Where $\sigma(\omega) = (\omega_2, \omega_3, \dots)$ aka *the shift*.

It follows that $S_2^K(\psi(\omega)) = \psi(\sigma^K \omega)$.

Also, if $\omega,\ \theta \in \Omega$ and $\omega_k = \theta_k\ \forall\ 1 \le k \le N$, then

$$\begin{aligned} |\psi(\omega) - \psi(\theta)| &= \left|\exp\left[2\pi i \sum_{n=1}^{\infty} \frac{\omega_n}{2^n}\right] - \exp\left[2\pi i \sum_{n=1}^{\infty} \frac{\theta_n}{2^n}\right]\right| \\ &= \left|\exp\left[2\pi i \sum_{n=N+1}^{\infty} \frac{\omega_n - \theta_n}{2^n}\right] - 1\right| \\ &\le \frac{\pi}{2^N}. \end{aligned}$$

We now use all this to prove the proposition.

We claim first that

(🚗) $$\exists\ \omega^* \in \Omega \text{ such that } \forall\ N \ge 1,\ \eta_1, \dots, \eta_N = 0, 1; \quad \exists\ \kappa \ge 1 \text{ such that } \sigma^\kappa(\omega^*)_j = \eta_j \quad \forall\ 1 \le j \le N.$$

To see this enumerate all the finite sequences of 0s and 1's and concatenate them to obtain $\omega^* \in \Omega$:

(i.e.) $$\Omega^* := \bigcup_{n=1}^{\infty} \{0,1\}^n = \{\eta^{(k)} = (\eta_1^{(k)}, \dots, \eta_{\nu_k}^{(k)}) :\ k \in \mathbb{N}\} \quad \& \quad \omega^* := (\eta^{(1)}, \eta^{(2)}, \dots);$$

then

$$\sigma^{\sum_{1 \le j \le k-1} \nu_j}(\omega^*)_\ell = \eta_\ell^{(k)} \quad \forall\ 1 \le \ell \le \nu_k$$

and ω^* satisfies (🚗).

Next we claim that $z = \psi(\omega^*)$ is as advertised.

Let $y \in \mathbb{S}^1,\quad y = \psi(\eta)$ and fix $\epsilon > 0$. We find $\kappa \ge 1$ so that

$$|S_2^\kappa(z) - y| \ < \ \epsilon.$$

To this end, choose $N \ge 1$ so that $\frac{\pi}{2^N} < \epsilon$ and find κ so that

$$\sigma^\kappa(\omega)_j = \eta_j \quad \forall\ 1 \le j \le N;$$

then, using the above

$$|S_2^\kappa(z) - y| = |S_2^\kappa \psi(\omega^*) - \psi(\eta)| = |\psi(\sigma^\kappa \omega^*) - \psi(\eta)| \le \frac{\pi}{2^N} < \epsilon. \quad ☑$$

Exercise 1.3.

Show that S_2^κ is

- transitive $\forall\ \kappa \in \mathbb{N}$;
- not minimal &
- ☆ $\exists\ x \in \mathbb{S}^1$ with $\overline{\{S_2^{\kappa n}(x) :\ n \geq 1\}} = \mathbb{S}^1\quad \forall\ \kappa \in \mathbb{N}$.

Week # 2, 23/10/2013.

Exercise. There is a gap in the proof of Picard's theorem. (i) Find it. (ii) Fill it.

NEWTON'S METHOD WHEN $f(x) > 0\ \forall\ x \in \mathbb{R}$?

Recall the example on p.6:

If $g(x) := x^2 - 1$, then $T_g x = \frac{x}{2} + \frac{1}{2x}$ and that $T_g^n(x) \to 1\ \forall\ x > 0$ and $T_g^n(x) \to -1\ \forall\ x < 0$.

We now check behaviour of $T_f^n(z)$ for $z \in \mathbb{C}$.

For $x \in \mathbb{R}$:

$$\frac{1}{i} T_g(ix) = \frac{1}{i}\left(\tfrac{ix}{2} + \tfrac{1}{2ix}\right) = \tfrac{x}{2} - \tfrac{1}{2x} = T_f(x)$$

where $f(x) := x^2 + 1 \geq 1$.

Consider $f(x) = x^2 + 1$, then $T_f(x) = \frac{1}{2}(x - \frac{1}{x})$.

Proposition 1.4

$\exists\ x \in \mathbb{R}$ *such that* $\overline{\{T_f^n x :\ n \geq 1\}} = \mathbb{R}$.

Proof sketch

We first show that $\exists$ a homeomorphism $\Phi : \mathbb{R} \to \mathbb{S}^1 \setminus \{1\}$, so that

$$T_f(x) = \Phi^{-1}(\Phi(x)^2).$$

To see this, define $\Psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by $\Psi(z) := \frac{z-i}{z+i}$ & $\Psi(\infty) = 1$, then (!) Ψ is a homeomorphism and

$$\Psi\left(\tfrac{1}{2}\left(z - \tfrac{1}{z}\right)\right) = \Psi(z)^2.$$

Moreover for $x \in \mathbb{R}$,

$$\Psi(x) = \frac{x - i}{x + i} = \frac{x^2 - 1}{x^2 + 1} + \frac{2xi}{x^2 + 1}\ \in\ \mathbb{S}^1 \setminus \{1\}.$$

Thus $\Phi := \Psi|_{\mathbb{R}}$ is as required and $T_f(x) = \Phi^{-1}(S_2(\Phi(x)))$ where $S_2(z) := z^2$ as before.

By proposition 1, $\exists\ z \in \mathbb{S}^1$ so that $\overline{\{S_2^n z :\ n \geq 1\}} = \mathbb{S}^1$. Evidently $S_2^n(z) \neq 1\ \forall\ n \geq 1$ and so if $x = \Phi^{-1}(z) \in \mathbb{R}$ then $T_f^n x = \Phi^{-1}(\{S_2^n z)\ \forall\ n \geq 1$ and

$$\overline{\{T_f^n x :\ n \geq 1\}} = \overline{\Phi^{-1}(\{S_2^n z :\ n \geq 1\})} = \mathbb{R}. \quad \boxtimes$$

Exercise 1.4.

For $f(x) = 1 + x^2$, show that no power of T_f can have an attractor.

Exercise 1.5. ★

For $p \in (0,1)$, $N \in \mathbb{N}$, set $f(x) = f_{p,N} := (1 + x^{2N})^{\frac{1}{2pN}}$, then $T_f x = (1-p)x - \frac{p}{x^{2N-1}}$. Show that $\exists\ p \in (0,1)$, $N \in \mathbb{N}$ so that $T^2_{f_{p,N}}$ has an attractor.

`Hint`: Find $x \in \mathbb{R}$ so that $|(T^2)'(x)| < 1$.

Exercise 1.6 (`Open Problem`).

Suppose that $f : \mathbb{R} \to \mathbb{R}_+$ is smooth (e.g. C^r, $r \geq 2$ or analytic), and that $\log f$ is strictly convex and satisfies $f(x) \underset{|x|\to\infty}{\longrightarrow} \infty$. Show that $\exists\ x \in \mathbb{R}$ so that $\overline{\{T_f^n x :\ n \geq 1\}} = \mathbb{R}$.

Complex dynamics: Fatou and Julia sets.

For a rational map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ a point $z \in \widehat{\mathbb{C}}$ is called *Fatou* if $\exists$ an open set $U \ni z$ on which $\{R^n :\ n \geq 1\}$ is a `normal family` in the sense that $\forall\ n_k \to \infty\ \exists\ m_\ell = n_{k_\ell} \to \infty$ & $\phi : U \to \widehat{\mathbb{C}}$ so that

$$\sup_{\omega\in K} \rho(R^{m_\ell}(\omega), \phi(\omega)) \underset{\ell\to\infty}{\longrightarrow} 0 \ \ \forall\ K \subset U \ \text{ compact},$$

where ρ is (equivalent to) the spherical metric on $\widehat{\mathbb{C}}$.

The *Fatou set* $F(R) := \{$ Fatou points of $R\}$. It is open and invariant: $(R^{-1}F(R) = F(R)$. The *Julia set* of R is $J(R) := \widehat{\mathbb{C}} \setminus F(R)$. It is closed and invariant.

It follows from proposition 1.2 that $J(S_2) = \mathbb{S}^1$ whence (!) for $f(x) = x^2 - 1$, $J(T_f) = i\mathbb{R}$.

Exercise 1.7.

Show that for $K \subset \widehat{C} \setminus \{0, \infty\}$ compact,

$$S_2^{-n}(K) := \{z \in \widehat{C} :\ S_2^n(z) \in K\} \xrightarrow[n\to\infty]{\mathcal{H}(\widehat{C})} \mathbb{S}^1.$$

so is $T^n\ \forall\ n \geq 1$.

`Hint`: $S_2^{-1}(J) = v_1(J) \cup v_2(J)$.

§2 Homeomorphisms of the circle.

One of the aims in dynamics is "classification" of dynamical systems up to "conjugacy". This section is devoted to the classification of homeomorphisms of the circle up to conjugacy by homeomorphism as done by Poincaré, Denjoy and Herman.

The *additive circle* is $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. It is represented by the metric space $([0,1), d)$ where $d(x,y) := \min_{n\in\mathbb{Z}} |x - y + n|$. The *multiplicative circle* is $\mathbb{S}^1 := \{z \in \mathbb{C} :\ |z| = 1\} \cong e^{2\pi i \mathbb{T}}$.

Lift of a continuous map of $\mathbb{T}$.

Let $T : \mathbb{T} \to \mathbb{T}$. A map $R : \mathbb{R} \to \mathbb{R}$ is called a *lift* of T if $R(x) + \mathbb{Z} = T(x + \mathbb{Z})$.

If R is a lift of T, then so is $R + N$ for any $N : \mathbb{R} \to \mathbb{Z}$.

Lifting theorem for $\mathbb{T}$

A continuous map of $\mathbb{T}$ has a continuous lift.

This is a special case of a more general proposition which we'll prove now. We'll need other special cases later.

Covering maps & **deck transformations.** Let X, Y be metric spaces.

A surjection $\pi : X \to Y$ is called a *covering map* if it is a local homeomorphism i.e.

- $\forall\ x \in X\ \exists\ \epsilon > 0$ so that $\pi : B(x, \epsilon) \to \pi(B(x, \epsilon))$ is a homeomorphism.

Let $\pi : X \to Y$ be a covering map. A homeomorphism $\gamma : X \to X$ is called a *deck transformation of* π if $\pi \circ \gamma = \pi$.

Let $\Gamma_\pi :=$ {deck transformations of π, then Γ_π is a group under composition.

The covering map $\pi : X \to Y$ is called a *regular* if

$$\pi^{-1}\{y\} = \{\gamma(x) :\ \gamma \in \Gamma_\pi\} \quad \forall\ x \in \pi^{-1}\{y\}.$$

Example. Let $X = \mathbb{R}$ & $Y = \mathbb{S}^1$, then $\pi : X \to Y$ defined by $\pi(x) = e^{ix}$ is a regular covering map with $\Gamma_\pi = \{\gamma_n :\ n \in \mathbb{Z}\}$ where $\gamma_n(x) := x + 2\pi n$.

Lifting Theorem *Suppose that X is a simply connected, separable metric space, Y is a compact metric space and $\pi : X \to Y$ is a regular cover.*

If $f : X \to Y$ is uniformly continuous, then $\exists\ F : X \to X$ continuous so that $\pi \circ F \equiv f$.

Proof of the Lifting Theorem

- $\exists\ \Delta > 0$ so that for any ball $B \subset Y$ of radius Δ, $\exists\ \phi_B : B \to X$ continuous with $\pi \circ \phi_B \equiv \mathrm{Id}|_B$.

`Proof of •`

$\exists$ an open covering $\{U_1, U_2, \ldots, U_N\}$ of Y so that for each $k\ \exists\ \phi_k : U_k \to X$ continuous so that $\pi \circ \phi_k = \mathrm{Id}_{U_k}$. It suffices to take $\Delta =$ the Lebesgue number of $\{U_1, U_2, \ldots, U_N\}$ so that for any ball $B \subset Y$ of radius $\Delta, \exists\ 1 \le k \le N$ with $B \subset U_k$. ☑

Path homotopy.

Let Z be a metric space. Two paths $P,Q:[0,1]\to Z$ with the same initial point $a=P(0)=Q(0)$ and endpoint $b=P(1)=Q(1)$ are *path homotopic* if $\exists\ h:[0,1]\times[0,1]\to Z$ continuous so that $h(0,t)=P(t)$ & $h(1,t)=Q(t)\ \forall\ t\in[0,1]$ (i.e. h is an *homotopy*) and in addition: $h(s,0)=a$ & $h(s,1)=b\ \forall\ s\in[0,1]$.

Such an h is called a *path homotopy* (from P to Q).

Lemma

(i) *If $P:[0,1]\to Y$ is a path and $q\in X$, $\pi(q)=P(0)$, then $\exists$ a path $Q:[0,1]\to X$ so that $Q(0)=q$ and $\pi\circ Q\equiv P$.*

(ii) *If $Q_1,\ Q_2:[0,1]\to X$ are paths with $\pi\circ Q_1\equiv\pi\circ Q_2$ and $Q_1(t)=Q_2(t)$ for some $t\in[0,1]$, then $Q_1\equiv Q_2$.*

(iii) *If $Q_1,\ Q_2:[0,1]\to X$ are paths satisfying: $Q_1(0)=Q_2(0)$ and $\pi\circ Q_1$ is path homotopic in Y to $\pi\circ Q_2$, then $Q_1(1)=Q_2(1)$ & Q_1 is path homotopic in X to Q_2.*

`Proof of (i)`

- $\exists\ 0=t_0<t_1<t_2<\cdots<t_n=1$ so that $P([t_{k-1},t_{k+1}])\subset B_k:=B(P(t_k),\Delta)$ for $1\le k\le n$ (where $t_{n+1}:=1$). Define $Q_1:[0,t_2]\to X$ by $Q_1:=\gamma_1\circ\phi_{B_1}\circ P$ where $\gamma_1\in\Gamma_\pi$ satisfies $\gamma_1(\phi_{B_1}(P(0)))=q$.

Next, define

- $Q_2:[t_1,t_3]\to X$ by $Q_2:=\gamma_2\circ\phi_{B_2}\circ P$ where $\gamma_2\in\Gamma_\pi$ satisfies $\gamma_2(\phi_{B_2}(P(t_2)))=Q_1(t_2)$;
- ...

⋮

- $Q_k:[t_{k-1},t_{k+1}]\to X$ by $Q_k:=\gamma_k\circ\phi_{B_k}\circ P$ where $\gamma_k\in\Gamma_\pi$ satisfies $\gamma_k(\phi_{B_k}(P(t_k)))=Q_{k-1}(t_k)$ for $k=3,\dots,n$

The required path is defined by

$$Q(x):=Q_k(x)\ \text{ for }\ x\in[t_{k-1},t_{k+1}]\ \ (1\le k\le n).\quad ☑$$

`Proof of (ii)`

Let $S:=\{t\in[0,1]:\ Q_1(t)=Q_2(t)\}$, then by assumption, $S\ne\varnothing$. By continuity, S is closed in $[0,1]$ and it suffices to show that S is open. To this end, suppose that $s\in S$ and set $u:=Q_1(s)=Q_2(s),\ z=\pi(u)$.

By continuity $\exists\ \epsilon>0$ so that $Q_i([s-\epsilon,s+\epsilon])\subset B:=B(z,\Delta)$ $(i=1,2)$. We can choose $\gamma\in\Gamma_\pi$ so that $\gamma\circ\phi_B(z)=u$.

It follows by continuity that for $t\in(s-\epsilon,s+\epsilon)$,

$$Q_1(t)=\gamma\circ\phi_B(\pi(Q_1(t)))=\gamma\circ\phi_B(\pi(Q_2(t)))=Q_2(t).\quad ☑$$

`Proof of (iii)`

Let $h : [0,1] \times [0,1] \to Y$ be a path homotopy, that is: a continuous map satisfying

$$h(s,j) = Q_i(j) \;\;\&\;\; h(i-1,t) = \pi(Q_i(t)), \quad (i = 1,2,\; j = 0,1,\; s,t \in [0,1]).$$

Fix $0 = t_0 < t_1 < \cdots < t_n = t_{n+1} = 1$ so that

$$h([t_{k-1}, t_{k+1}] \times [t_{\ell-1}, t_{\ell+1}]) \subset B_{k,\ell} := B(h(t_k, t_\ell), \Delta) \;\forall\; 1 \le k,\; \ell \le n.$$

By (i) for $s \in [0,1]$ $\exists$ a path $Q_s : [0,1] \to X$ so that

$$Q_s(0) = Q_1(s) \;\&\;\; \pi(Q_s(t)) = h(s,t) \;\forall\; s,t \in [0,1].$$

We claim that $H : [0,1] \times [0,1] \to X$ defined by $H(s,t) := Q_s(t)$ is the required path homotopy.

To see that H is continuous, noting that $\pi \circ H \equiv h$, choose $\gamma_{k,\ell} \in \Gamma_\pi$ so that

$$H(s,t) = \gamma_{k,\ell} \circ \phi_{B_{k,\ell}}(h(s,t)) \;\forall\; s,t \in [t_{k-1}, t_{k+1}] \times [t_{\ell-1}, t_{\ell+1}],\; 1 \le k,\; \ell \le n.$$

This ensures continuity of H on each $R_{k,\ell} = [t_{k-1}, t_{k+1}] \times [t_{\ell-1}, t_{\ell+1}]$. Global continuity follows also because

$$R_{k,\ell} \cap R_{k',\ell'} \ne \varnothing \text{ whenever } |k - k'| \;\&\; |\ell - \ell'| \le 1.$$

It remains to show that $H(1,t) = Q_2(t) \;\forall\; t \in [0,1]$. To see this we note that $H(1,0) = Q_1(0) = Q_2(0)$ and $\pi \circ H(1,\cdot) \equiv \pi \circ Q_2$, which forces $H(1,t) = Q_2(t) \;\forall\; t \in [0,1]$ by (ii). ☑

`Proof of the Lifting Theorem`

For Z a metric space and $z \in Z$, let

$$\mathfrak{p}(Z,z) := \{P : [0,1] \to Z : \text{a path},\; P(0) = z\}.$$

Let $\alpha \in X$ with $\pi(\alpha) = a \in Y$.

By (i) & (ii) of the lemma, $\exists\,!\; \psi_\alpha : \mathfrak{P}(Y,a) \to \mathfrak{P}(X,\alpha)$ so that

$$\pi \circ \psi_\alpha(P) = P \;\;\&\;\; \psi_\alpha(P)(0) = \alpha \quad \forall\; P \in \mathfrak{P}(Y,a).$$

Evidently (!) ψ_α is continuous in the sense that $\forall\; \epsilon > 0 \;\exists\; \delta > 0$ so that

$$\sup_{t\in[0,1]} d_Y(P(t), P'(t)) < \delta \implies \sup_{t\in[0,1]} d_X(\psi_\alpha(P)(t), \psi_\alpha(P')(t)) < \epsilon.$$

Now fix $a,\; b \in X$ satisfying $\pi(b) = f(a)$ and define

$$\Psi : \mathfrak{p}(X,a) \to \mathfrak{p}(X,b) \;\text{ by }\; \Psi(P) := \psi_{f(a)}(f \circ P).$$

Using uniform continuity of $f : X \to Y$, it is not hard to show (!) that Ψ is continuous in the sense that $\forall\ \epsilon > 0\ \exists\ \delta > 0$ so that

$$\sup_{t\in[0,1]} d_X(P(t), P'(t)) < \delta \implies \sup_{t\in[0,1]} d_X(\Psi(P)(t), \Psi(P')(t)) < \epsilon.$$

If $P,\ P' \in \mathfrak{p}(X, a)$ are path homotopic, then so are $f \circ P$ & $f \circ P'$ and by (iii), so are $\Psi(P)$ & $\Psi(P')$. In particular $\Psi(P)(1) = \Psi(P')(1)$.

Since X is simply connected, if $P,\ P' \in \mathfrak{p}(X, a)$ & $P(1) = P'(1)$, they are path homotopic. Thus $\exists\ F : X \to X$ so that

$$\Psi(P)(1) = F(P(1))\ \forall\ P \in \mathfrak{p}(X, a).$$

To show that F is the advertised lifting, it remains to show its continuity, which follows because if $x, y \in X$ are close then $\exists\ P_x,\ P_y \in \mathfrak{p}(X, a)$ close, with $P_x(1) = x,\ P_y(1) = y$. ☑

Exercise. Can you prove the lifting theorem for $\mathbb{T}$ without using the general lifting theorem?

Orientation

The triple $(x, y, z) \in \mathbb{T}^3$ is in *positive order* if $\exists$ points $x^* \le y^* \le z^* \in \mathbb{R},\ z^* - x^* \le 1$ such that $x^* + \mathbb{Z} = x, y^* + \mathbb{Z} = y, z^* + \mathbb{Z} = z$. Note that if (x, y, z) is in positive order, then so is (y, z, x). The triple $(x, y, z) \in \mathbb{T}^3$ is in *negative order* if (z, y, x) is in positive order.

A map $T : \mathbb{T} \to \mathbb{T}$ is called

- *orientation preserving* at $w \in \mathbb{T}$ if $\exists\ \epsilon > 0$ so that $(x, y, z) \in B(w, \epsilon)^3$ in positive order $\implies$ (Tx, Ty, Tz) in positive order and
- *orientation reversing* at $w \in \mathbb{T}$ if $\exists\ \epsilon > 0$ so that $(x, y, z) \in B(w, \epsilon)^3$ in positive order $\implies$ (Tx, Ty, Tz) in negative order.

A map is called orientation preserving/reversing if it is orientation preserving/reversing at every point.

Examples. (i) The maps $R_\alpha\ \ (\alpha \in \mathbb{R})$ are orientation preserving, as is $x \mapsto qx \mod 1\ \ (q \in \mathbb{N})$.

(ii) The map $x \mapsto -x$ is orientation reversing.

(iii) Concatenations of orientation preserving/reversing maps are also orientation preserving/reversing according to the formulae

$$\text{preserving} \circ \text{preserving} = \text{reversing} \circ \text{reversing} = \text{preserving}$$

and

$$\text{preserving} \circ \text{reversing} = \text{reversing} \circ \text{preserving} = \text{reversing}.$$

(iv) The continuous map $f : \mathbb{T} \to \mathbb{T}$ defined by

$$f(x) := \begin{cases} 2x & 0 \le x \le \frac{1}{2}; \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$

is orientation preserving on $(0, \frac{1}{2})$, orientation reversing on $(\frac{1}{2}, 1)$ and neither preserving nor reversing orientation at $\frac{1}{2}$.

Proposition

Let $T : \mathbb{T} \to \mathbb{T}$ be a homeomorphism and let $R : \mathbb{R} \to \mathbb{R}$ be its lifting, then $|R(x+1) - R(x)| = 1$ and either:

- *R is strictly increasing and T is orientation preserving; or*
- *R is strictly decreasing and T is orientation reversing.*

Proof Evidently $\exists\ N \in \mathbb{Z}$ so that

$$R(x+1) = R(x) + N\ \forall\ x \in \mathbb{R}.$$

- If $N = 0$, then $\exists\ 0 < u < v < 1$ so that $R(u) = R(v)$ whence $T(\pi(u)) = T(\pi(v))$ and T is not 1-1 (as $\pi(u) \ne \pi(v)$)). ⊠
- If $N = \epsilon\nu$ with $\nu \ge 2$ & $\epsilon = \pm 1$, then by the intermediate value theorem, $\exists\ \theta \in (0,1)$ so that $R(x+\theta) = R(x) + \epsilon$, whence $T(\pi(0)) = T(\pi(\theta))$ and T is not 1-1 (as $\pi(0) \ne \pi(\theta)$)). ⊠

Thus $\epsilon = \pm 1$.

If $\epsilon = 1$ then R is strictly increasing (else T is not 1-1) and T is orientation preserving; and if $\epsilon = -1$ then R is strictly decreasing and T is orientation reversing. ☑

Rotation number

Let $T : \mathbb{T} \to \mathbb{T}$ be an orientation preserving homeomorphism, and let $R : \mathbb{R} \to \mathbb{R}$ be a continuous lift of T.

Proposition 2

1) $\exists\ \rho(R)$ such that $\frac{R^n(x)}{n} \to \rho(R)\ \forall\ x \in \mathbb{R}$;
2) $\exists\ \rho(T) \in \mathbb{T}$ such that $\rho(R) + \mathbb{Z} = \rho(T)$ for every continuous lift R of T.

The *rotation number* of T is $\rho(T) \in \mathbb{T}$.

Proof

We claim first that $\forall\ n \ge 1\ \exists\ k_n \in \mathbb{R}$ such that

$$R^n(x) - x \in [k_n - 1, k_n + 1]\ \forall\ x \in \mathbb{R}.$$

To see this with $k_n = R^n(0)$, set $F(x) := R^n(x) - x$; then $F(x+1) = F(x)$ and for $0 \le x \le 1$,

$$F(x) - F(0) = R^n(x) - x - R^n(0) \in [-1, 1-x] \subset [-1, 1].$$

Thus

$$R^{mn}(0) = \sum_{k=0}^{m-1} R^n(R^{kn}(0)) - R^{kn}(0) \in [mk_n - m, mk_n + m]$$

and

$$\left|\frac{R^{mn}(0)}{mn} - \frac{k_n}{n}\right| \le \frac{1}{n}.$$

Consequently,

$$\begin{aligned}
&\left|\frac{R^m(0)}{m} - \frac{R^n(0)}{n}\right| \\
&\le \left|\frac{R^m(0)}{m} - \frac{k_m}{m}\right| + \left|\frac{k_m}{m} - \frac{R^{mn}(0)}{mn}\right| + \left|\frac{R^{mn}(0)}{mn} - \frac{k_n}{n}\right| + \left|\frac{k_n}{n} - \frac{R^n(0)}{n}\right| \\
&\le \frac{2}{m} + \frac{2}{n}.
\end{aligned}$$

Thus $\exists \lim_{n\to\infty} \frac{R^n(0)}{n} =: \rho(R)$.

Evidently $\left|\frac{R^n(x)}{n} - \frac{R^n(0)}{n}\right| \le \frac{1+|x|}{n}\ \forall\ x \in \mathbb{R}$ so $\frac{R^n(x)}{n} \to \rho(R)\ \forall\ x \in \mathbb{R}$.

If R, S are continuous lifts of T, then $S \equiv R + N$ (some $N \in \mathbb{Z}$), whence $S^n \equiv R^n + nN\quad (n \in \mathbb{N})$ and $\rho(S) = \rho(R) + N$. □

Week # 3, 30/10/2013.

Exercises.

1) For $q > 0$, consider the map $T_q : \mathbb{T} \to \mathbb{T}$ defined by $T_q(x) := qx \mod 1$. Find the set

$$\{x \in \mathbb{T} : T_q \text{ is orientation preserving at } X\}.$$

2) Let $T : \mathbb{T} \to \mathbb{T}$ be an orientation preserving homeomorphism and let $R : \mathbb{R} \to \mathbb{R}$ be a continuous lift of T. Show that for $k \in \mathbb{Z}$, R^k is a continuous lift of T^k and $\rho(R^k) = k\rho(R)$, whence $\rho(T^k) = k\rho(T) \mod 1$.

Proposition 3

Let $T : \mathbb{T} \to \mathbb{T}$ be an orientation preserving homeomorphism, then $\rho(T) \in \mathbb{Q}/\mathbb{Z}$ iff $\exists$ a periodic point for T in $\mathbb{T}$.

Proof

Let $R : \mathbb{R} \to \mathbb{R}$ be a continuous lift of T.

Suppose first that $T^q(x + \mathbb{Z}) = x + \mathbb{Z}$, then $\exists\ p \in \mathbb{Z}$ such that $R^q(x) = x + p$. Evidently, $R^{aq}(x) = x + ap \ \ (a \in \mathbb{Z})$ and

$$\rho(R) = \lim_{N\to\infty} \frac{R^N(x)}{N} = \lim_{a\to\infty} \frac{R^{aq}(x)}{aq} = \frac{p}{q} \in \mathbb{Q}.$$

Now suppose that $\rho(R) = \frac{p}{q} \in \mathbb{Q}$, then $\rho(R^q) = p \in \mathbb{Z}$. We claim that $\exists\ x \in \mathbb{T},\ T^q(x) = x$.

To prove this, it suffices to show that

$$(🚂) \qquad \rho(S) \in \mathbb{Z} \implies \exists\ x \in \mathbb{R},\ S(x) - x \in \mathbb{Z}$$

where $S = R^q$.

Proof of (🚂)

The map $z \mapsto S(z) - z$ is periodic and uniformly continuous on $\mathbb{R}$.

Thus, assuming $S(x) - x \notin \mathbb{Z}\ \forall\ x \in \mathbb{R}$, we have that $\exists\ p \in \mathbb{Z}\ \&\ \epsilon \in (0, \frac{1}{2})$ such that

$$p + \epsilon \le S(z) - z \le p + 1 - \epsilon\ \forall\ z \in \mathbb{R}.$$

Iterating,

$$\frac{S^N(0)}{N} = \frac{1}{N}\sum_{k=0}^{N-1}(S(S^k(0)) - S^k(0)) \in [p + \epsilon, p + 1 - \epsilon]\ \forall\ N \ge 1$$

contradicting $\rho(S) \in \mathbb{Z}$. ⊠

Proposition 4

Suppose that $\rho(T) \notin \mathbb{Q}$, then $\forall\ x \in \mathbb{R},\ m_1, m_2, n_1, n_2 \in \mathbb{Z}$,

$$R^{n_1}(x) + m_1 < R^{n_2}(x) + m_2 \iff n_1\rho(T) + m_1 < n_2\rho(T) + m_2.$$

Proof
¶1 For $x, y \in \mathbb{R},\ m_1, m_2, n_1, n_2 \in \mathbb{Z}$,

$$R^{n_1}(x) + m_1 < R^{n_2}(x) + m_2 \iff R^{n_1}(y) + m_1 < R^{n_2}(y) + m_2.$$

Else $\exists\ x, y \in \mathbb{R},\ m_1, m_2, n_1, n_2 \in \mathbb{Z}$ with

$$R^{n_1}(x) + m_1 < R^{n_2}(x) + m_2 \ \ \&\ \ R^{n_1}(y) + m_1 > R^{n_2}(y) + m_2.$$

Evidently $n_1 \neq n_2$ (otherwise this is impossible) and so by continuity $\exists\ z \in \mathbb{R}$ such that $R^{n_1}(z) + m_1 = R^{n_2}(z) + m_2$ whence if $n_2 > n_1$ and $w := R^{n_1}(z)$, then $R^{n_2-n_1}(w) - w \in \mathbb{Z}$ contradicting $\rho(T) \notin \mathbb{Q}$.
¶2 $R^{n_1}(0) + m_1 < R^{n_2}(0) + m_2 \implies n_1\rho(T) + m_1 < n_2\rho(T) + m_2$.
Note first that

$$\begin{aligned}
&R^{n_1}(0) + m_1 < R^{n_2}(0) + m_2 \\
&\iff R^{n_1-n_2}(R^{n_2}(0)) - R^{n_2}(0) = R^{n_1}(0) - R^{n_2}(0) < m_2 - m_1 \\
&\overset{¶1}{\iff} R^{n_1-n_2}(x) - x < m_2 - m_1\ \forall\ x \in \mathbb{R}
\end{aligned}$$

It follows that if $R^{n_1}(0) + m_1 < R^{n_2}(0) + m_2$, then

$$\begin{aligned}
R^{N(n_1-n_2)}(0) &= \sum_{k=0}^{N-1} (R^{(k+1)(n_1-n_2)}(0) - R^{k(n_1-n_2)}(0)) \\
&= \sum_{k=0}^{N-1} (R^{n_1-n_2}(R^{k(n_1-n_2)}(0)) - R^{k(n_1-n_2)}(0)) \\
&< N(m_2 - m_1)
\end{aligned}$$

whence

$$\rho(T) \leftarrow \frac{R^{N(n_1-n_2)}(0)}{N(n_1 - n_2)} \overset{(!)}{<} \frac{m_2 - m_1}{n_1 - n_2}.$$

¶3 $n_1\rho(T) + m_1 < n_2\rho(T) + m_2 \implies R^{n_1}(0) + m_1 < R^{n_2}(0) + m_2$ is shown as in ¶2, but with the logic reversed. □

Exercises on rational rotation numbers.
Suppose that $f : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism with $\rho(f) = \frac{p}{q} \in \mathbb{Q}$ with $p,\ q$ relatively prime.
(i) For any periodic point $x \in \mathbb{T}$ there is an order preserving bijection $f^j(x) \mapsto \frac{j}{q}$ defines an order preserving bijection between $\{f^j(x)\}_{j=0}^{q-1}$ and $\{\frac{j}{q}\}_{j=0}^{q-1}$.
(ii) If f has a unique periodic point z then $f^{nq}(x) \xrightarrow[|n|\to\infty]{} z\ \forall\ z \in \mathbb{T}$.
(iii) If f has more than one periodic point, then for any nonperiodic point $x \in \mathbb{T}$, $\exists$ periodic points $z_- \neq z_+$ so that $f^{nq}(x) \xrightarrow[\pm n\to\infty]{} z_\pm$.
(iv) Show that f is not topologically transitive.

Proposition 5

Suppose that $T : \mathbb{T} \to \mathbb{T}$ *is an orientation preserving homeomorphism with* $\rho(T) \notin \mathbb{Q}$*, then* $\exists\ h : \mathbb{T} \to \mathbb{T}$ *continuous and orientation preserving, with* $h \circ T = r_{\rho(T)} \circ h$*.*

If, in addition, T *is topologically transitive then* $T \cong r_{\rho(T)}$*, and* T *is minimal.*

Proof

Let $T : \mathbb{T} \to \mathbb{T}$ be a continuous lift of T. Given $u \in \mathbb{R}$, write $\Gamma_0(u) := \{R^n(u) + m : \ n, m \in \mathbb{Z}\}$ and $\Gamma_1 := \{n\rho(R) + m : \ n, m \in \mathbb{Z}\}$. By proposition 4, if $\pi : \Gamma_0(u) \to \Gamma_1$ is defined by $\pi(R^n(u)+m) := n\rho(R)+m$, then π is an order preserving bijection. Evidently $\pi(x+1) = \pi(x) + 1,\ \pi \circ R = \pi + \rho(T)$.

We need the

Claim

If $a \le b,\ a, b \in \overline{\Gamma}_0(u)$ and $(a, b) \cap \overline{\Gamma}_0(u) = \varnothing$, then

$$\pi(a-) := \sup_{y \in \Gamma_0(u),\ y<a} \pi(y) = \inf_{z \in \Gamma_0(u),\ z<b} \pi(y) =: \pi(b+).$$

If the claim is false, then by irrationality of $\rho(R)$ (denseness of Γ_1), $\exists\ t \in \Gamma_1 \cap (\pi(a-), \pi(b+))$. It follows that $\exists\ s \in \Gamma_0(u),\ t = \pi(s)$, but this is impossible since by order preservation of $\pi,\ s \in \Gamma_0(u) \cap (a, b) = \varnothing$.

The claim with $a = b$ (where $(a, b) = (a, a) = \varnothing$) shows that $\exists\ !\ \tilde{\pi} : \overline{\Gamma}_0(u) \to \mathbb{R}$ continuous, strictly increasing, with $\tilde{\pi}|_{\Gamma_0(u)} \equiv \pi$.

The claim with $a < b,\ a, b \in \overline{\Gamma}_0(u)$ and $(a, b) \cap \overline{\Gamma}_0(u) = \varnothing$ shows that in this situation, $\tilde{\pi}(a) = \tilde{\pi}(b)$, whence $\exists\ !\ \hat{\pi} : \mathbb{R} \to \mathbb{R}$, continuous, non-decreasing such that $\hat{\pi}|_{\overline{\Gamma}_0(u)} \equiv \tilde{\pi}$.

Evidently $\hat{\pi}(x+1) = \hat{\pi}(x) + 1$ and $\hat{\pi} \circ R = \hat{\pi} + \rho(R)$. The required continuous $h : \mathbb{T} \to \mathbb{T}$ is defined by $h(x + \mathbb{Z}) := \hat{\pi}(x) + \mathbb{Z}$.

In case T is topologically transitive, $\exists\ u \in \mathbb{R}$ with $\overline{\Gamma}_0(u) = \mathbb{R}$ and the maps $\hat{\pi}$ and h are homeomorphisms. □

Denjoy's Examples

For any $\alpha \notin \mathbb{Q}\ \exists\ T : \mathbb{T} \to \mathbb{T}$ *a* C^1 *orientation preserving homeomorphism with* $\rho(T) = \alpha$ *and which is not minimal.*

Construction sketch `a bit different!`

Choose $\lambda_n > 0\ \ (n \in \mathbb{Z})$ such that $\sum_{n \in \mathbb{Z}} \lambda_n = 1$ and $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ as $|n| \to \infty$. Set $\alpha_n := R_\alpha^n(0)\ \ (n \in \mathbb{Z})$.

We claim first that

¶1 $\exists$ a disjoint collection $\{I_n : n \in \mathbb{Z}\}$ of open subintervals of $(0,1)$ such that $|I_n| = \lambda_n \quad (n \in \mathbb{Z})$ and for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$:

$$I_{n_1} + \lfloor n_1\alpha \rfloor + m_1 < I_{n_2} + \lfloor n_2\alpha \rfloor + m_2 \iff n_1\alpha + m_1 < n_2\alpha + m_2.$$

To see this, define $B : (0,1) \to (0,1)$ by

$$B(x) := \sum_{n\in\mathbb{Z},\ \alpha_n \le x} \lambda_n$$

and let

$$I_n := (B(\alpha_n -), B(\alpha_n)) =: (a_n, b_n).$$

Evidently[1], the collection $\{I_n : n \in \mathbb{Z}\}$ is as advertised. ☑¶1

¶2 Next, $\forall\ n \in \mathbb{Z}$ we construct a C^∞ orientation preserving diffeomorphism $f_n : \overline{I}_n \to \overline{I}_{n+1}$ such that $f_n'|_{\partial I_n} \equiv 1$ and $\sup_{I_n} |\log f_n'| \to 0$ as $|n| \to \infty$. For such a diffeomorphism

$$(\text{🖂}) \qquad f_n(x) = f_n(a_n) + \int_{a_n}^x g_n(t)dt \ \text{ where } \ g_n = f_n',$$

and we construct $g_n : \overline{I}_n \to \mathbb{R}_+$ C^∞ so that

$$(\text{✈}) \qquad \int_{I_n} g_n(t)dt = \lambda_{n+1},\ g_n(a_n) = g_n(b_n) = 1 \ \ \& \ \sup_{I_n} |\log g_n| \xrightarrow[|n|\to\infty]{} 0$$

and define $f_n : \overline{I}_n \to \overline{I}_{n+1}$ by (🖂).

Evidently[2]

$$g_n(x) := 1 + \frac{6(\lambda_{n+1} - \lambda_n)e}{\lambda_n^3} \cdot (b_n - x)(x - a_n)$$

satisfies (✈).

¶3 Define $g : U := \bigcup_{n\in\mathbb{Z}} I_n \to \mathbb{R}_+$ by $g|_{I_n} \equiv g_n$ and define $f : \mathbb{T} \to \mathbb{T}$ by

$$f(x) := a_1 + \int_0^x g(t)dt \quad \text{mod } 1.$$

Since $a_0 = 0$, we have that $f|_{I_n} \equiv f_n$.(!)

Moreover, f is differentiable on $U := \bigcup_{n\in\mathbb{Z}} I_n$ with $f' = g$.

Extend the definition of g to $[0,1]$ by defining $g|_{[0,1]\setminus E} \equiv 1$. It follows from (✈) that $g : [0,1] \to \mathbb{R}_+$ is continuous, whence (!) $f : \mathbb{T} \to \mathbb{T}$ is a C^1 diffeomorphism, evidently orientation preserving.

To calculate the rotation number of f, let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f. If $z \in \mathbb{R}$ projects onto $w \in I_0$, then by ¶1,

$$F^{n_1}(z) + m_1 > F^{n_2}(z) + m_2 \ \Leftrightarrow\ n_1\alpha + m_1 > n_2\alpha + m_2$$

and it follows that $\rho(F) = \alpha$. ☑

[1] `i.e.: this proof is an exercise`

[2] `see the previous footnote`

Exercises.

1) Prove (and/or correct) lemmas and (!)'s.

2) Let $\mathcal{H}$ denote the lifts of orientation preserving homeomorphisms of $\mathbb{T}$
equipped with the metric $d(S,T) := \sup_{x\in\mathbb{R}}(|S(x) - T(x)| + |S^{-1}(x) - T^{-1}(x)|)$. Show that the rotation number $\rho : \mathcal{H} \to \mathbb{R}$ is continuous.

3) Let $T : \mathbb{T} \to \mathbb{T}$ be an orientation preserving homeomorphism without periodic points. Show that:

a) $\exists\ K \subset \mathbb{T}$ closed and T-invariant such that

$$\omega(x) := \bigcap_{n\geq 1} \overline{\{T^k x :\ k \geq n\}} = K\ \forall\ x \in \mathbb{T}.$$

(Hint: Prove that $\omega(y) \subset \omega(x)\ \forall\ x, y \in \mathbb{T}$.)

b) Either $K = \mathbb{T}$, or K is homeomorphic to the (classical) Cantor set.

Remark

In the sequel, we'll prove Denjoy's theorem:

If $T : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous orientation preserving homeomorphism, $\rho(T) \notin \mathbb{Q}$ and $\bigvee_{\mathbb{T}} \log DT < \infty$, then T is topologically transitive.

Orientation preserving homeomorphisms of $\mathbb{T}$ as interval maps. Let $T : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism with $\rho(T) \notin \mathbb{Q}$, and consider the induced mapping $f : I := [0,1] \to I$. There is a point $c = c_f \in (0,1)$ such that f is continuous and strictly increasing on $[0,c]$ and $[c,1]$. Also $f(0) = f(1)$ and $f(c-) = 1,\ f(c+) = 0$. Moreover (!), f has no periodic point. Denote the collection of such maps by $\mathcal{S}([0,1])$. For another compact interval $J \subset \mathbb{R}$, denote by $\mathcal{S}(J) = h^{-1}\mathcal{S}([0,1])h$ where $h : J \to [0,1]$ is the increasing affine homeomorphism between the intervals.

The "1st return time renormalisation". Let $f \in \mathcal{S}(J)$ be aperiodic. Let $\{J', J''\}$ be the partition into open intervals defined by $(J \setminus \{c_f\} = J' \cup J''$.

Either $f(J') \subset J''$, or $f(J'') \subset J'$. Order the partition so that $f(J') \subset J''$, define $\mathfrak{n}(f) := \min\{j \geq 1 :\ \ J' \cap f^{j+1}J' \neq \varnothing\}$ and set $J(f) := \overline{J' \cup f^{\mathfrak{n}(f)+1}J}$.

Define the *return time function* $\varphi = \varphi_{J(f)} \to \mathbb{N}$ by $\varphi(x) := \min\{n \geq 1 : f^n x \in J(f)\} \leq \infty$ and the *return time-* or *induced map* $f_{J(f)} = \mathcal{R}(f) : J(f) \to J(f)$ by $f_{J(f)}(x) := f^{\varphi(x)}(x)$.

Renormalisation Proposition 6 $f_{J(f)} \in \mathcal{S}(J(f))$ *and*

$$\varphi(x) = \begin{cases} 1 & x \in J'' \cap J(f), \\ \mathfrak{n}(f) + 1 & x \in J'. \end{cases}$$

Proof Examine the cases $J' = [a, c)$, $J' = (c, b]$ in detail. □

Week # 4, 6/11/2013.

Renormalization of irrational rotations

Fix $0 < \alpha < \frac{1}{2}$, $\alpha \notin \mathbb{Q}$ and let $f_\alpha \in \mathcal{S}([0,1])$ represent R_α, then

$$f_\alpha(x) = \begin{cases} x+\alpha & x \in [0, 1-\alpha), \\ x+\alpha-1 & x \in [1-\alpha, 1). \end{cases}$$

Here $c = 1-\alpha$, $J' = (1-\alpha, 1)$ and $J" = (0, 1-\alpha)$. We have that

$$f_\alpha^i(J') = ((i-1)\alpha, i\alpha) \quad 1 \le i \le \tfrac{1}{\alpha},$$

whence

$$\mathfrak{n}(f_\alpha) = \max\{j \ge 1 : j\alpha \le 1-\alpha\} = [\tfrac{1}{\alpha}] - 1.$$

It follows that

$$J(f_\alpha) := \overline{J' \cup f_\alpha^{\mathfrak{n}(f_\alpha)+1}(J')} = [1-\alpha, 1] \cup [\mathfrak{n}(f_\alpha)\alpha, (\mathfrak{n}(f_\alpha)+1)\alpha] = [([\tfrac{1}{\alpha}]-1)\alpha, 1],$$

that $|J(f_\alpha)| = \alpha + \alpha\{\frac{1}{\alpha}\}$, and that

$$(f_\alpha)_{J(f_\alpha)}|_{J' \cap J(f_\alpha)} = f_\alpha|_{J"}^{\mathfrak{n}(f_\alpha)} \circ f_\alpha|_{J'}, \ (f_\alpha)_{J(f_\alpha)}|_{J" \cap J(f_\alpha)} = f_\alpha$$

whence ($J' \cap J(f_\alpha) = [1-\alpha, 1]$ and $J" \cap J(f_\alpha) = [([\frac{1}{\alpha}]-1)\alpha, 1-\alpha]$)

$$(f_\alpha)_{J(f_\alpha)}(x) = \begin{cases} x+1-[\frac{1}{\alpha}]\alpha & x \in [([\frac{1}{\alpha}]-1)\alpha, 1-\alpha], \\ x+[\frac{1}{\alpha}]\alpha - 1 x \in [1-\alpha, 1]. \end{cases}$$

Proposition 7

1) If $\alpha \in (0, \frac{1}{2}) \setminus \mathbb{Q}$, then

$$\mathfrak{n}(f_\alpha) = [\tfrac{1}{\alpha}] - 1 \ \& \ (f_\alpha)_{J(f_\alpha)} \cong f_{\frac{1}{1+G(\alpha)}} \ \text{where } G(\alpha) := \{\tfrac{1}{\alpha}\}.$$

2) If $\alpha \in (\frac{1}{2}, 1) \setminus \mathbb{Q}$, then

$$\mathfrak{n}(f_\alpha) = [\tfrac{1}{1-\alpha}] - 1 \ \& \ (f_\alpha)_{J(f_\alpha)} \cong f_{\frac{G(1-\alpha)}{1+G(1-\alpha)}}.$$

Proof In exercises (below).

Denjoy's theorem

Omega limit set.

Let (X, T) be a topological dynamical system. The *omega limit set* of T at $x \in X$ is

$$\omega_T(x) := \{y \in X, \ \exists \ n_k \to \infty \text{ such that } T^{n_k}x \to y\}.$$

For T a homoeomorphism, the *alpha limit set* of T at $x \in X$ is

$$\alpha_T(x) := \omega_{T^{-1}}(x).$$

If X is a compact metric space, then (!) $\omega_T(x)$ is a non-empty closed set $\forall\ x \in X$.

Proposition: Uniqueness of ω limit set for circle maps

Suppose that $T : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism with $\rho(T) = \alpha \notin \mathbb{Q}$, then
(i) *there is a perfect subset of $\mathbb{T}$ so that $\omega_T(x) = E\ \forall\ x \in \mathbb{T}$;* &
(ii) *either $E = \mathbb{T}$ or E is nowhere dense.*

Proof

¶1 For $x \in \mathbb{T}$, $m \ne n \in \mathbb{Z}$, let $I \subset \mathbb{T}$ be a closed interval with $\partial I = \{T^m x, T^n x\}$ (there are 2 such), then

$$\bigcup_{\ell \ge 0} T^{-\ell} I = \mathbb{T}.$$

`Proof of ¶1` Let

$$I_k := T^{-k(m-n)} I \overset{!}{=} [T^{-m(k-1)-nk}x, T^{-mk-n(k-1)}x] =: [a_k, b_k].$$

Since $a_{k+1} = b_k$, $\bigcup_{k=0}^N I_k$ is an interval $\forall\ N \ge 1$ and either $\bigcup_{k=0}^N I_k \uparrow\ \mathbb{T}$; or $\exists\ \lim_{k\to\infty} T^{-mk-n(k-1)}x =: z \in \mathbb{T}$.

In the second case, by continuity of T, $T^{-k(m-n)}z = z$ contradicting irrationality of $\rho(T)$. ☑¶1

¶2 $\omega_T(y) = \omega_T(x)\ \forall\ x,\ y \in \mathbb{T}$.

`Proof of ¶2` It suffices to show that $\omega_T(y) \subset \omega_T(x)$. Let $z \in \omega_T(y)$, then $\exists\ \ell_n \to \infty$ with $T^{\ell_n}(x) \to z$. By ¶1, for each $n \ge 1$, $\exists\ k_n \ge 1$ so that $T^{k_n}(x) \in [T^{\ell_n}y, T^{\ell_{n+1}}y]$. It follows (!) that $T^{k_n}(x) \to z$ whence $z \in \omega_T(x)$. ☑¶2

¶3 Either $E := \omega_T(0) = \mathbb{T}$ or E is nowhere dense.

`Proof of ¶3` The set $E := \omega_T(0)$ is a closed T-invariant subset of $\mathbb{T}$ and, by ¶2, T is minimal on E. ∂E is a closed T-invariant subset of E. By minimality of (E, T), either $\partial E = \varnothing$ in which case $E = \mathbb{T}$ (being both open and closed), or $\partial E = E$ in which case E is nowhere dense. ☑¶3

To see that E is perfect, let $z \in E$, then $\exists\ n_k \to \infty$ so that $T^{n_k}z \to z$. The points $\{T^{n_k}z :\ k \ge 1\}$ are distinct as otherwise there would be a period for T contradicting $\rho(T) \notin \mathbb{Q}$. Thus $z \in E'$. ☐

Denjoy's theorem

If $T : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous orientation preserving homeomorphism, $\rho(T) \notin \mathbb{Q}$ and $\bigvee_{\mathbb{T}} \log DT < \infty$, then T is m.

The proof is in a series of steps:

¶1 `(Rokhlin interval tower I)` Let $T:\mathbb{T}\to\mathbb{T}$ is an orientation preserving homeomorphism with $\rho(T)=\alpha\notin\mathbb{Q}$, then $\exists\ q_n\uparrow\infty$ so that $\forall\ x\in\mathbb{T}$, the intervals $\{(T^k(x),T^{k-q_n}(x)):\ 0\le k\le q_n\}$ are disjoint.

`Proof of ¶1` By minimality of R_α, $\exists\ q_n\to\infty$ so that

$$d(R_\alpha^{q_n}(0),0)<d(R_\alpha^k(0),0)\ \forall\ |k|<q_n.$$

It follows that the intervals $\{(R_\alpha^k(0),R_\alpha^{k-q_n}(0)):\ 0\le k\le q_n\}$ are disjoint.

By proposition 4 the intervals $\{(T^k(x),T^{k-q_n}(x)):\ 0\le k\le q_n\}$ are disjoint $\forall\ x\in\mathbb{T}$. ☑

¶2 Suppose that $J\subset\mathbb{T}$ is an interval & $\{T^jJ:\ 0\le j\le q_n\}$ are disjoint, then

$$|\log T^{q_n\prime}(x)-\log T^{q_n\prime}(y)|\le\bigvee\log T'\ \forall\ x,y\in\overline{J}.$$

`Proof of ¶2` Since $(T^k(x),T^k(y))\subset T^kJ$, we have

$$\begin{aligned}\bigvee\log T'&\ge\sum_{j=0}^{q_n}|\log T'(T^{q_n}(x))-\log T'(T^{q_n}(y))|\\&\ge|\sum_{j=0}^{q_n}(\log T'(T^{q_n}(x))-\log T'(T^{q_n}(y)))|\\&=|\log T^{q_n\prime}(x)-\log T^{q_n\prime}(y)|.\quad ☑¶\end{aligned}$$

¶3 For q_n as in proposition 8,

$$T^{q_n\prime}(x)T^{-q_n\prime}(x)\ge e^{-\bigvee\log T'}\quad\forall\ x\in\mathbb{T}.$$

`Proof of ¶3` Fix $x\in\mathbb{T}$.

By ¶1, the assumptions for ¶2 hold for q_n as in ¶1, with with $J=[T^{-q_n}x,x]$ for x & $y=T^{-q_n}x$.

Thus, using ¶2:

$$\begin{aligned}|\log(T^{q_n\prime}(x)T^{-q_n\prime}(x))|&=|\log(T^{q_n\prime}(x)+\log T^{-q_n\prime}(x))|\\&=|\log(T^{q_n\prime}(x)-\log T^{q_n\prime}(T^{-q_n}(x)x))|\\&\le\bigvee\log T'\end{aligned}$$

and $T^{q_n\prime}(x)T^{-q_n\prime}(x)\ge e^{-\bigvee\log T'}$. ☑¶3

To finish, if Denjoy's theorem fails, then T is not minimal and $\exists\ x\in\mathbb{T}$ with $\overline{\{T^nx:\ n\in\mathbb{Z}\}}=K\subsetneq\mathbb{T}$. Let $U:=\mathbb{T}\setminus K$, then $TU=U$ is open.

Let $I \subset U$ be a maximal interval, then so is $T^n I \ \forall \ n \in \mathbb{Z}$. Irrationality of $\rho(T)$ means that the $T^n I$ are disjoint (else the endpoints would be periodic). Thus for $q_n \to \infty$ as in ¶1,

$$\begin{aligned}
1 &\ge \sum_{n=1}^{N} (|T^{q_n} I| + |T^{-q_n} I|) \\
&= \sum_{n=1}^{N} \int_I (T^{q_n}{}'(x) + T^{-q_n}{}'(x)) dx \\
&\ge \sum_{n=1}^{N} \int_I \sqrt{T^{q_n}{}'(x) T^{-q_n}{}'(x)} dx \\
&\ge N|I| \exp[-\frac{1}{2} \bigvee \log T'] \\
&\xrightarrow[N\to\infty]{} \infty. \quad \boxtimes
\end{aligned}$$

□

Denjoy-Koksma Inequality

Interval tower lemma

For each $\alpha \notin \mathbb{Q}$, $\exists\ q_n < q_{n+1} \uparrow \infty$ (aka the `principal denominators` *of α) so that whenever $T : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism with $\rho(T) = \alpha \notin \mathbb{Q}$, we have*

$$1 \le \sum_{j=0}^{q_{n+1}-1} 1_{J_n(x)} \circ T^j(x) \le 2 \quad \forall \quad x \in \mathbb{T}$$

where $J_n := [T^{-q_n}(x), T^{q_n}(x)]$.

Proof See exercises.

Corollary

Suppose that $T : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism with $\rho(T) = \alpha \notin \mathbb{Q}$, then $\forall\ n \ge 1,\ x, y \in \mathbb{T},\ \exists$ a permutation $k = k_{x,y} : \{0, 1 \dots, q_n - 1\} \to \{0, 1 \dots, q_n - 1\}$ such that $\sum_{j=0}^{q_n-1} 1_{(T^j(x), T^{k(j)}(y))} \le 2$.

Denjoy-Koksma Inequality

Suppose that $\alpha \notin \mathbb{Q}$, then for $F : \mathbb{T} \to \mathbb{R}$ integrable with $\int_{\mathbb{T}} F dm = 0$,

$$|\sum_{k=0}^{q_n-1} F(R_\alpha^k x)| \le 2 \bigvee F \ \forall\ x \in \mathbb{T},\ n \ge 1.$$

Proof

Setting $F_n := \sum_{k=0}^{n-1} F \circ R_\alpha^k$, we see using the corollary that for $x, y \in \mathbb{T},\ n \geq 1$:

$$|F_{q_n}(x) - F_{q_n}(y)| \leq \sum_{k=0}^{q_n-1} |F(R_\alpha^j(x) - F(R_\alpha^{k_{x,y}(j)}(y))| \leq 2 \bigvee F.$$

To finish

$$\begin{aligned} |F_{q_n}(x)| &= |F_{q_n}(x) - \int_{\mathbb{T}} F_{q_n}(y)dy| \\ &\leq \int_{\mathbb{T}} |F_{q_n}(x) - F_{q_n}(y)|dy \\ &\leq 2 \bigvee F. \end{aligned}$$

□

Ergodicity

Suppose that $T : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous, orientation preserving homeomorphism, then T is *non-singular* with respect to Lebesgue measure m in the sense that for

$$A \in \mathcal{B}, \quad m(T^{-1}A) = 0 \iff m(A) = 0.$$

The measure theoretic analogue of transitivity is *ergodicity*:

- T is *ergodic* if

$$A \in \mathcal{B}, \quad T^{-1}A = A \implies m(A) = 0 \text{ or } m(\mathbb{T} \setminus A) = 0.$$

Theorem (Herman)

If $T : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous, orientation preserving homeomorphism with $\bigvee \log T' < \infty$ and $\rho(T) = \alpha \notin \mathbb{Q}$, then T is ergodic w.r.t. m.

Proof

Let $A \in \mathcal{B}(\mathbb{T}),\ TA = A$ with $m(A) > 0$. Let $x \in \mathbb{T}$ be a density point of A, and set for $n \geq 1$, $J_n := (T^{-q_n}(x), T^{q_n}(x))$.

By the interval tower lemma,

$$1 \leq \sum_{j=0}^{q_{n+1}-1} 1_{J_n} \circ T^j \leq 2.$$

Consequently, for $F : \mathbb{T} \to \mathbb{R}$,

$$|F_\ell(y) - F_\ell(z)| \leq 2 \bigvee F \quad \forall\ n \geq 1,\ y, z \in J_n,\ 0 \leq \ell \leq q_{n+1} - 1$$

where $F_\ell := \sum_{j=0}^{\ell-1} F \circ T^j$. In particular,

$$\frac{T^{\ell\,\prime}(y)}{T^{\ell\,\prime}(z)} \le E := e^{2\bigvee \log T'} \quad \forall\ n \ge 1,\ y, z \in J_n,\ 0 \le \ell \le q_{n+1} - 1$$

whence fixing $y_0 \in J_n$:

$$\begin{aligned}
m(\mathbb{T} \setminus A) &\le \sum_{j=0}^{q_{n+1}-1} m(T^j(J_n) \setminus A) \\
&= \sum_{j=0}^{q_{n+1}-1} m(T^j(J_n \setminus A)) \\
&= \sum_{j=0}^{q_{n+1}-1} \int_{J_n \setminus A} T^{j\prime}(y)dy \\
&\le Em(J_n \setminus A) \sum_{j=0}^{q_{n+1}-1} T^{j\prime}(y_0) \\
&\le E^2 \tfrac{m(J_n \setminus A)}{m(J_n)} \sum_{j=0}^{q_{n+1}-1} m(T^j J_n) \\
&\le 2E^2 \tfrac{m(J_n \setminus A)}{m(J_n)} \to 0.
\end{aligned}$$

□

Topological Recurrence

Suppose that (X, T) is a continuous map of a Polish space.

- An open set $U \subset X$ is a *wandering neighborhood* if $U \cap T^{-n}U = \varnothing\ \forall\ n \in \mathbb{N}$. Let $\mathfrak{W}$ denote the collection of wandering neighborhoods.
- A point is called *wandering* if it belongs to a wandering neighborhood. Let $\mathtt{W}$ denote the set of wandering points, then $\mathtt{W}$ is open and T-invariant. The collection of *nonwandering points* is $\mathtt{NW} := X \setminus \mathtt{W}$ (which is closed and T-invariant).

Exercise 1.7. Show that

(i) if $T : X \to X$ is continuous and X is compact, then $\mathtt{NW} \ne \varnothing$.

Hint If $T^{n_k}x \to z$, then $z \notin \mathtt{NW}$.

(ii) $\exists\ (X, T)$, a homeomorphism of a Polish space with $\mathtt{NW} = \varnothing$.

Proposition 1.5

If (X, T) is a homeomorphism of a Polish space, then $\exists$ a wandering neighborhood U so that $\mathtt{W}\Delta(\biguplus_{n\in\mathbb{Z}} T^n U)$ is meagre.

Proof By separability $\exists$ wandering neighborhoods U_n $(n \ge 1)$ so that $\mathtt{W} = \bigcup_{n=1}^{\infty} U_n$.

- Denote $\widehat{A} := \bigcup_{n\in\mathbb{Z}} T^n A$ (for $A \subset X$) and define sets V_n $(n \ge 1)$ by

$$V_1 = U_1,\ V_{n+1} := V_n \cup (U_{n+1} \setminus \overline{\widehat{V_n}}).$$

Evidently, each V_k is open and $V_k \subset V_{k+1}$.

- We claim that the V_k are wandering neighborhoods.

To see this by induction, assume that V_k is a wandering neighborhood and let $n \ne 0$, then

$$V_{k+1} \cap T^n V_{k+1} = A \cup B \cup C \cup D$$

where

$$A = V_k \cap T^n V_k = \varnothing,\ B = V_k \cap T^n (U_{k+1} \setminus \overline{\widehat{V_k}}) = \varnothing,$$

$$C = (U_{k+1} \setminus \overline{\widehat{V_k}}) \cap T^n V_k = \varnothing,\ D = (U_{k+1} \setminus \overline{\widehat{V_k}}) \cap T^n (U_{k+1} \setminus \overline{\widehat{V_k}}) = \varnothing.$$

It follows that $U := \bigcup_{n\ge 1} V_k$ is a wandering neighborhood, and that $\widehat{U} \Delta \mathtt{W} \subset \bigcup_{k\ge 1} \partial \widehat{V_k}$ which is meagre. ☐

Recurrence.

The continuous $T : X \to X$ is called *regionally recurrent* if $\mathtt{W} = \varnothing$, ie if $\forall\ U$ open, nonempty $\exists\ n \ge 1,\ U \cap T^{-n} U \ne \varnothing$.

A *recurrent point* for T is a point $x \in X$ so that $\exists\ n_k \to \infty,\ T^{n_k} x \to x$. Let $\mathfrak{R} = \mathfrak{R}_T := \{\mathtt{recurrent\ points\ for}\ T\}$.

Proposition 1.6

Suppose that $T : X \to X$ is continuous, regionally recurrent and X is Polish space, then $\mathfrak{R}$ is residual[3] *in X.*

Proof Next time.

Exercises: Interval tower lemma

1. Continued fractions and Denominators. Define the *denominators* of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ by

$$\mathfrak{D}_\alpha := \{q \in \mathbb{N} :\ \|q\alpha\| < \tfrac{1}{q}\}$$

where $\|x\| := \min_{n\in\mathbb{Z}} |x + n|$ for $x \in \mathbb{R}$.

[3]Residual set = קבוצה שמנה = contains a dense G_δ

It is not hard to show that $\#\mathfrak{D}_\alpha = \infty\ \forall\ \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Indeed[4] consider the *Farey sequences* $F_Q := \{\frac{p}{q} :\ 0 \le p < q \le Q,\ (p,q) = 1\}$. If $\frac{p}{q}$ and $\frac{p'}{q'}$ are adjacent in some F_Q then $|\frac{p}{q} - \frac{p'}{q'}| = \frac{1}{qq'}$, and the next element to come between them is $\frac{p+p'}{q+q'}$. Thus $\forall\ \alpha \in (0,1) \setminus \mathbb{Q}$, $\exists$ infinitely many $\frac{p}{q} \in (0,1)$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$, i.e. $\|q\alpha\| < \frac{1}{q}$, whence $\#\mathfrak{D}_\alpha = \infty$.

The Gauss map. $G : (0,1] \to [0,1]$ is defined by $G(x) := \{\frac{1}{x}\}$. Note that $x = \frac{1}{a(x)+G(x)}$ where $a(x) := [\frac{1}{x}]$.

If $\frac{p}{q} \in (0,1) \cap \mathbb{Q}$, then (!) $\exists\ n \ge 1$ such that $G^n(\frac{p}{q}) = 0$. Setting $r_k := G^k(\frac{p}{q})$ and $a_k := a(r_{k-1})$, we have $r_{k-1} = \frac{1}{a_k + r_k}$, whence

$$\frac{p}{q} = \frac{1}{a_1+r_1} = \frac{1}{a_1+\frac{1}{a_2+r_2}} = \cdots = \frac{1}{a_1+\frac{1}{a_2+\frac{1}{\ddots+\frac{1}{a_n}}}}.$$

If $\alpha \in (0,1) \setminus \mathbb{Q}$ then $r_k := G^k(\alpha) \ne 0\ \forall\ k \ge 1$ and

$$\alpha = \frac{1}{a_1+\frac{1}{a_2+\frac{1}{\ddots+\frac{1}{a_n+r_n}}}}\ \ \forall\ n \ge 1$$

where $a_n := a(r_{n-1})$.

Exercise 1.

1) Suppose that $\alpha \in (0,1) \setminus \mathbb{Q}$ and let $f_\alpha : [0,1] \to [0,1]$ be defined by

$$f_\alpha(x) := \begin{cases} x + \alpha & 0 \le x \le 1-\alpha, \\ x + \alpha - 1 & x \ge 1 - \alpha. \end{cases}$$

a) Suppose that $\alpha \in (0, \frac{1}{2}) \setminus \mathbb{Q}$ and let $h : [0,1] \to J(f_\alpha)$ be the increasing affine homeomorphism. Prove that

$$h^{-1} \circ (f_\alpha)_{J(f_\alpha)} \circ h = f_{\frac{1}{1+G(\alpha)}}$$

where $G(\alpha) := \{\frac{1}{\alpha}\}$.

b) Show that if $\alpha \in (\frac{1}{2}, 1) \setminus \mathbb{Q}$, then $\mathfrak{n}(f_\alpha) = [\frac{1}{1-\alpha}] - 1$ and that $(f_\alpha)_{J(f_\alpha)} \cong f_{\frac{G(1-\alpha)}{1+G(1-\alpha)}}$.

2) For $n \ge 1$, define $f_n : \mathbb{R}_+^n \to \mathbb{R}_+$ by

$$f_n(x_1, \ldots, x_n) := \frac{1}{x_1+\frac{1}{x_2+\frac{1}{\ddots+\frac{1}{x_n}}}}.$$

Show that $f_n \uparrow$ as $x_{2k} \uparrow$ and $f_n \downarrow$ as $x_{2k+1} \uparrow$.

[4] as in Hardy, G. H.; Wright, E. M. An introduction to the theory of numbers. Fifth edition. The Clarendon Press, Oxford University Press, New York, 1979.

3) Suppose that $a_n \in \mathbb{N}$ $(n \in \mathbb{N})$and set

$$q_0 = 1,\ q_1 = a_1,\ q_{n+1} = a_{n+1}q_n + q_{n-1};$$

$$p_0 = 0,\ p_1 = 1,\ p_{n+1} = a_{n+1}p_n + p_{n-1}.$$

Show that $p_{n-1}q_n - p_nq_{n-1} = (-1)^n,\ f_n(a_1,\dots,a_n) = \frac{p_n}{q_n}$, whence

$$f_n(a_1,\dots,a_n+1) - f_n(a_1,\dots,a_n) = \tfrac{(-1)^n}{q_n(q_n+q_{n-1})},$$

and

$$f_{n+1}(a_1,\dots,a_n,j) - f_n(a_1,\dots,a_n) = \tfrac{(-1)^n}{q_n(jq_n+q_{n-1})} \quad (j \ge 1).$$

4) Now suppose that $\alpha \in (0,1) \setminus \mathbb{Q}$, set $r_k := G^k(\alpha) \neq 0,\ a_k := a(r_{k-1})$ $(k \ge 1)$. Show that

$$\tfrac{p_{2n}}{q_{2n}} < \alpha < \tfrac{p_{2n+1}}{q_{2n+1}}$$

and

$$\tfrac{p_n}{q_n} - \tfrac{p_{n+1}}{q_{n+1}} = \tfrac{(-1)^{n+1}}{q_nq_{n+1}}.$$

5) Show that $\{jq_n + q_{n-1} :\ n \ge 1,\ 1 \le j \le a_{n+1}\} \subset \mathfrak{D}_\alpha$.

6) The *regular continued fraction expansion* of $\alpha \in (0,1) \setminus \mathbb{Q}$ is given by

$$\alpha = \cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\ddots}}}} := \lim_{n\to\infty} f_n(a_1,a_2,\dots,a_n)$$

where $a_n = a(G^{n-1}\alpha)$ (the *partial quotients* of α). Show that

$$(a_1,a_2,\dots) \mapsto \cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\ddots}}}}$$

is a homeomorphism $\mathbb{N}^{\mathbb{N}} \leftrightarrow (0,1) \setminus \mathbb{Q}$.

2. Renormalization.

Translations. Given a compact interval $J = [a,b] \subset \mathbb{R}$ consider $\mathcal{T}(J) := \{f \in \mathcal{S}([a,b]) : \exists\, f' \equiv 1 \text{ on } J \setminus \{c_f\}\}$. Evidently $\forall\, c \in (a,b),\ \exists\,!\, f \in \mathcal{T}(J)$ with $c_f = c$, namely

$$f(x) = \begin{cases} x + b - c & x \in (a,c), \\ a - c + x & x \in (c,b). \end{cases}$$

We'll write $f = (a,c,b)$. To exercise this notation, note that rotation by $\alpha \in (0,1)$ in $\mathbb{T}$ is represented by $f_\alpha = (0, 1-\alpha, 1)$, and that $(a,c,b) \cong (0, \frac{c-a}{b-a}, 1)$. Thus (a,c,b) has no periods iff $\frac{c-a}{b-a} \notin \mathbb{Q}$.

If $f \in \mathcal{T}(J)$, then $f_{J(f)} \in \mathcal{T}(J(f))$. If $f = (a,c,b)$ set $f_{J(f)} = (a',c',b')$. It follows from previous propositions and exercises (!) that

$$(a',c',b') = \begin{cases} (a + [\frac{c-a}{b-c}](b-c), c, b) & c > \frac{a+b}{2}, \\ (a, c, b - [\frac{b-c}{c-a}](c-a)) & c < \frac{a+b}{2}; \end{cases}$$

and

$$\varphi_{(a,c,b)} = \begin{cases} 1_{(a',c')} + ([\frac{c-a}{b-c}] + 1)1_{(c',b')} = 1_{(a',c')} + (\mathfrak{n}(f) + 1)1_{(c',b')} & c > \frac{a+b}{2}, \\ ([\frac{b-c}{c-a}] + 1)1_{(a',c')} + 1_{(c',b')} = (\mathfrak{n}(f) + 1)1_{(a',c')} + 1_{(c',b')} & c < \frac{a+b}{2} \end{cases}$$

where $f_{J(f)}(x) = f^{\varphi_{(a,c,b)}(x)}(x)$. Also if $\frac{c-a}{b-a} \notin \mathbb{Q}$, then $\frac{c'-a'}{b'-a'} \notin \mathbb{Q}$, (a',c',b') has no periods and

$$c < \tfrac{a+b}{2} \iff c' > \tfrac{a'+b'}{2}.$$

Renormalisation and the tower lemma.

Fix $\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} \in (0,1) \setminus \mathbb{Q}$ and define $J_0 := [0,1]$ and $\phi_0 := f_\alpha = f = (0, 1-\alpha, 1) \in \mathcal{T}(J_0)$. Set

$$J_1 = \begin{cases} J(\phi_0) & \alpha < \frac{1}{2}, \\ J_0 & \alpha > \frac{1}{2} \end{cases},$$

$$\phi_1 := ((a_1 - 1)\alpha, 1-\alpha, 1) = (\phi_0)_{J_1} = \begin{cases} (\phi_0)_{J(\phi_0)} & \alpha < \frac{1}{2}, \\ \phi_0 & \alpha > \frac{1}{2} \end{cases}$$

and for $n \ge 1$, set $\phi_{n+1} := (\phi_n)_{J(\phi_n)}$. By the above, ϕ_n has no periods $\forall\, n \ge 1$ and the process may be continued. Set $J_n := J(\phi_n)$.

Evidently

$$J'_{n+1} = J_{n+1} \cap J''_n, \ \&\ J''_{n+1} = J_{n+1} \cap J'_n = J'_n.$$

Thus $\phi_n : J'_n \to J''_n$ and

$$\phi_n|_{J''_n} = (\phi_{n-1}|_{J''_{n-1}})^{\mathfrak{n}(\phi_{n-1})} \circ (\phi_{n-1}|_{J'_{n-1}}), \ \&\ \phi_n|_{J'_n} = \phi_{n-1}|_{J''_{n-1}}.$$

Now define $Q_0 = 0$, $Q_1 = 1$ if $\alpha > \frac{1}{2}$ and $Q_1 := \mathfrak{n}(f_\alpha) + 1$ if $\alpha > \frac{1}{2}$ (equivalently, $Q_1 = a_1 := [\frac{1}{\alpha}]$). Then define $Q_{n+1} := \mathfrak{n}(\phi_n)Q_n + Q_{n-1}$ for $n \geq 1$.

By induction,

$$\phi_n|_{J_n'} = f^{Q_{n-1}}, \;\&\; \phi_n|_{J_n''} = f^{\mathfrak{n}(\phi_{n-1})Q_{n-1}+Q_{n-2}} = f^{Q_n}.$$

It follows that (!)

$$J_n' = (c, f^{Q_n}(c)), \; J_n'' = (c, f^{Q_{n-1}}(c))$$

and

$$\{0 \leq j \leq Q_{n+1} : \; f^j(c) \in J_n\} = \{Q_{n-1} + iQ_n : \; 0 \leq j \leq a_{n+1}\}.$$

Tower lemma 0

Up to boundary overlap,

$$\biguplus_{i=0}^{Q_{n-1}-1} f^i(J_n') \uplus \biguplus_{i=0}^{Q_n-1} f^i(J_n'') = [0,1].$$

`Sketch proof` To see disjointness, $f^{Q_{n-1}}J_n' \cap f^{Q_n}J_n'' = f_{J_n}(\varnothing) = \varnothing$ whence $\biguplus_{i=0}^{Q_{n-1}-1} f^i(J_n') \cap \biguplus_{i=0}^{Q_n-1} f^i(J_n'') = \varnothing$, else $\exists\, x \in f^i(J_n') \cap f^j(J_n'')$ $(0 \leq i < Q_{n-1}$, $0 \leq j < Q_n$ whence $f^{Q_{n-1}-i}(x) \in f^{Q_{n-1}}(J_n') \cap f^{Q_n}(J_n'')$.

To see that the tower covers, fix $x \in J_0$ and let $\kappa = \kappa_n := \min\{k \geq 0 : f^{-k}(x) \in \overline{J_n}$. If $f^{-\kappa}(x) \in \overline{J_n'}$, then $k < Q_{n-1}$ since $\varphi_{J_n} = Q_{n-1}$ on J_n'. If not then $f^{-\kappa}(x) \in \overline{J_n''}$, then $k < Q_n$ since $\varphi_{J_n} = Q_n$ on J_n''. ☐

Exercise Lemma 1

$$\phi_n \cong f_{\alpha(n)} \in \mathcal{T}([0,1]) \; \textit{where } \alpha(n) = \begin{cases} \frac{1}{1+G^n(\alpha)} & n \textit{ odd}, \\ \frac{G^n(\alpha)}{1+G^n(\alpha)} & n \textit{ even}; \end{cases}$$

and

$$\mathfrak{n}(\phi_n) = [\tfrac{1}{G^n(\alpha)}] = a_{n+1} \;\forall\; n \geq 1,$$

whence $Q_n = q_n \;\forall\; n \geq 1$.

Fix $\alpha \in (0,1) \smallsetminus \mathbb{Q}$ and define for $n \geq 1$ the collections of intervals:

$$\mathfrak{T}_n := \{R_\alpha^j[0, \{q_{2n}\alpha\}) : \; 0 \leq j < q_{2n+1}\} \cup \{R_\alpha^k[\{q_{2n+1}\alpha\}, 1) : \; 0 \leq k < q_{2n}\}.$$

Tower lemma 1

For $n \geq 1$, $\mathfrak{T}_n$ *is a disjoint collection and that* $\bigcup_{J \in \mathfrak{T}_n} J = \mathbb{T}$.

`Sketch proof` Follows from tower lemma 0 via the exercise lemma (which proves the statement for $1 - \alpha + \mathfrak{T}_n$). ☐

Tower lemma 2

Suppose that $T : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism with $\rho(T) = \alpha \notin \mathbb{Q}$. Fix $x \in \mathbb{T}$ and show that $\forall\ n \ge 1$,

$\mathfrak{P}_n := \{T^j[x, T^{q_{2n}}(x)) :\ 0 \le j < q_{2n+1}\} \cup \{T^k[T^{q_{2n+1}}(x), x) :\ 0 \le k < q_{2n}\}$

is a disjoint collection and that $\bigcup_{J \in \mathfrak{P}_n} J = \mathbb{T}$.

`Sketch proof` The truth of the n^{th} statement depends only on the order of $\{T^j(x)\}_{j=0}^{q_{2n+1}}$ in $\mathbb{T}$. By proposition 4, this is the same as the order of $\{R_\alpha^j(0)\}_{j=0}^{q_{2n+1}}$ in $\mathbb{T}$. The lemma therefore follows from tower lemma 1. □

Interval Tower lemma *Suppose that $T : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism with $\rho(T) = \alpha \notin \mathbb{Q}$ and let $n \ge 1,\ x \in \mathbb{T}$, then*

- $\{T^j(x, T^{q_{n-1}}(x)) :\ 0 \le j \le q_n - 1\}$ *are disjoint;*
- $\sum_{j=0}^{q_n-1} 1_{T^j(T^{-q_{n-1}}(x), T^{q_{n-1}}(x))} = 1, 2.$

Week # 5, 13/11/2013.

Proposition 1.6

Suppose that $T : X \to X$ is continuous, regionally recurrent and X is Polish space, then $\mathfrak{R}$ is residual[5] *in X.*

Proof For $k \geq 1$, let $\mathcal{U}_k$ be a countable open cover of X by open balls of radius $\frac{1}{k}$, and let $\mathfrak{R}_k := \bigcup_{U \in \mathcal{U}_k} U \cap \bigcup_{n=1}^{\infty} T^{-n}U$. Evidently each $\mathfrak{R}_k$ is open.

- We claim that $\overline{\mathfrak{R}_k} = X \ \forall \ k \geq 1$.

To see this, let $x \in X$, $\epsilon > 0$. $\exists \ x \in U \in \mathcal{U}_k$ and $\exists \ 0 < \delta \leq \epsilon$ so that $B_o(x,\delta) \subset U$. Since T is regionally recurrent, $\varnothing \neq B_o(x,\delta) \cap \bigcup_{n=1}^{\infty} T^{-n}B_o(x,\delta) \subset B_o(x,\delta) \cap \mathfrak{R}_k$. Thus $\overline{\mathfrak{R}_k} = X$. ☐

- By Baire's theorem, $\bigcap_{k\geq 1} \mathfrak{R}_k$ is dense, whence residual in X.
- To finish, we claim that $\mathfrak{R} = \bigcap_{k\geq 1} \mathfrak{R}_k$.

Evidently $\mathfrak{R} \subset \bigcap_{k\geq 1} \mathfrak{R}_k$. To see the reverse inclusion, suppose $x \in \bigcap_{k\geq 1} \mathfrak{R}_k$, then $\forall \ k \geq 1$, $\exists \ n_k \geq 1$, $U_k \in \mathcal{U}_k$ with $x, T^{n_k}x \in U_k$ whence $d(x, T^{n_k}x) \leq \frac{1}{k} \to 0$ and $x \in \mathfrak{R}$. ☐

Exercise 1.8.

(i) Suppose that $T : X \to X$ is a homeomorphism of a perfect Polish space[6]. Show that if (X,T) has an attractor, then it has a wandering neighborhood.

`Hint`: Suppose that (a) $\mathtt{W} = \varnothing$; and (b) $U \subset X$ is open and $T^n x \to a \ \forall \ x \in U$, whence $\exists \ x \in \mathfrak{R} \cap U \setminus \{a\}$ ....(!).... & $x = a$e.

(ii) Let $T : X \to X$ be continuous map of a Polish space. Show that if T is regionally recurrent, then so is $T^n \ \forall \ n \geq 1$.

Two sided & forward

For a topological dynamical system (X,T) the *forward T-orbit* (aka *forward semi-orbit*) of $x \in X$ is $\mathcal{O}_+^{(T)}(x) := \{T^n x : \ n \in \mathbb{N}\}$; and

for an invertible topological dynamical system (X,T) the (two-sided) *T-orbit* of $x \in X$ is $\mathcal{O}^{(T)}(x) := \{T^n x : \ n \in \mathbb{Z}\}$.

The topological dynamical system (X,T) is called

- *forwards transitive* if $\exists \ x \in X$, $\overline{\mathcal{O}_+^{(T)}(x)} = X$ and

[5] Residual set = קבוצה שמנה = contains a dense G_δ

[6] i.e. no isolated points

- *forwards minimal* if $\overline{\mathcal{O}_+^{(T)}(x)} = X \ \ \forall \ \ x \in X$.

The invertible topological dynamical system (X,T) is called

- *two-sided transitive* if $\exists\ x \in X,\ \overline{\mathcal{O}^{(T)}(x)} = X$ and
- *two-sided minimal* if $\overline{\mathcal{O}^{(T)}(x)} = X \ \ \forall \ \ x \in X$.

MINIMALITY.

Proposition 2.1

A homeomorphism $T : X \to X$ of a metric space X is two-sided minimal iff

there are no non-trivial, T-invariant, closed subsets of X, i.e.

⚽ $$E \subset X \text{ closed},\ T^{-1}E = E \implies E = \varnothing \text{ or } X.$$

Proof

For each $x \in X$, $E_x := \overline{\mathcal{O}^{(T)}(x)}$ is a T-invariant, non-empty, closed subset of X. Thus ⚽ $\implies$ T minimal.

Each T-invariant, non-empty, closed subset of X contains some E_x and so the converse implication is also valid. ☑

Proposition 2.2

If a continuous map of a compact, metric space is minimal, then it is forward minimal.

Proof

Let (X,T) be a minimal continuous map of a compact, metric space. For each $x \in X$, the *ω-limit set of x under T*:

$$\omega(T,x) := \{y \in X :\ \exists\ n_k \to \infty,\ T^{n_k}x \to y\}$$

is a closed T-invariant, subset of X. By compactness, $\omega(T,x) \neq \varnothing\ \forall\ x \in X$. By minimality, $\omega(T,x) = X\ \forall\ x \in X$. Forward minimality follows from this. ☑

Proposition 2.3

Let (X,T) be a continuous map of a compact, metric space, then (X,T) is minimal iff $\forall\ U \subset X$, open and non-empty, $\exists\ N_U \geq 1$ so that $X = \bigcup_{k=1}^{N_U} T^{-k}U$.

Proof

Evidently, (X,T) is minimal iff $\forall\ x \in X$ & $U \subset X$ open and non-empty, $\exists\ n \geq 1,\ T^n x \in U$; equivalently $X = \bigcup_{k=1}^{\infty} T^{-k}U\ \forall\ U \subset X$ open and non-empty. The finite union statement follows from compactness. ☑

Almost periodic points.

A subset $K \subset \mathbb{N}$ is called *syndetic* if it has bounded gaps, i.e. $\exists\ L > 0$ so that K intersects with every interval in $\mathbb{N}$, longer than L.

For $T : X \to X$ continuous, a point $x \in X$ is *almost periodic* (for T) if for every non-empty open set $U \subset X$, $\{n \in \mathbb{N} :\ T^n(x) \in U\}$ is either empty, or syndetic.

For example, `periodic points` (i.e. $T^N x = x$ for some $N \geq 1$) are almost periodic.

Proposition 2.4

Let (X, T) be a continuous map of a compact, metric space.

(i) *If (X, T) is minimal then all points are almost periodic for T.*

(ii) *If there is an almost periodic point with dense forward orbit, then T is minimal.*

`Proof of` (ii)

Let $x \in X$ be an almost periodic point with dense forward orbit.

We'll show that if $y \in X$ and $\varnothing \neq U \subset X$ is open, then $\exists\ k \geq 1,\ T^k(y) \in U$.

Proof

- WLOG $x \in U$.
- $\exists$ open sets $U' \subset X,\ V \subset X \times X$ so that

$$x \in U' \subset U,\ V \supset \Delta(X \times X)\ \&\ \ (U' \times X) \cap V) \subset X \times U.$$

Here $\Delta(X \times X) := \{(x, x) :\ x \in X\}$.

- $\exists\ K(U')$ s.t. $\forall\ n \geq 1\ \exists\ k \in [0, K(U'))$ with $T^{n+k}(x) \in U'$.
- By continuity of $T \times T$, $\ \exists\ V' \supset \Delta(X \times X)$ so that

$$\bigcup_{j=0}^{K(U')} (T \times T)^j V' \subset V,$$

- $\exists\ y \in W$ open so that $W \times W \subset V'$.

$\exists\ n \geq 1$ such that $T^n(x) \in W$ & $\exists\ 0 \leq k < K(U')$ such that$T^{n+k}(x) \in U'$

thus

$(T^{n+k}x, T^k(y)) \subset (U' \times X) \cap (T \times T)^k(W \times W) \subset (U \times X) \cap V \subset X \times U.$ ☑

Exercise 2.3.

Show that proposition 2.4 is true for a continuous map of a compact Hausdorff space.

Minimal sets.

Let (X,T) be a minimal continuous map of a compact, metric space. A closed subset $\varnothing \neq M \subset X$ is a *minimal set* for T if $T^{-1}M = M$ and (M,T) is minimal.

Proposition 2.5

A continuous map of a compact, metric space has a minimal set.

Proof

Let (X,T) be a minimal continuous map of a compact, metric space and let

$$\mathfrak{M} := \{\text{closed, non-empty, } T\text{-invariant subsets of } X\}.$$

Order $\mathfrak{M}$ by inclusion. A set $M \in \mathfrak{M}$ is a minimal set iff it is a minimal element of $\mathfrak{M}$.

Existence of these follows from Zorn's lemma because

- *every chain $\mathcal{C} \subset \mathfrak{M}$ has a non-empty intersection in $\mathfrak{M}$.*

This is because an arbitrary intersection of closed invariant sets is a closed invariant set e.g. $\bigcap_{M\in\mathcal{C}} M$. Also, $\mathcal{C}$ has the finite intersection property and so by compactness, $\varnothing \neq \bigcap_{M\in\mathcal{C}} M \in \mathfrak{M}$. ☑

Corollary *Every continuous map of a compact, metric space has an almost periodic point.*

`Exercises on Minimality`

Exercise M1. Let $(X,T) := (\{0,1\}^{\mathbb{Z}}, \texttt{shift})$. Show that there is an almost periodic, nonperiodic point for T.

Exercise M2 "`Cycle of fifths`".

According to music theory, the operation of raising pitch by a "perfect fifth" is periodic:

$$C \mapsto G \mapsto D \mapsto A \mapsto E \mapsto B \mapsto F^\sharp \mapsto C^\sharp \mapsto G^\sharp \mapsto D^\sharp \mapsto A^\sharp = B^\flat \mapsto F \mapsto C.$$

See e.g.
`http://tamingthesaxophone.com/jazz-cycle-of-5ths.html`

According to "Pythagorean music theory", raising pitch by a perfect fifth is attained by increasing the frequency by $\frac{3}{2}$. Lowering pitch by an octave is attained by halving the frequency.

Show that the collection of frequencies obtained by raising pitch by perfect fifths and lowering by octaves is dense in $\mathbb{R}_+$.

See http://en.wikipedia.org/wiki/Well_temperament

Exercise M3.

For $d \geq 1$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ define $R_\alpha : \mathbb{T}^d \to \mathbb{T}^d$ by $R_\alpha(x) := x + \alpha \mod 1$ (i.e. $R_\alpha(x)_k := x_k + \alpha_k \mod 1 \ \forall\ 1 \leq k \leq d$).

Show that $(\mathbb{T}^d, R_\alpha)$ is minimal iff $(1, \alpha_1, \alpha_2, \ldots, \alpha_d)$ are linearly independent over $\mathbb{Q}$.

TRANSITIVITY.

Proposition 2.7 `(two sided transitivity)`

Let T be an homeomorphism of a Polish (i.e. complete, separable) metric space X. TFAE*:*

(i) T *is two-sided topologically transitive;*

(ii) `(topological ergodicity)`

$\forall\ U \subset X$ *open and non-empty,* $\overline{\bigcup_{n\in\mathbb{Z}} T^n U} = X$*;*

(iii) $\exists\ X_0 \subset X$ *a dense* G_δ *so that* $T^{-1}X_0 = X_0$ *and so that* (X_0, T) *is minimal.*

Proof of (i) $\Rightarrow$ (ii):

Suppose that $\overline{\{T^n x :\ n \in \mathbb{Z}\}} = X$ and let $U \subset X$ be open and non-empty, then $\exists\ n_0,\ T^{n_0} x \in U$ whence $T^{n+n_0} x \in T^n U\ \forall\ n \geq 1$ and

$$\overline{\bigcup_{n\in\mathbb{Z}} T^n U} \supset \overline{\{T^{n+n_0} x \in \mathbb{Z}\}} = X. \boxtimes$$

Proof of (ii) $\Rightarrow$ (iii)

Let $\mathcal{U}$ be a countable base of open sets for the topology of X. By assumption, $\forall\ U \in \mathcal{U},\ U \neq \varnothing\ \Delta_U := \{x \in X :\ \exists\ n \in \mathbb{Z},\ T^n x \in U\} = \bigcup_{n\in\mathbb{Z}} T^n U$ is open and dense in X. By Baire's theorem,

$$\Delta := \{x \in X :\ \overline{\{T^n x :\ n \in \mathbb{Z}\}} = X\} = \bigcap_{U\in\mathcal{U},\ U\neq\varnothing} \Delta_U$$

is a dense G_δ set of transitive points in X. So is $X_0 := \bigcap_{n\in\mathbb{Z}} T^n \Delta$ which is also T-invariant. $\boxtimes$

Proposition 2.8 `(forward transitivity)`

Suppose that X is a perfect Polish space, and that $T : X \to X$ is continuous. TFAE*:*

(i) T *is forward topologically transitive;*

(ii) $\forall\ U \subset X$ *open and non-empty,* $\overline{\bigcup_{n\in\mathbb{N}} T^{-n} U} = X$*;*

(iii) $\exists\ X_0 \subset X$ *a dense* G_δ *so that* $T^{-1}X_0 = X_0$ *and so that* (X_0, T) *is forward minimal.*

Proof

Proof of (i) ⇒ (ii):

Suppose that $\overline{\{T^n x:\ n \ge 1\}} = X$. Since X has no isolated points,

$$\overline{\mathcal{O}_+^{(T)}(x)} = \omega(T, x) = \overline{\{T^n x:\ n \ge N\}} = X\ \forall\ N \ge 1.$$

Now let $U,\ V \subset X$ be open and non-empty. We claim that $\exists\ n \ge 1,\ T^{-n}U \cap V \ne \varnothing$ (which proves $\overline{\bigcup_{n\ge 1} T^{-n}U} = X$). To establish the claim, $\exists\ N \ge 1,\ T^N x \in V$. Since $\overline{\{T^n x:\ n \ge N+1\}} = X,\ \exists\ n \ge 1,\ T^{N+n}x \in U$. Thus $T^N x \in V \cap T^{-n}U \ne \varnothing$. ☑

Proof of (ii) ⇒ (iii):

As above, let $\mathcal{U}$ be a countable base of open sets for the topology of X. By assumption, $\forall\ U \in \mathcal{U},\ U \ne \varnothing\ \Delta_U := \{x \in X:\ \exists\ n \ge 1,\ T^n x \in U\} = \bigcup_{n\ge 1} T^{-n}U$ is open and dense in X. By Baire's theorem,

$$X_0 := \{x \in X:\ \overline{\{T^n x:\ n \ge 1\}} = X\} = \bigcap_{U\in\mathcal{U},\ U\ne\varnothing} \Delta_U$$

is a dense G_δ set in X, clearly T-invariant and (X_0, T) is minimal. ☑

Proposition 2.9 (recurrence and transitivity)

Suppose that $T : X \to X$ *is a regionally recurrent, topologically transitive homeomorphism of a Polish space, then* T *is forward topologically transitive.*

Proof We claim first that $\overline{\bigcup_{n=1}^\infty T^{-n}V} = X\ \forall\ \varnothing \ne V$ open. To see this, we fix $\varnothing \ne U, V$ open and show $\exists\ n \ge 1,\ U \cap T^{-n}V \ne \varnothing$. Indeed by topological transitivity of T, $\exists\ N \in \mathbb{Z}$ with $W := U \cap T^N V \ne \varnothing$. By regional recurrence, $\exists\ n > |N|,\ W \cap T^{-n}W \ne \varnothing$, whence

$$\varnothing \ne W \cap T^{-n}W = U \cap T^N V \cap T^{-n}U \cap T^{-(n-N)}V \subset U \cap T^{-(n-N)}V. \quad \square$$

Now fix a countable base $\mathcal{U}$ for the topology on X, then by Baire's theorem

$$\Delta := \{x \in X:\ \overline{\{T^n x:\ n \ge 1\}} = X\} = \bigcap_{\varnothing\ne U\in\mathcal{U}} \bigcup_{n=1}^{\infty} T^{-n}U \ne \varnothing$$

and T is forward topologically transitive. $\square$

Transitivity Exercises

Exercise T1.

Show that a topologically transitive homeomorphism of a Polish space either

- has a residual orbit $\mathcal{O}^{(T)}(x)$;
- is a permutation of a finite set; or
- the Polish space is perfect and the homeomorphism is regionally recurrent.

Exercise T2.

(i) Exhibit a compact metric space X with a continuous map $T : X \to X$ so that (X,T) is positively transitive but for which $\exists\ \varnothing \neq U \subset X$ open with $\overline{\bigcup_{n\geq 0} T^{-n}U} \neq X$.

(ii) Let X be a perfect Polish space, and let $T : X \to X$ be a regionally recurrent, topologically transitive homeomorphism.

Show that

$$\exists\ x \in X \text{ so that } \overline{\mathcal{O}_+^{(T)}(x)} = \overline{\mathcal{O}_+^{(T^{-1})}(x)} = X.$$

(iii) Show that an isometry of a perfect metric space is forward minimal iff it is forward topologically transitive.

Generic ergodicity

For X a polish space, let

$$\mathcal{B}(X) := \{\texttt{Borel subsets of } X\} \ \&\ \mathcal{N}(X) := \{A \in \mathcal{B}(X) :\ A\,\texttt{meagre}\}.$$

Recall from topological measurability theory:

🚂 $$\forall\ A \in \mathcal{B}(X),\ \exists\ U \subset X \text{ open s.t. } \quad A \Delta U \in \mathcal{N}(X).$$

A Polish dynamical system (X,T) is called *genericically ergodic* if

$$A \in \mathcal{B}(X)\ \ T^{-1}A = A \quad \Longrightarrow \quad A \in \mathcal{N}(X) \text{ or } \ X \setminus A \in \mathcal{N}(X).$$

Proposition 3.1

Let X be a perfect, polish space and let $T : X \to X$ be a forward transitive, continuous map, then T is generically ergodic.

Proof

Let $X_0 \subset X$ be a T-invariant, dense G_δ so that (X_0, T) is minimal, and suppose that

$$A \in \mathcal{B}(X) \setminus \mathcal{N}(X), \quad T^{-1}A = A.$$

We'll show that $X \setminus A \in \mathcal{N}(X)$.

Let $U \subset X$ be open so that $A\Delta U \in \mathcal{N}(X)$. Since $A \notin \mathcal{N}(X)$, we have that $U \neq \varnothing$ whence $U \cap X_0 \neq \varnothing$.

Now let

$$X_1 := X_0 \setminus \bigcup_{n\geq 0} T^{-n}(A\Delta U),$$

then X_1 is a dense G_δ and $T : X_1 \to X_1$.

Moreover, $A \cap X_1 = U \cap X_1 \neq \varnothing$, whence, by minimality of (X_1, T), A is open and dense in X_1 proving that

$$X \setminus A \subset (X \setminus X_1) \cup (X_1 \setminus A) \in \mathcal{N}(X). \quad ☑$$

Remark. It can be shown analogously (!) that any two sided transitive, invertible, polish dynamical system is also generically ergodic.

Example. `of a continuous map of a perfect, compact, metric space which is generically ergodic but not regionally recurrent.`

Let $\widehat{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ be the one point compactification of $\mathbb{Z}$, let $X := \{1,2\}^{\mathbb{N}} \times \widehat{\mathbb{Z}}$ and define $T : X \to X$ by

$$T(x,y) := \begin{cases} (Sx, y + x_1) & y \in \mathbb{Z}; \\ (Sx, y) & y = \infty \end{cases}$$

where $x = (x_1, x_2, \dots)$, $Sx := (x_2, x_3, \dots)$, then X is a perfect, compact, metric space, and T is continuous, onto. The recurrent points of T are given by $\mathfrak{R}(T) = \mathfrak{R}(S) \times \{\infty\}$. This is not a residual set in X and so (X, T) is not regionally recurrent.

Week # 6, 20/11/2013.

Proposition 6.1
(X,T) is generically ergodic.

Idea of Proof
We'll exhibit a countable group $\Gamma \subset$ `Homeo.` (X) which is generically ergodic and so that $T^{-1}A = A \implies \gamma A = A\ \forall\ \gamma \in \Gamma$.
Generic ergodicity will bew established using:

Proposition 6.2 `(group action transitivity)`
Let Γ be a countable group of homeomorphisms of the Polish metric space X. `TFAE`*:*

(i) $\exists\ x \in X$ *with* $\overline{\{\gamma(x) :\ \gamma \in \Gamma\}} = X$;

(ii) `(topological ergodicity)`
$\forall\ U \subset X$ *open and non-empty,* $\overline{\bigcup_{\gamma\in\Gamma} \gamma U} = X$;

(iii) $\exists\ X_0 \subset X$ *a dense* G_δ *so that* $\gamma X_0 = X_0\ \ \forall\ \gamma \in \Gamma$ *and so that* (X_0, Γ) *is minimal* ($\overline{\{\gamma(x) :\ \gamma \in \Gamma\}}_{X_0} = X_0\ \ \forall\ x \in X_0$;

(iv) $A \in \mathcal{B}(X),\ \Gamma A = A \implies A \in \mathcal{N}(X)$ *or* $X \setminus A \in \mathcal{N}(X)$.

Proof Exercise.
`Hint`: See propositions 2.7 & 3.1.

Proof of propn. 6.1
Suppose that $k \geq 1$ and $v = (v_1, \dots, v_k),\ w = (w_1, \dots, w_k) \in \{1,2\}^k$. Write $s_k(v) := \sum_{j=1}^k v_j$,

$$[b] := \{x \in \{1,2\}^{\mathbb{N}} :\ x_j = b_j\ \forall\ 1 \leq j \leq k\} \qquad (b = v, w)$$

and define

$$\pi_{v,w} : [v] \times \mathbb{Z} \to [w] \times \mathbb{Z}$$

by

$$\pi_{v,w}((v,x),n) := ((w,x), n + s_k(v) - s_k(w)).$$

It follows that $\pi_{v,w} : [v] \times \{N\} \to [w] \times \{N + s_k(v) - s_k(w)\}$ is a homeomorphism $\forall\ N \in \mathbb{Z}$ and that $\pi_{w,v} = \pi_{v,w}^{-1}$.
Moreover,

(♗) For $\xi \in [v] \times \mathbb{Z},\ \zeta \in [w] \times \mathbb{Z}$:

$$\pi_{w,v}(\xi) = \zeta \iff T^k(\xi) = T^k(\zeta).$$

Now define $\Phi_{v,w} : X \to X$ by

$$\Phi_{v,w}(z) = \begin{cases} \pi_{v,w}(z) & z \in [v] \times \mathbb{Z}, \\ \pi_{w,v}(z) & z \in [w] \times \mathbb{Z}, \\ z & \text{else.} \end{cases}$$

Evidently, each $\Phi_{v,w} \in \texttt{Homeo.}\,(X)$ and $\Phi_{v,w}^2 \equiv \texttt{Id}$.

Let $\Gamma := \langle \gamma \rangle$ be the group of homeomorphisms of X generated under composition.

It follows from (♝) that for $\xi,\ \zeta \in X_0 := \Omega \times \mathbb{Z}$,

$$\exists\ \gamma \in \Gamma,\ y = \gamma(x) \iff \exists\ N \geq 1,\ T^N(x) = T^N(y).$$

Thus (!) for $A \subset X_0$,

$$T^{-1}A = A \implies \gamma A = A\ \forall\ \gamma \in \Gamma$$

and topological ergodicity of T follows from that of Γ.

By proposition 6.2, to establish this, it suffices to show

(⚽) $$\overline{\bigcup_{\gamma \in \Gamma} \gamma U} = X\ \forall\ \ U \subset\ X \ \text{ open, nonempty.}$$

To prove (⚽), it suffices to show that if $U,\ W \subset X$ are non-empty, open sets, then

(🦇) $$\exists\ k \geq 1,\ \ u,\ w \in \{1,2\}^k,\ N \in \mathbb{Z} \text{ such that}$$
$$[u] \times \{N\} \subset\ U\ \ \&\ \ [w] \times \{N + s_k(u) - s_k(w)\} \subset\ W.$$

Proof of (🦇)

Fix $i \in \mathbb{N},\ a,\ b \in \{1,2\}^i,\ K, L \in \mathbb{Z}$ so that

$$[a] \times \{K\} \subset\ V\ \ \&\ \ [b] \times \{L\} \subset\ W.$$

Next, $\exists\ j \in \mathbb{N},\ c,\ d \in \{1,2\}^j$ so that

$$\begin{aligned} L &= (K + s_i(a) - s_i(b)) + s_j(c) - s_j(d) \\ &= K + s_{i+j}(a,c) - s_{i+j}(b,d). \end{aligned}$$

Setting $k = i + j,\ v = (a,c)\ \&\ w = (b,d)$ establishes (🦇). ☑

Exercise 6.1 (`generical exactness`).

The topological dynamical system (X,T) is *generically exact* if

$$\mathfrak{T}(X) := \bigcap_{n \geq 1} T^{-n}\mathcal{B}(X) \subseteq \{A :\ \text{either } A \in \mathcal{N}(X) \text{ or } X \smallsetminus A \in \mathcal{N}(X)\}.$$

(i) Show that generical exactness ⇒ generical ergodicty but not conversely.

(ii) Show that

$$\mathfrak{T}(X) = \{A \in \mathcal{B}(X) : \ x \in A, \ (x,y) \in \mathcal{T}_T \ \Rightarrow \ y \in A\}$$

where

$$\mathcal{T}_T := \{(x,y) \in X \times X : \ \exists \ N \geq 1 \text{ such that } T^N(x) = T^N(y)\}.$$

(iii) Show that (X,T) as in the example is generically exact.

Exercise 6.2. Let (X,T) be a [`forward`] topologically transitive homeomorphism of the metric space (X,d). Show that if $\{T^n : \ n \in \mathbb{Z}\}$ [$\{T^n : \ n \in \mathbb{N}\}$] is an equicontinuous family (of continuous maps $X \to X$), then (X,T) is [`forward`] minimal.

Structure

Homomorphism of topological dynamical systems. Suppose that $X, \ Y$ are topological spaces and that $S : X \to X, \ T : Y \to Y$ are continuous maps.

A *topological homomorphism* $\pi : (X,S) \to (Y,T)$ is a continuous, surjective map $\pi : X \to Y$ satisfying $\pi \circ S = T \circ \pi$
aka : *topological: -factor map, -extension map,* and -semiconjugacy.

In this case, (Y,T) is known as a *topological factor* or *image* of (X,S) which itself is known as a *topological extension* of (Y,T).

A *topological isomorphism* (aka *conjugacy*) is an invertible homomorphism i.e. a homomorphism $\pi : (X,S) \to (Y,T)$ with $\pi : X \to Y$ a homeomorphism.

- Two dynamical systems are called *weakly topologically isomorphic* if they are both factors of each other.

- For Polish dynamical systems there is also a *generic homomorphism* $\pi : (X,S) \to (Y,T)$ where there are residual subsets $X_0 \subset X, \ Y_0 \subset Y$, invariant under S & T respectively so that $\pi : (X_0,S) \to (Y_0,T)$ is a topological homomorphism. Also, analogously, generic isomorphism & generic weak isomorphism.

Invertible extensions and inverse limits

The question here is to find a "`canonical`" invertible extension of (X,T), a continuous map of a metric space: i.e. $(\widetilde{X},\widetilde{T})$ a homeomorphism of a metric space together with $\pi : \widetilde{X} \to X$ continuous, onto st $\pi \circ \widetilde{T} = T \circ \pi$.

The inverse limit construction. Given (X,T), a surjective, continuous map of a metric space define

$$\widetilde{X} := \{(x_1,x_2,\dots)\in X^{\mathbb{N}} :\ Tx_{n+1}=x_n\ \forall\ n\ge 1\},$$

then $\widetilde{X}$ is a closed subset of $X^{\mathbb{N}}$ (with respect to the product topology). Equip $\widetilde{X}$ with the inherited product topology.

- If X is Polish (compact) then so is $\widetilde{X}$.

Define $\widetilde{T}:\widetilde{X}\to\widetilde{X}$ by $\widetilde{T}(x_1,x_2,\dots):=(Tx_1,x_1,x_2,\dots)$, then $\widetilde{T}:\widetilde{X}\to\widetilde{X}$ is a homeomorphism (with $\widetilde{T}^{-1}$=shift).

The map $(x_1,x_2,\dots)\mapsto x_1$ is a semiconjugacy $(\widetilde{X},\widetilde{T})\to(X,T)$.

- $(\widetilde{X},\widetilde{T})$ is "`smallest`" as an invertible extension of (X,T) in the following sense,:

Proposition 6.3 *If (Y,R) is an invertible extension of (X,T), then it is also an extension of $(\widetilde{X},\widetilde{T})$.*

Proof Suppose that $\phi:Y\to X$ is a semiconjugacy $(Y,R)\to(X,T)$ and define $\psi:Y\to X^{\mathbb{N}}$ by $\psi(y)_n:=\phi(R^{-(n-1)}y)$. Evidently $\psi:Y\to X^{\mathbb{N}}$ is continuous. To see that $\psi:Y\to\widetilde{X}$,

$$T(\psi(y)_{k+1})=T(\phi(R^{-k}y)=\phi(R^{-k+1}y)=\psi(y)_k.$$

This last equation also shows that $\psi\circ R=\widetilde{T}\circ\psi$. ☐

- This property of "`smallness`" defines $(\widetilde{X},\widetilde{T})$ up to weak isomorphism and it is called the *natural extension* of (X,T).

Proposition 6.4

Suppose that $T:X\to X$ is a continuous map of a Polish space.

- *If (X,T) is regionally recurrent, then so is $(\widetilde{X},\widetilde{T})$.*
- *If (X,T) is topologically transitive, then so is $(\widetilde{X},\widetilde{T})$.*

Proof Both claims follow easily from the following

Lemma 3.4

If $\varnothing\ne U\subset\widetilde{X}$ is open, then $\exists\ N\ge 0$, $\varnothing\ne W\subset X$ open so that $U\supseteq\widetilde{T}^N\pi^{-1}W$ where $\pi:(x_1,x_2,\dots)\mapsto x_1$ $(\widetilde{X}\to X)$.

Proof By the definition of the product topology, $\exists\ k\ge 1$, $\varnothing\ne U_1,U_2,\dots U_k\subset X$ open so that

$$U\supseteq[U_1,U_2,\dots,U_k]:=\{x\in\widetilde{X}:\ x_j\in U_j\ \forall\ 1\le j\le k\}\ \ne\ \varnothing.$$

Now

$$\begin{aligned}[U_1, U_2, \dots, U_k] &:= \{x \in \widetilde{X} : \ x_j \in U_j \ \forall \ 1 \le j \le k\} \\ &\overset{!}{=} [\underbrace{X, \dots, X}_{k-1 \text{ times}}, \bigcap_{j=1}^{k} T^{-(k-j)} U_j] \\ &= \widetilde{T}^{k-1} [\bigcap_{j=1}^{k} T^{-(k-j)} U_j] \\ &=: \widetilde{T}^{k-1} \pi^{-1} W. \quad \square\end{aligned}$$

Subshifts

The *two-* [one-] *sided full shift* over the *state space* S is $S^{\mathbb{Z}}$ [$S^{\mathbb{N}}$]. If S is countable, it is equipped with the product discrete topology which is always Polish and compact when $\#S < \infty$. The shift is defined by $(\sigma x)_n := x_{n+1}$.

A *two-sided subshift* Σ (of $S^{\mathbb{Z}}$) is a closed, σ-invariant subset. A *one-sided subshift* Σ_+ (of $S^{\mathbb{N}}$) is a closed, σ-invariant subset.

Let $\Sigma \subset S^{\mathbb{Z}}$ be a subshift. The associated *language* is $L(\Sigma) := \{x_a^b := (x_a, x_{a+1}, \dots, x_b) : \ a \le b, \ x \in \Sigma\} \subset S^* := \bigcup_{n=1}^{\infty} S^n$ (here $x_a^b := (x_a, x_{a+1}, \dots, x_b)$ for $a \le b$). Write $|w| := n$ for $w = (w_1, w_2, \dots, w_n) \in L$.

The associated one- and two-sided subshifts are

$$\Sigma_+(L) := \{x \in S^{\mathbb{N}} : \ x_a^b \in L \ \forall \ a \le b\} \ \& \ \Sigma_\pm(L) := \{x \in S^{\mathbb{Z}} : \ x_a^b \in L \ \forall \ a \le b\}.$$

Exercise 6.3.

Suppose that $\Sigma \subset S^{\mathbb{N}}$ is a one-sided subshift and that σ is the shift on Σ. Show that $(\widetilde{\Sigma}, \widetilde{\sigma}) \cong (\Sigma_\pm(L(\Sigma)), \sigma)$ where $\cong$ denotes topological isomorphism.

Topological Markov shift. The subshift $\Sigma \subset S^{\mathbb{Z}}$ ($S^{\mathbb{N}}$) is a *topological Markov shift* (`TMS`) if there is a matrix $A : S \times S \to \{0,1\}$ so that $\Sigma_A = \{x \in S^{\mathbb{Z}} : \ A(x_n, x_{n+1}) = 1 \ \forall \ n \ge 1\}$.

Exercise 6.4.

(i) Show that a `TMS` (Σ_A, σ) is forward topologically transitive $\iff$ $\forall \ s, t \in S, \ \exists \ n \ge 1$ such that $A^n(s,t) > 0$ where $A^1 := A$ and $A^{n+1}(s,t) := \sum_{u \in S} A(s,u) A^n(u,t)$.

(ii) Let $S_2 : \mathbb{S}^1 \to \mathbb{S}^1$ be $S_2(z) := z^2$. Show that the compact dynamical systems $(\mathbb{S}^1, S_2)$ and $(\{0,1\}^{\mathbb{N}}, \texttt{shift}\}$ are Baire isomorphic but not topologically isomorphic.

Exercise 6.5 `the solenoid.` The *solenoid* is

$$\widetilde{\mathbb{S}} := \{(z_1, z_2, \dots) \in \mathbb{S}^{1\mathbb{N}} : \ z_n = S_2(z_{n+1}) \ \forall \ n \geq 1\}.$$

(i) Show that $\widetilde{\mathbb{S}}$ is homeomorphic with $\mathbb{T} \times \{0,1\}^{\mathbb{N}}$ via

$$(t, (\epsilon_1, \epsilon_2, \dots) \ \mapsto \ \pi(t, (\epsilon_1, \epsilon_2, \dots)) := (t, \tfrac{\epsilon_1 + t}{2}, \tfrac{\epsilon_1}{2} + \tfrac{\epsilon_2 + t}{2^2}, \dots).$$

(ii) Show that $\widetilde{\mathbb{S}}$ is connected but not pathwise connected.

(iii) Define a group structure on $\widetilde{\mathbb{S}}$ so that it
(a) it is a compact, abelian topological group and
(b) $\widetilde{S_2} : \widetilde{\mathbb{S}} \to \widetilde{\mathbb{S}}$ is a group endomorphism.

Invariant probabilities

Given a measurable space $(X, \mathcal{B})$ and a measurable transformation $T : X \to X$, set $\mathcal{M}(X,T) := \{\mu \in \mathcal{P}(X, \mathcal{B}) : \ \mu \circ T^{-1} = \mu\}$.

Proposition 6.5 *If X is a compact metric space and $T : X \to X$ is continuous, then $\mathcal{M}(X,T) \neq \varnothing$.*

Proof Fix $x_n \in X \ \ (n \geq 1)$ and set $\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j x_n}$.

- If $\mu_{n_k} \to \nu \in \mathcal{P}(X)$ weak $*$ in $C(X)^*$, then $\nu \in \mathcal{M}(X,T)$.
- The Banach-Alaoglu theorem ensures such an $n_k \to \infty$. ☐

Example Let $X := (0,1)$ and $Tx := x^2$, then $\mathcal{M}(X,T) = \varnothing$. To see this, note that $\forall \ x \in (0,1)$, the sets $\{T^n(Tx, x]\}_{n \in \mathbb{Z}}$ are disjoint and $\biguplus_{n \in \mathbb{Z}} T^n(Tx, x] = (0,1)$. If $\mu \in \mathcal{M}(X,T)$, then

$$1 = \mu(\biguplus_{n \in \mathbb{Z}} T^n(Tx, x]) = \sum_{n \in \mathbb{Z}} \mu(T^n(Tx, x]) = \infty \cdot \mu((Tx, x]) = 0, \infty \neq 1.$$

Week # 7, 27/11/2013.

¶3 `Unique ergodicity` A measurable transformation $T : X \to X$ of the measurable space $(X, \mathcal{B})$ is called *uniquely ergodic* if $\#\mathcal{M}(X,T) = 1$.

Proposition 7.1 *Suppose that X is a compact metric space, that $T : X \to X$ is continuous and that $\mu \in \mathcal{M}(X,T)$, then*

$\mathcal{M}(X,T) = \{\mu\} \iff$

$$\tfrac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \to \int_X f d\mu \quad \forall\ f \in C(X),\ x \in X.$$

In this case, the convergence is uniform on X $\forall\ f \in C(X)$.

Proof

$\Leftarrow$) Let $p \in \mathcal{M}(X,T)$, then $\forall\ f \in C(X)$

$$\int_X f dp = \int_X \left(\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\right) dp \to \int_X f d\mu.$$

$\Rightarrow$ and uniform convergence) Suppose that $f \in C(X)$ but that $\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$ does not converge uniformly to $\int_X f d\mu$, then $\exists\ \epsilon > 0$ and $x_k \in X$, $n_k \to \infty$ with

$$|\tfrac{1}{n_k}\sum_{j=0}^{n_k-1} f(T^j x_k) - \int_X f d\mu| \geq \epsilon \ \forall\ k \geq 1.$$

As before, set $\mu_k := \frac{1}{n_k}\sum_{j=0}^{n_k-1} \delta_{T^j x_k}$. If $\mu_{k_\ell} \to \nu \in \mathcal{P}(X)$ weak $*$ in $C(X)^*$, then (!) $\nu \in \mathcal{M}(X,T)$. The Banach-Alaoglu theorem ensures this for some subsequence $k_\ell \to \infty$. But this time, we also get that

$$|\int_X f d\nu - \int_X f d\mu| \leftarrow |\tfrac{1}{n_k}\sum_{j=0}^{n_k-1} f(T^j x_k) - \int_X f d\mu| \geq \epsilon$$

so $\nu \neq \mu$. ☐

Convex analysis of $\mathcal{M}$.

• Note that $\mathcal{M}(X,T)$ is convex; and a closed subset of $\mathcal{P}(X)$ (equipped with the weak $\star$ topology. A measure $\nu \in \mathcal{M}(X,T)$ is called *extreme* if

$$p_1,\ p_2 \in \mathcal{P}(X),\ 0 \leq t \leq 1,\ \nu = tp_1 + (1-t)p_2 \ \Rightarrow\ p_1 = p_2 = \nu.$$

• Let $\texttt{Ext}\,\mathcal{M}(X,T) = \{\texttt{extreme points of}\ \ \mathcal{M}\}$ and $\mathcal{M}_e(X,T) = \{p \in \mathcal{M}(X,T) :\ (X, \mathcal{B}(X), p, T)\ \ \texttt{ergodic}\}$.

Theorem 7.2 *Let (X,T) be a compact dynamical system, then $\mathcal{M}(X,T)$ is a compact convex set (in $\mathcal{P}(X)$) and* $\texttt{Ext}\,\mathcal{M}(X,T) = \mathcal{M}_e(X,T)$.

`Proof of` $\texttt{Ext}\,\mathcal{M}(X,T) \subseteq \mathcal{M}_e(X,T)$

Suppose that $\mu \in \mathcal{M}(X,T) \setminus \mathcal{M}_e(X,T)$, then $\exists\ A \in \mathcal{B}(X)$ so that

$$T^{-1}A = A \ \ \&\ \mu(A) = p \in (0,1).$$

We have that

$$\mu = p\mu_A + (1-p)\mu_{A^c}$$

where $\mu_B(C) := \frac{\mu(B\cap C)}{\mu(B)}$.

Since $T^{-1}A = A$ we have that $\mu_A,\ \mu_{A^c} \in \mathcal{M}(X,T)$ whence $\mu \notin \mathtt{Ext}\,\mathcal{M}(X,T)$. ☑

Proof that $\mathtt{Ext}\,\mathcal{M}(X,T) \supseteq \mathcal{M}_e(X,T)$ for T invertible

Suppose that $\mu \in \mathcal{M}_e(X,T)$. If $p,\ q \in \mathcal{M}(X,T)$ & $t \in (0,1)$ are so that $\mu = tp + (1-t)q$, then $p,\ q \ll \mu$.

By the Radon-Nikodym theorem $\exists\ h \in L^1(\mu)$ so that $p(A) = \int_A hd\mu\ \ (A \in \mathcal{B})$. Thus

$$\int_A h \circ T^{-1} d\mu = \int_{T^{-1}A} hd\mu = p(T^{-1}A) = p(A) = \int_A hd\mu$$

and $h = h \circ T$ a.s.. By ergodicity of μ, $h = \int_X hd\mu = p(X) = 1$ and $p = \mu$. ☑

Remarks.

The proof of $\mathtt{Ext}\,\mathcal{M}(X,T) \supseteq \mathcal{M}_e(X,T)$ is uses the

Proposition *If $(X,\mathcal{B},m,T)$ is an ergodic, probability preserving transformation* (EPPT) *and if $\mu \in \mathcal{P}(X,\mathcal{B})$, $\mu\circ T^{-1} = \mu$ & $\mu \ll m$, then $\mu = m$.*

This proposition is proved for T an invertible EPPT and its proof uses the

Lemma *Let $(X,\mathcal{B},m,T)$ be an* EPPT. *If $h : X \to \mathbb{R}$ is measurable and $h \circ T = h$ a.e., then $\exists\ c \in \mathbb{R}$ so that $h = c$ a.e..*

Sketch proof of the Lemma

¶ If $A \in \mathcal{B}(X)$ & $T^{-1}A \overset{m}{=} A$ (i.e. $m(T^{-1}A\Delta A) = 0$), then $m(A) = 0,1$.

Proof $A \overset{m}{=} \bigcap_{n\ge1} \bigcup_{k\ge n} T^{-k}A =: B$ & $T^{-1}B = B$.

Now let $\alpha_n := \{[\frac{k}{2^n}, \frac{k+1}{2^n}) :\ k \in \mathbb{Z}_+\}$, and for $n \ge 1,\ k \in \mathbb{Z}_+$ let $A_n(k) := [h \in [\frac{k}{2^n}, \frac{k+1}{2^n})$.

Since $h \circ T = h$ a.e., we have $T^{-1}A_n(k) \overset{m}{=} A_n(k)\ \forall\ n \ge 1,\ k \in \mathbb{N}$ and by ¶:

$$\exists\ k : \mathbb{N} \to \mathbb{Z}_+ \text{ such that } m(A_n(k(n))) = 1\ \forall\ n \ge 1.$$

Evidently $A_n(k(n) \downarrow$ as $n \uparrow$ whence

- $m(\bigcap_{n\ge 1} A_n(k(n))) = 1$;
- $\frac{k(n)}{2^n} \xrightarrow[n\to\infty]{} c$;
- $h = c$ on $\bigcap_{n\ge 1} A_n(k(n))$. ☑

Exercise 7.1 (`almost invariant functions`).

(i) Let $(X, \mathcal{B}, m, T)$ be an ergodic non-singular transformation, and let Y be a separable metric space.

If $f : X \to Y$ is measurable and $f \circ T = f$ a.e., then $\exists\ y \in Y,\ f = y$ a.e..

(ii) Suppose that $T : X \to X$ is a regionally recurrent, forward transitive, continuous map of a Polish space X, Y is a separable metric space and $f : X \to Y$ is Borel measurable and $f \circ T = f$ on a residual set, then then f is constant on a residual set.

EXAMPLES

1. The Dyadic Integers. :

$\Omega = \{0,1\}^{\mathbb{N}}$, $(x+y)_n = x_n + y_n + \delta_n \mod 2$ where $\delta_1 = 0$, $\delta_{n+1} := [\frac{x_n+y_n+\delta_n}{2}]$.

The reason for the name "dyadic integers" is that

$$\sum_{k=1}^{\infty} 2^{k-1}(n(1,\underline{0}))_k = n \quad \forall\ n \ge 1$$

The adding machine

Define the *adding machine* $\tau : \Omega \to \Omega$ by $\tau(x) := x + (1, \overline{0})$, i.e.

$$\tau(1, ..., 1, 0, \epsilon_{n+1}, \epsilon_{n+2}, ...) = (0, ..., 0, 1, \epsilon_{n+1}, \epsilon_{n+2}, ...).$$

The Odometer Property.

$$\{((\tau^k x)_1, ..., (\tau^k x)_n) : 0 \le k \le 2^n - 1\} = \{0,1\}^n\ \forall\ x \in \Omega,\ n \ge 1.$$

Proposition 7.3

τ is uniquely ergodic (with $\mathcal{M}(\Omega, \tau) = \{m\}$).

Proof It suffices to prove that

$$(\text{✔}) \quad \frac{1}{N}\sum_{k=0}^{N-1} f \circ \tau^k \to \int_\Omega f dm \text{ as } N \to \infty \text{ uniformly on } \Omega\ \forall\ \ f \in C(\Omega).$$

Proof of (✔):

If $n \in \mathbb{N}$ is fixed, and $g : \{0,1\}^n \to \mathbb{R}$, and $f : \Omega \to \mathbb{R}$ is defined by $f(x) = g(x_1, ..., x_n)$, then by the odometer property,

$$\frac{1}{2^n} \sum_{k=0}^{2^n - 1} f \circ \tau^k \equiv \int_\Omega f dm,$$

whence (!)

$$\frac{1}{N} \sum_{k=0}^{N-1} f \circ \tau^k \to \int_\Omega f dm \text{ as } N \to \infty \text{ uniformly on } \Omega$$

and (✓) follows since functions of this form are uniformly dense in $C(\Omega)$. ☑

Exercise 7.2. Show that Ω is a compact topological group with Haar measure $m \in \mathcal{P}(\Omega)$ given by $m([\epsilon_1, ..., \epsilon_n]) = (\frac{1}{2})^n$..

2. Rotations of $\mathbb{T}^d$.

Proposition 7.4

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$ and $\{1, \alpha_1, \ldots, \alpha_d)$ are linearly independent over $\mathbb{Q}$, then $(\mathbb{T}^d, R_\alpha)$ is uniquely ergodic with $\mathcal{M}(\mathbb{T}^d, R_\alpha) = \{m = \mathtt{Leb}\}$.

Proof

For $k \in \mathbb{Z}^d$ & $x \in \mathbb{T}^d$, define $\chi_k(x) := e^{2\pi i \langle k, x \rangle}$. The condition on α ensures that $\chi_k(\alpha) \neq 1 \ \forall \ k \neq 0$. Thus

$$\sum_{j=0}^{N-1} \chi_k \circ R_\alpha^j(x) = \begin{cases} 1 & k = 0; \\ \chi_k(x) \frac{1 - \chi_k(\alpha)^N}{1 - \chi_k(\alpha)} & k \neq 0 \end{cases}$$

with the consequence that for $f = \chi_k, \ k \in \mathbb{Z}$,

$$(\text{✗}) \qquad \frac{1}{N} \sum_{j=0}^{N-1} f \circ R_\alpha^j \to \int_{\mathbb{T}^d} f dm \text{ as } N \to \infty \text{ uniformly on } \mathbb{T}^d.$$

Now, (✗) persists for linear combinations of χ_k's and their uniforma limits which are unifornmly dense in $C(\mathbb{T}^d)$ by the Stone-Weierstrass theorem. By proposition 7.1, $\mathcal{M}(\mathbb{T}^d, R_\alpha) = \{m\}$. ☑

3. The one-sided full shift.

Let $\Omega := \{0,1\}^{\mathbb{N}}$ and let $S = \mathtt{shift}$.

Proposition 7.5

$$|\mathcal{M}_e(\Omega, T)| = \mathfrak{c}.$$

Proof sketch We exhibit an injection $t \mapsto \mu_t$ $(\ (0,1) \to \mathcal{M}_e(\Omega, T)\ )$. To this end, fix $t \in (0,1)$ and define

$$\mu_t : \{\texttt{cylinders}\} \to [0,1]$$

by

$$\mu_t([a_1, \dots, a_N]) := \prod_{j=1}^{N} p_t(j) \quad \text{for } a_1, \dots, a_N = 0.1$$

$$\text{where } \; p_t(0) = 1 - t \ \&\ p_t(1) = t.$$

It follows that μ_t is extends to an additive and T-invariant set function on $\mathcal{A} := \{\texttt{finite unions of cylinders}\}$ whence by Caratheodory theory $\exists$ an extension (also denoted) μ_t to $\mathcal{B}(\Omega)$. By T-invariance on $\mathcal{A}$, we have $\mu_t \in \mathcal{M}(\Omega, T)$.

To prove ergodicity we prove a stronger property called *mixing*

$$(\text{🛈}) \qquad \mu_t(A \cap T^{-n}B) \xrightarrow[n\to\infty]{} \mu_t(A)\mu_t(B) \quad \forall\ A,\ B \in \mathcal{B}(\Omega).$$

Note first that (🛈) holds for A, B cylinders whence for A, $B \in \mathcal{A}$. Since $\mathcal{A}$ is dense in $\mathcal{B}(X)$ with respect to the sermi-metric $\rho(A, B) := \mu_t(A \Delta B)$, (🛈) holds $\forall\ A,\ B \in \mathcal{B}(\Omega)$. ☑

Ergodicity $\not\Rightarrow$ mixing.

For $\alpha \notin \mathbb{Q}$, $(\mathbb{T}, R_\alpha, m)$ is ergodic but $\exists\ q_n \to \infty$ so that

$$m(A \Delta R_\alpha^{-q_n} A) \xrightarrow[n\to\infty]{} 0 \quad \forall\ A \in \mathcal{B}(\mathbb{T}).$$

Anzai skew products

For $\psi : \mathbb{T} \to \mathbb{T}$ continuous and $\alpha \in \mathbb{T}$ define the *Anzai skew product* $T = T_{\alpha,\psi} : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T(x, y) := (x + \alpha, y + \psi(x)) \mod 1,$$

a Haar measure preserving homeomorphism.

Proposition 7.6

The following are equivalent for $\alpha \notin \mathbb{Q}$:
1) $T_{\alpha,\psi}$ is minimal,
2) $T_{\alpha,\psi}$ is topologically transitive,
3) $\nexists\ k : \mathbb{T} \to \mathbb{T}$ continuous and $q \geq 1$ such that $q\psi = k \circ T - k$.

Proof

Evidently 1) $\implies$ 2).

To see that 2) $\implies$ 3), assume that $\exists\ k : \mathbb{T} \to \mathbb{T}$ continuous and $q \geq 1$ such that $q\psi = k \circ R_\alpha - k$. Define $f : \mathbb{T}^2 \to \mathbb{T}$ by $f(x, y) := e^{2\pi i k(x) - qy}$.

Evidently, f is continuous, non-constant and T-invariant, so T is not topologically transitive.

To see that 3) $\implies$ 1), suppose that T is not minimal, and let $M \subsetneq \mathbb{T}^2$ be minimal (ie closed, T-invariant and such that $T|_M$ is minimal).

For $\beta \in \mathbb{T}$ define $q_\beta : \mathbb{T}^2 \to \mathbb{T}^2$ by $q_\beta(x, y) = (x, y + \beta)$, then:

- $q_\beta \circ T = T \circ q_\beta$ whence (!) $q_\beta M$ is minimal $\forall\ \beta \in \mathbb{T}$; thus
- if $\beta \in \mathbb{T},\ q_\beta M \cap M \neq \varnothing$ then $q_\beta M = M$.

Set $H := \{\beta \in \mathbb{T} :\ q_\beta M = M\}$, then

- H is a subgroup of $\mathbb{T}$ and closed since $\beta \mapsto q_\beta M$ is continuous $\mathbb{T} \to \mathcal{H}(\mathbb{T}^2)$ where

$$\mathcal{H}(\mathbb{T}^2) := \{\text{non-empty closed subsets of } \mathbb{T}^2\}$$

equipped with the Hausdorff metric (a compact metric space); and

- $M_x := \{y \in \mathbb{T} :\ (x, y) \in M\} = j(x) + H$ where $j : \mathbb{T} \to \mathbb{T}$.

It follows that

$$j(x) + H = M_x = (T^{-1}M)_x = M_{x+\alpha} - \psi(x) = j(x + \alpha) - \psi(x) + H.$$

- We have that $H \neq \mathbb{T}$ since otherwise $M = \mathbb{T}^2$ contradicting non-minimality of T, thus
- $\exists\ q \geq 1$ such that $qH = \{0\}$

whence setting $k := qj$ we obtain $q\psi(x) = k(x + \alpha) - k(x)$.

To establish continuity of $k : \mathbb{T} \to \mathbb{T}$, define $Z_q : \mathbb{T}^2 \to \mathbb{T}^2 \to$ by $Z_q(x, y) := (x, qy)$, then Z_q is continuous,

$$Z_q M = \{(x, qy) :\ (x, y) \in M\} = \{(x, k(x)) :\ x \in \mathbb{T}\},$$

is closed and (!) $k : \mathbb{T} \to \mathbb{T}$ is continuous. ☐

Proposition 7.7

For $\alpha \notin \mathbb{Q}$ and $\psi : \mathbb{T} \to \mathbb{T}$ measurable,

$T_{\alpha,\psi}$ is ergodic iff $\nexists\ k : \mathbb{T} \to \mathbb{T}$ measurable and $q \in \mathbb{N}$ such that $q\psi = k \circ R_\alpha - k$.

Proof

Assume first that $\exists\ k : \mathbb{T} \to \mathbb{T}$ measurable and $q \in \mathbb{N}$ such that $q\psi = k \circ R_\alpha - k$. Define $f : \mathbb{T}^2 \to \mathbb{T}$ by $f(x, y) := e^{2\pi i(k(x) - qy)}$. It follows that f is not a.e. constant and that $f \circ T = f$ whence T is not ergodic.

Conversely, suppose that T is not ergodic and let $f : \mathbb{T}^2 \to \mathbb{R}$ be bounded, measurable, not constant and T-invariant. For $n \in \mathbb{Z}$ define

$f_n : \mathbb{T} \to \mathbb{C}$ by

$$f_n(x) := \int_{\mathbb{T}} f(x,y) e^{-2\pi iny} dy.$$

By T-invariance of f,

$$\begin{aligned} f_n(x) &:= \int_{\mathbb{T}} f \circ T(x,y) e^{-2\pi iny} dy \\ &= \int_{\mathbb{T}} f(x+\alpha, y+\psi(x)) e^{-2\pi iny} dy \\ &= e^{2\pi in\psi(x)} f_n(x+\alpha). \end{aligned}$$

Evidently, $|f_n|$ is R_α-invariant, whence constant a.e.. Since f is not constant, $\exists\ q \in \mathbb{N}$ such that $|f_q(x)| > 0$[7]; whence $q\psi = k \circ R_\alpha - k$ where $f_q = re^{-2\pi ik}$. □

[7] else $f(x,y) = g(x)$ a.e. with $g \circ R_\alpha = g$ a.e. $\Rightarrow$ g constant

Week # 8, 4/12/2013.

Proposition 8.1 (Furstenberg)

For $\alpha \notin \mathbb{Q}$ and $\psi : \mathbb{T} \to \mathbb{T}$ continuous,
if $T_{\alpha,\psi}$ is ergodic, then it is uniquely ergodic.

Proof

We'll use

von Neumann's ergodic theorem

If $(X, \mathcal{B}.m, T)$ is an invertible, ergodic probability preserving transformation then

$$A_n^{(T)} f \xrightarrow[n\to\infty]{L^2(m)} \int_X f dm \qquad \forall \ f \in L^2(m)$$

where $A_n^{(T)} f := \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k$.

Sketch of proof Let $\mathcal{H} := L^2(m)_0 = \{f \in L^2(m) : \ \int_X f dm = 0\}$ a T-invariant, closed linear subspace and define $U : \mathcal{H} \to \mathcal{H}$ by $Uf = f \circ T$.

It suffices to show that

(⚕) $$\|A_n(f)\| \xrightarrow[n\to\infty]{\mathcal{H}} 0 \ \ \forall \ f \in \mathcal{H}.$$

Let $\mathcal{H}_0 := \{g - Ug : \ g \in \mathcal{H}\}$. We claim first that (⚕) holds for $f \in \mathcal{H}_0$, indeed if $f = g - Ug$, then

$$A_n(f) = A_n(g - Ug) = \frac{g - U^n g}{n} \xrightarrow[n\to\infty]{\mathcal{H}} 0.$$

By approximation, we see that (⚕) also holds for $f \in \overline{\mathcal{H}_0}$:

For $f \in \overline{\mathcal{H}_0} g \in \mathcal{H}_0$ with $\|f - g\| < \epsilon$,

$$\|A_n(f)\| \le \|A_n(f - g)\| + \|A_n(g)\| \le \epsilon + o(1).$$

Lastly, by ergodicity:

$$\mathcal{H}_0^{\perp} = \{f \in \mathcal{H} : \ \langle f, g - Ug\rangle = 0 \ \forall \ g \in \mathcal{H}\} = \{f \in \mathcal{H} : \ U^{-1} f = f\} = \{0\}$$

and $\overline{\mathcal{H}_0} = \mathcal{H}$. ☑

Proof of theorem 8.1

Evidently $m_{\mathbb{T}^2} \in \mathcal{M}_e(\mathbb{T}^2, T_{\alpha,\psi})$.

¶1 Every sequence has a subsequence $n_k \to \infty$ so that for $m_{\mathbb{T}}$-a.e. $x \in \mathbb{T}$, $\forall \ f \in C(\mathbb{T}^2), \ y \in \mathbb{T}$,

(🚂) $$A_{n_k}^{(T_{\alpha,\psi})} f(x, y) \xrightarrow[n\to\infty]{} \int_X f dm_{\mathbb{T}^2}.$$

Proof Let $\Gamma \subset C(\mathbb{T}^2)$ be countable and uniformly dense.

By von Neumann's theorem, for each $f \in \Gamma$ and for every subsequence, $\exists$ a subsequence so that the convergence (🚂) holds for f at each m-a.e. $(x, y) \in \mathbb{T}^2$. Ordering Γ and performing a Cantor-type diagonalization yields a subsequence $n_k \to \infty$ and $M \in \mathcal{B}(\mathbb{T}^2),\ m_{\mathbb{T}^2}(M) = 1$ so that the convergence (🚂) holds for every $f \in \Gamma$ at each $(x, y) \in M$.

Since Γ is uniformly dense, the convergence (🚂) holds for every $f \in C(\mathbb{T}^2)$ at each $(x, y) \in M$.

For $t \in \mathbb{T}$, define $q_t : \mathbb{T}^2 \to \mathbb{T}^2$ by $q_t(x, y) = (x, y + t)$, then $q_t \circ T_{\alpha,\psi} = T_{\alpha,\psi} \circ q_t$ and $m_{\mathbb{T}^2} \circ q_t = m_{\mathbb{T}^2}$.

For $(x, y) \in M,\ f \in C(\mathbb{T}^2),\ t \in \mathbb{T}$,

$$\begin{aligned} A_n^{(T_{\alpha,\psi})} f(x, y + t) &= A_n^{(T_{\alpha,\psi})} f \circ q_t(x, y) \\ &\xrightarrow[n\to\infty]{} \int_X f \circ q_t dm_{\mathbb{T}^2} \\ &= \int_X f dm_{\mathbb{T}^2}. \end{aligned}$$

Now let $M_0 := \{x \in \mathbb{T} :\ \exists\ y \in \mathbb{T},\ (x, y) \in M\}$, then $m_{\mathbb{T}}(M_0) = 1$ and the convergence (🚂) holds for every $f \in C(\mathbb{T}^2)$ at each $(x, y) \in M_0 \times \mathbb{T}$. ☑¶1

Now suppose that $\mu \in \mathcal{M}_e(\mathbb{T}^2, T_{\alpha,\psi})$. We'll show $\mu = m_{\mathbb{T}^2}$.

¶2 $\exists$ a subsequence $n_k \to \infty$ satisfying (🚂) and also so that $\exists\ Q \in \mathcal{B}(\mathbb{T}^2),\ \mu(Q) = 1$ so that $\forall\ f \in C(\mathbb{T}^2),\ (x, y) \in Q$,

$$A_{n_k}^{(T_{\alpha,\psi})} f(x, y) \xrightarrow[n\to\infty]{} \int_X f d\mu. \tag{☕}$$

Proof Using the first paragraph of the proof of ¶1, show that there is a subsequence of the one satisfying (🚂) satisfying (☕). ☑

¶3 $\mu \circ \pi^{-1} = m_{\mathbb{T}}$ where $\pi(x, y) = x$.

Proof Since $\pi \circ T_{\alpha,\psi} = R_\alpha \circ \pi$ we have $\mu \circ \pi^{-1} \in \mathcal{M}(\mathbb{T}, R_\alpha) = \{m_{\mathbb{T}}\}$. ☑

To finish, we see that $\mu(\pi^{-1}M_0 \cap Q) = 1$ with the conclusion that $\mu = m_{\mathbb{T}^2}$. ☑

Example: An ergodic Anzai skew product. Consider $\psi(x) = x$ and $T = T_{\alpha,\psi}$ defined by $T(x, y) := (x + \alpha, y + x)$ where $\alpha \in \mathbb{T} \smallsetminus \mathbb{Q}$. To see that T is ergodic, suppose that $N \geq 1$ and $k : \mathbb{T} \to S^1$ measurable such that $e^{2\pi i N x} = k(x + \alpha)\overline{k}(x)$.

Fix $q_k \to \infty$ such that $q_k\alpha \to 0$ in $\mathbb{T}$, then $f \circ R_\alpha^{q_k} \xrightarrow[k\to\infty]{L^2(m)} f\ \forall\ f \in L^2(m)$ whence:

- $e^{2\pi i N q_k x} e^{\pi i N q_k (q_k - 1)\alpha} = e^{2\pi i N \psi_{q_k}} = k(x + q_k\alpha)\overline{k}(x) \xrightarrow[k\to\infty]{m} 1$ whence

- $0 = e^{\pi i N q_k(q_k-1)\alpha}\widehat{m}(Nq_k) = \int_{\mathbb{T}} k(x+q_k\alpha)\overline{k}(x)dx \to 1.$

This contradiction establishes ergodicity. ☑

Minimality ⇏ Ergodicity

Essential continuity. Let X be a metric space and let $m \in \mathcal{P}(X)$. A measurable function $f : X \to \mathbb{C}$ is called *m-essentially continuous* (e.c.) if $\exists\ g : X \to \mathbb{C}$ continuous such that $g = f$ m-a.e.

Given $f : X \to \mathbb{R}$ measurable, set

$$G_f := \{a \in \mathbb{R} : \ e^{iaf} \text{ is essentially continuous } \}.$$

Lemma 8.1

For $f : X \to \mathbb{R}$ measurable, if $G_f = \mathbb{R}$, then f is essentially continuous.

Proof

Set $dP(a) := \frac{1}{\sqrt{2\pi}}e^{-\frac{a^2}{2}}da$.

By assumption $\exists\ G : \mathbb{R} \times X \to \mathbb{S}^1$ satisfying $e^{iaf} = G(a,\cdot)$ m-a.e. $\forall\ a \in \mathbb{R}$ and $x \mapsto G(a,x)$ is continuous $(X \to \mathbb{S}^1)$ $\forall\ a \in \mathbb{R}$.

It follows from Fubini's theorem that for $P \times m$-a.e. $(a,x) \in \mathbb{R} \times X$, $G(a,x) = e^{iaf(x)}$, whence

- G is $P \times m$-Lebesgue measurable;
- for m-a.e. $x \in X$ and $\forall\ t \in \mathbb{R}$,

$$\int_{\mathbb{R}} G(a,x)e^{iat}dP(a) = \int_{\mathbb{R}} e^{ia(f(x)+t)}dP(a) = e^{-\frac{(f(x)+t)^2}{2}}.$$

Write $g_t(x) := \int_{\mathbb{R}} G(a,x)e^{iat}dP(a)$. If $x_n \xrightarrow[n\to\infty]{X} x$, then $G(a,x_n) \xrightarrow[n\to\infty]{} G(a,x)$ $\forall\ t \in \mathbb{R}$ and, by bounded convergence, $g_t(x_n) \xrightarrow[n\to\infty]{} g_t(x)$, Thus $g_t : X \to \mathbb{C}$ is continuous $\forall\ t \in \mathbb{R}$.

It follows that $F : X \to \mathbb{R}$ defined by $F(x) := \log \frac{g_0(x)}{g_{\frac{1}{2}}(x)} - \frac{1}{4}$ is continuous. But (!)

$$F = f \quad m - \text{a.e.} \quad ☑$$

Lemma 8.2

Let m be Lebesgue measure on $\mathbb{T}$. Suppose that $f : \mathbb{T} \to \mathbb{R}$ is measurable, but not m-essentially continuous; then $\exists\ a \in \mathbb{R}$ such that e^{ianf} is not essentially continuous $\forall\ n \in \mathbb{N}$.

Proof

For $f:\mathbb{T}\to\mathbb{R}$ measurable, set

$$G_f := \{a\in\mathbb{R}:\ e^{iaf} \text{ is essentially continuous }\}.$$

Evidently G_f is a subgroup of $\mathbb{R}$.

We claim that $G_f\in\mathcal{B}(\mathbb{R})$

To see this, define the linear operators $P_N: L^1(\mathbb{T})\to C(\mathbb{T})\quad (N\ge 1)$ by

$$P_N h(t) := \frac{1}{N}\sum_{n=1}^{N}\sum_{|k|<n}\widehat{h}(k)e^{2\pi ikt}.$$

Each operator P_N is continuous and we have that $t\mapsto P_N e^{itf}$ is continuous $\mathbb{R}\to C(\mathbb{T})\ \ \forall\ N\ge 1$.

Recall (!) that $h\in L^1(\mathbb{T})$ is essentially continuous iff $\{P_N h:\ N\ge 1\}$ is a Cauchy sequence in $C(\mathbb{T})$.

Now consider

$$Y := C(\mathbb{T})^{\mathbb{N}} = \{y=(y_1,y_2,\dots):\ y_n\in C(\mathbb{T})\ \forall\ n\ge 1\}$$

which becomes a Polish space (!) when metrized by

$$d(x,y) := \sum_{n=1}^{\infty}\frac{\|x_n-y_n\|_{C(\mathbb{T})}\wedge 1}{2^n},$$

and define $\pi:\mathbb{R}\to Y$ by $\pi(t)_n = P_n e^{itf}$, then π is continuous and $G_f=\pi^{-1}\mathcal{C}$ where $\mathcal{C} := \{\text{Cauchy sequences in } C(\mathbb{T})\}$.

To see measurability,

$$\begin{aligned}\mathcal{C} &= \{y\in Y:\ \exists\ C(\mathbb{T})-\lim_{n\to\infty} y_n\}\\ &= \bigcap_{N=1}^{\infty}\bigcup_{q=1}^{\infty}\bigcap_{k,\ell\ge q}\{y\in Y:\ \|y_k-y_\ell\|_{C(\mathbb{T})}<\frac{1}{N}\}\in\mathcal{B}(Y)\end{aligned}$$

whence $G_f=\pi^{-1}\mathcal{C}\in\mathcal{B}(\mathbb{R})$.

Now that G_f is a Borel subgroup of $\mathbb{R}$, we claim that either $G_f=\mathbb{R}$, or G_f is meagre in $\mathbb{R}$. To see this suppose that G_f is not meagre in $\mathbb{R}$, then $\exists\ U\ne\varnothing$ open in $\mathbb{R}$ so that $G_f\cap U$ is residual in U. It follows that $\exists\ \epsilon>0$ such that

$$(G_f\cap U)\cap(G_f\cap U+x)\ne\varnothing\ \forall\ |x|<\epsilon,$$

whence $(-\epsilon,\epsilon)\subset G_f$ and $G_f=\mathbb{R}$.

Thus, if $f:\mathbb{T}\to\mathbb{R}$ is measurable and not essentially continuous, then G_f^c is residual in $\mathbb{R}$ and $\exists\ a\in\bigcap_{q=1}^{\infty}\frac{1}{q}G_f^c$ which is as required. ☑

Example: Minimality $\not\Rightarrow$ ergodicity

Proposition 8.3 (Furstenberg, Kolmogorov)
For each $\alpha \in \mathbb{T} \setminus \mathbb{Q}$, $\exists\ \psi : \mathbb{T} \to \mathbb{T}$ continuous so that $T_{\alpha,\psi}$ is minimal and not ergodic.

Proof
Fix a sequence $q_n \in \mathbb{N}$, $q_n \uparrow \infty$ so that $|1 - e^{2\pi i q_n \alpha}| \le 2^{-n}\ \forall\ n \ge 1$.
Define $\Psi = \Psi^{(\alpha)} : \mathbb{R} \to \mathbb{R}$ by the Fourier series with coefficients

$$\widehat{\Psi}(\pm|k|) = \begin{cases} \frac{1-e^{\pm 2\pi i q_n \alpha}}{n} & |k| = q_n, \\ 0 & \text{else.} \end{cases}$$

This function is continuous as the Fourier series converges absolutely and since $\widehat{\Psi}(-k) = \overline{\widehat{\Psi}(k)}$,

$$\Psi(x) := \sum_{n\ge 1} \frac{1 - e^{2\pi i q_n \alpha}}{n} e^{2\pi i q_n x} \in \mathbb{R}.$$

For $r > 0$, $r\Psi \mod 1 : \mathbb{T} \to \mathbb{T}$ is continuous. We'll show that for suitable $r > 0$, $r\Psi \mod 1$ is as advertised.

Next, let

$$c_k := \frac{\widehat{\Psi}(k)}{1 - e^{2\pi i k\alpha}} = \begin{cases} \frac{1}{n} & |k| = q_n, \\ 0 & \text{else,} \end{cases}.$$

then $\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty$ and by the Riesz-Fischer theorem there is a function $g \in L^2(\mathbb{T})$ such that $\widehat{g}(k) = c_k$.
Evidently $\widehat{g - g \circ R_\alpha} = \widehat{\Psi}$, whence $\Psi = g - g \circ R_\alpha \mod m$.
By proposition 7.7 $T_{\alpha, r\Psi \bmod 1}$ is non-ergodic $\forall\ r > 0$.
The rest of this proof is to show that $\exists\ r > 0$ so that $T_{\alpha, r\Psi \bmod 1}$ is minimal.

Since

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{|k|\le n} \widehat{g}(k) = \frac{2}{N}\sum_{n=1}^{N}\sum_{\{k\ge 1:\ q_k \le n\}} \frac{1}{k} \xrightarrow[N\to\infty]{} \infty$$

it follows that g is not essentially continuous. By lemma 8.2, $\exists\ r_0 > 0$ such that $e^{2\pi i r_0 n g}$ is not essentially continuous $\forall\ n \in \mathbb{N}$.
Define $\psi : \mathbb{T} \to \mathbb{T}$ by $\psi(x) := r_0\psi(x) \mod 1$. This is continuous and $\psi = k \circ R_\alpha - k$ where $k := r_0 g \mod 1$.
If $T_{\alpha,\psi}$ is not minimal then by proposition 7.6 $\exists\ K : \mathbb{T} \to \mathbb{T}$ continuous and $q \ge 1$ such that $q\psi = K \circ R_\alpha - K$. By ergodicity of R_α, $K - q r_0 g$

is constant a.e., contradicting non essential continuity of $e^{2\pi i q r_0 g}$. Thus $T_{\alpha,\psi}$ is minimal. ☑

Exercise 8.1.

(i) Show that $\exists\ \alpha \in \mathbb{T}$ such that $\exists\ q_n \in \mathbb{N},\ q_n \uparrow \infty$ with so that

$$|1 - e^{2\pi i q_n \alpha}| \asymp 2^{-q_n}\ \ \forall\ n \geq 1.$$

(ii) Show that (for this α) $\Psi^{(\alpha)} : \mathbb{R} \to \mathbb{R}$ is a real analytic function and that the skew product of proposition 6.1 is real analytic.

(iii) Suppose that $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ has "bad approximation" in the sense that $\exists\ \epsilon > 0$ so that $|\alpha - \frac{p}{q}| \geq \frac{\epsilon}{q^3}$, and let $\Psi : \mathbb{T} \to \mathbb{T}$ be twice continuously differentiable ($C^2(\mathbb{T})$). Define $\phi := \Psi \mod 1 : \mathbb{T} \to \mathbb{T}$. Show that $T_{\alpha,\phi}$ is not ergodic.

PERIODIC POINTS

Let $T : X \to X$. A point $x \in X$ is called a *periodic point* if $\exists\ p \in \mathbb{N}$ such that $T^p x = x$. In this case, $p \in \mathbb{N}$ is called the *period* of x and the collection $\{T^k x :\ 0 \leq k \leq p-1\}$ is called (the associated) *periodic orbit*. The *minimal period* of x is the smallest period, or the size of x's periodic orbit.

Define

$$\Pi_n(T) := \{x \in X :\ T^n x = x\},\ \Pi(T) := \bigcup_{n=1}^{\infty} \Pi_n(T),$$

$$P_n(T) := |\Pi_n(T)|,\quad p(T) := \overline{\lim_{n\to\infty}}\ \frac{\log(P_n(T)+1)}{n}$$

and the *(dynamical) zeta function* of T:

$$\zeta_T(z) := e^{\sum_{n=1}^{\infty} \frac{P_n(T)}{n} z^n} \qquad (|z| < e^{-p(f)}).$$

Example 0

Consider $R_\alpha : \mathbb{T} \to \mathbb{T}$ defined by $R_\alpha x = x + \alpha$.

$$\Pi_n(R_\alpha) = \begin{cases} \mathbb{T} & n\alpha \in \mathbb{Z}, \\ \varnothing & \text{else}, \end{cases} \quad \Pi(R_\alpha) = \begin{cases} \mathbb{T} & \alpha \in \mathbb{Q}, \\ \varnothing & \text{else}. \end{cases}$$

Example 1

Consider $E_q : \mathbb{T} \to \mathbb{T}$ defined by $E_q x := qx \mod 1$ (for $q \in \mathbb{N}$). Evidently

$$\Pi_n(E_q) = \mathrm{Ker}(E_q^n - 1) = \{\frac{k}{q^n - 1} :\ 0 \leq k \leq q^n - 2\},$$

$$P_n(E_q) = q^n - 1,\ p(E_q) = \log q,\ \zeta_{E_q}(z) = \frac{1-z}{1-qx}.$$

Example 2

Let $T \in \texttt{Aut}(\mathbb{T}^d) = \{T : \mathbb{T}^d \to \mathbb{T}^d :\ \text{cts} \ \&\ T(x+y) = T(x) + T(y)\}$.

¶0 $m \circ T^{-1} = m$

Proof Since T is an automorphism, $m \circ T^{-1}$ is translation invariant.....

¶1 $\exists$ a $d \times d$ matrix A with integer entries so that

$$\det A = \pm 1 \ \&\ T(x + \mathbb{Z}^d) = A(x) + \mathbb{Z}^d.$$

Proof *sketch*: Use the lifting theorem (on p.13).

¶2 If $(\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d), m, T)$ is ergodic then $A^t : \mathbb{Z}^d \to \mathbb{Z}^d$ has no non-zero periodic points.

Proof For $n \in \mathbb{Z}^d$, let $\chi_n : \mathbb{T}^d \to \mathbb{S}^1$ be defined by $\chi_n(x) := e^{2\pi i \langle n, x\rangle}$. We have that (!)

$$\chi_n \circ T = \chi_{A^t n}.$$

Thus if $N \in \mathbb{Z}^d \setminus \{0\}$ & $p \in \mathbb{N}$ are so that $A^{tp}N = N$ & $A^{tk}N \ne N\ \forall\ 0 \le k < p$, then $0 \ne \sum_{k=1}^{p} \chi_N \circ T^k =: F$ is not constant ($\because \{\chi_j :\ j \in \mathbb{Z}^d\}$ are orthogonal) and T-invariant. ☑

¶3 (`Exercise 8.2`): Show that if $A^t : \mathbb{Z}^d \to \mathbb{Z}^d$ has no non-zero periodic points, then

(a) $A^{tn}N \xrightarrow[n\to\infty]{} \infty\ \forall\ N \in \mathbb{Z}^d \setminus \{0\}$;

(b) $\chi_N \circ T^n \xrightarrow[n\to\infty]{\text{weakly in } L^2(m)} 0$;

(c) $m(A \cap T^{-n}B) \xrightarrow[n\to\infty]{} m(A)m(B)\ \forall\ A,\ B \in \mathcal{B}(\mathbb{T}^2)$;

whence $(\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d), m, T)$ is ergodic.

¶4 $\Pi(T) \supseteq\ \mathbb{T}^d \cap \mathbb{Q}^d$

Proof Set

$$X_q := \{\tfrac{1}{q} \cdot x \in \mathbb{Q}^d :\ x \in \mathbb{Z}^d\} \cap \mathbb{T}^d \quad (q \in \mathbb{N}).$$

Since $T(x) = A(x) \mod 1$, $T(X_q) \subset X_q$ and since T is injective and $|X_q| = q^d < \infty$, $T : X_q \to X_q$ is a bijection.

Thus $\forall\ x \in X_q,\ \exists\ k > \ell \ge 1$ such that $T^k x = T^\ell(x) =: y)$, whence if $\ell - k = p \ge 1$ then $T^p y = y\ \Rightarrow\ T^p x = x \in \Pi(T)$. ☑

¶5 If $A^t : \mathbb{Z}^d \to \mathbb{Z}^d$ has no non-zero periodic points, then $\Pi(T) \subseteq \mathbb{T}^d \cap \mathbb{Q}^d$.

Proof Suppose that $x \in \Pi(T)$ and that $T^p x = x$, then $\exists k \in \mathbb{Z}^d$ such that $A^p x = x + k$. By aperiodicity of A^t, 1 is not an eigenvalue of A^p and $k \neq 0$, whence we have $x = (A^p - I)^{-1} k \in \mathbb{T}^d \cap \mathbb{Q}^d$. $\square$

¶6 If $A^t : \mathbb{Z}^d \to \mathbb{Z}^d$ has no non-zero periodic points, then $P_n(T) = |\det(A^n - I)|$.

Proof To see this, note that as above $\Pi_n(T) = T^{-1}_{A^n - I}\{0\}$ whence $P_n(T) = |\det(A^n - I)|$. $\square$

Exercise 8.3.

(i) Consider $T : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $T(x, y) = (2x + y, x + y) \mod 1$. Show that

(a) $P(T) = \log \lambda_+$;

and

(b) $\zeta_{T_A}(z) = \frac{(1-z)^2}{(1-\lambda_+ z)(1-\lambda_- z)}$

where $\lambda_\pm := \frac{3 \pm \sqrt{5}}{2}$.

(ii) Consider $E : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $E(x, y) = (4x + 2y, 2x + 2y) \mod 1$. Show that $\Pi(T) \subsetneqq \mathbb{T}^d \cap \mathbb{Q}^d$.

Show that for $T \in \mathtt{End}(\mathbb{T}^d)$,

(iii) $P_n(T) = \Delta(A^n - I)$;

(iv) T is ergodic iff T is topologically transitive;

(v) T is not minimal.

Week # 9, 18/12/2013.

Subshifts via "grammar"

Let S be a finite set. The *word set* of S is

$$S^* := \bigcup_{n\ge 1} S^n.$$

For $\Gamma \subset S^*$, the *subshift with forbidden word-set* Γ is

$$X_\Gamma := \{x \in S^{\mathbb{Z}} :\ x_a^b := (x_a, \dots, x_b) \notin \Gamma\ \forall\ -\infty < a < b < \infty\}$$

if this set is non-empty.

Exercise 9.1. Show that the subshift with forbidden word-set $\Gamma \subset S^*$ is a subshift (as defined on page 49) and that any subshift is a subshift with some forbidden word-set.

Subshift of finite type.

A subshift is a *subshift of finite type* (`SFT`) if it is a subshift with a finite forbidden word-set. For example a `topological Markov shift` (TMS – as defined on page 49) is a SFT.

Exercise 9.2. Show that a SFT is topologically isomorphic to some TMS.

Calculations

$$\Pi_n(T) \cong \{(x_1, x_2, \dots, x_n) \in S^n :\ a_{x_k, x_{k+1}} = 1\ \forall\ 1 \le k \le n-1,\ a_{x_n, x_1} = 1\}.$$

$$P_n(\Sigma_A, T) = \mathtt{Tr}(A^n).$$

$$\zeta_{\Sigma_A, T}(z) = e^{\sum_{n=1}^\infty \frac{\mathtt{Tr}(A^n) z^n}{n}} = e^{\mathtt{Tr}(\sum_{n=1}^\infty \frac{A^n z^n}{n})} = \frac{1}{\det(1 - Az)}.$$

The asymptotics of $\mathtt{Tr}(A^n)$ are given by the

Frobenius-Perron Theorem

Suppose that $P \in M_{d\times d}$ $(:= d\times d$ `matrices`$)$ *is such that* $p_{i,j} \ge 0\ \forall\ i,j$ *and* $\exists\ N \ge 1$ *such that* $p_{i,j}^{(N)} > 0\ \forall\ i,j$.

Let $\lambda_{\max} := \max\{|\lambda| :\ \lambda \in \mathbb{C} :\ \exists\ x \in \mathbb{C}^d,\ Px = \lambda x\}$, *then*

`(A)` $\exists\ x_+ \in \mathbb{R}_+^d,\ Px_+ = \lambda_{\max} x_+$;

`(B)` $\{x \in \mathbb{C}^d :\ Px = \lambda_{\max} x\} = \{cx_+ :\ c \in \mathbb{C}\}$;

`(C)` $\lambda \in \mathbb{C} \smallsetminus \{\lambda_{\max}\},\ x \in \mathbb{C}^d$ *such that* $Px = \lambda x \implies |\lambda| < \lambda_{\max}$ *and* $x \notin \mathbb{R}_+^d$.

Before proceeding with the proof, we recall some basics of linear dynamics:

(1) For $A \in \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^d)$ the *spectral radius* of A is

$$r(A) = \max\{|\lambda| : \exists\ x \in \mathbb{C}^d,\ Ax = \lambda x\}.$$

(2) **Gelfand's formula** For any norm $\|\cdot\|$ on $\mathbb{R}^d$, $\exists\ \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = r(A)$ where $\|A\| := \max\{\|Ax\| : \|x\| = 1\}$, whence $r(A) \le \|A\|$.

(3) $\forall\ A \in \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^d)$ and $\epsilon > 0$, $\exists$ a norm $\|\cdot\|$ on $\mathbb{R}^d$ such that $\|A\| \le r(A) + \epsilon$.

For $\lambda \in \mathbb{C}$ an eigenvalue of $A \in \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^d)$, let

$$E_\lambda := \{x \in \mathbb{R}^d : \exists\ n \ge 1\ (A - \lambda I)^n x = 0\},$$
$$\overline{E}_\lambda := \{x \in \mathbb{C}^d : \exists\ n \ge 1\ (A - \lambda I)^n x = 0\}$$

and

$$\widetilde{E}_\lambda := (\overline{E}_\lambda \oplus \overline{E}_{\overline{\lambda}}) \cap \mathbb{R}^d.$$

Note that $E_\lambda = \{0\}$ if $\lambda \notin \mathbb{R}$, and for $\lambda \in \mathbb{R}$, $\widetilde{E}_\lambda = E_\lambda$.

(4) If $\lambda = \rho e^{i\theta} \notin \mathbb{R}$, then $\dim \widetilde{E}_\lambda = 2$ and there is a basis of $\widetilde{E}_\lambda$ such that

$$A|_{\widetilde{E}_\lambda} = \rho R_\theta := \begin{pmatrix} \rho\cos\theta & \rho\sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{pmatrix},$$

whence $\|A^n x\| \asymp \rho^n\ \forall\ x \in \widetilde{E}_\lambda \setminus \{0\}$.

Proof of the Frobenius-Perron Theorem

Let $\Pi := [0,\infty)^d$. Evidently $P\Pi \to \Pi$. Let $\Sigma := \{x \in \Pi : \|x\|_1 = 1\}$. Evidently, Σ is convex. We claim that

¶0 $P^N \Pi \setminus \{0\} \subset \Pi^o$.

Proof Let $x_j \ge 0\ \forall \jmath$ & $x_{j_0} > 0$, then for any i,

$$(P^N x)_i = \sum_j p_{i,j}^{(N)} x_j \ge p_{i,j_0}^{(N)} x_{j_0} > 0. \quad ☑$$

¶1 $0 \notin P\Sigma$.

Proof By ¶0, $0 \notin P^N\Sigma$. If $N > 1$ and $0 = Py,\ y \in \sigma$, then $0 = P^{N-1}0 = P^N y \ne 0$.⊠

Define $T : \Sigma \to \Sigma$ by $T(x) := \|Px\|^{-1} Px$.

- We claim that

¶2 $T : [x,y] \to [Tx, Ty]$ is continuous $\forall\ x, y \in \Sigma,\ x \ne y$ where $[x,y] := \{tx + (1-t)y : t \in [0,1]\}$ and a homeomorphism iff $Tx \ne Ty$.

- `This property is called` *weak convexity*.

Proof

$$T(tx + (1-t)y) = \|tPx + (1-t)Py\|_1^{-1}(tPx + (1-t)Py) = sTx + (1-s)Ty$$

where

$$s = s(t) := \frac{t\|Px\|_1}{t\|Px\|_1 + (1-t)\|Py\|_1}.$$

Evidently $s : [0,1] \to [0,1]$ is continuous and a homeomorphism iff $Tx \neq Ty$. ☐

By ¶2, if $C \subset \Sigma$ is convex, then so are TC and $T^{-1}C$. We claim that for $C \subset \Sigma$ convex:

¶3 $T(\text{Ext } C) \supseteq \text{Ext } TC$.

Proof

$$T(\int_C x d\mu(x)) = \int TCTx d\nu(x). \quad ☑$$

Let $\Sigma_0 := \bigcap_{n\geq 1} T^n\Sigma$, then $\Sigma_0 \subset \Sigma$ is closed, convex and T-invariant.

¶4 $\Sigma_0 \subset \Sigma^o$.

Proof By ¶0, $P^N : \Pi \setminus \{0\} \to \Pi^o$ and we have $T^N : \Sigma \to \Sigma^o$, whence $\Sigma_0 \subset T^N\Sigma \subset \Sigma^o$. ☐.

¶5 $\#\text{Ext } \Sigma_0 \leq d$.

Proof We have that $\text{Ext } \Sigma = \{e_1, \dots, e_d\}$ where $(e_k)_j = \delta_{k,j}$. $\exists\ n_k \to \infty,\ E_1, \dots, E_d \in \Sigma_0$ so that $T^{n_k}e_j \to E_j\ \forall\ 1 \leq j \leq d$. It follows from $\Sigma_0 \subset T^{n_k}\Sigma$ and weak convexity of T that

$$\forall\ x \in \Sigma_0\ \exists\ p^{(k)} \in \mathcal{P}(\{1, \dots, d\}),\ x = \sum_{j=1}^{d} p_j^{(k)} T^{n_k} e_j,$$

whence for some $k_\ell \to \infty,\ p^{(k_\ell)} \to p \in \mathcal{P}(\{1, \dots, d\})$, $x = \sum_{j=1}^d p_j E_j$ and $x \notin \{E_1, \dots, E_d\} \implies x \notin \text{Ext}\Sigma_0$.

In other words, $\text{Ext}\Sigma_0 \subset \{E_1, \dots, E_d\}$.☑

`Connection with positive eigenvalues.`

Since $\text{Ext}\Sigma_0$ is finite, we have by ¶3 that $T : \text{Ext}\Sigma_0 \to \text{Ext}\Sigma_0$ is bijective. Thus $\forall\ e \in \text{Ext}\Sigma_0\ \exists\ k_e \geq 1$ such that $T^{k_e}e = e$. Multiplying the k_e's,

- $\exists\ \kappa \geq 1$ so that $\forall\ e \in \text{Ext}\Sigma_0\ \exists\ \lambda = \lambda_e > 0$ such that $P^\kappa e = \lambda e$.

¶6 $\#\Sigma_0 = 1$.

Proof If not $\exists\ e \neq f \in \text{Ext}\Sigma_0$ and $\lambda_e, \lambda_f > 0$ such that $P^\kappa e = \lambda_e e,\ P^\kappa f = \lambda_f f$.

In case $\lambda_e = \lambda_f$, choose $a, b \geq 0$ such that $g := ae - bf \in \partial\Sigma$, then $P^\kappa(ae - bf) = \lambda_e(ae - bf)$, whence $\Sigma^o \ni T^{\kappa N}(ae - bf) = (ae - bf) \in \partial\Sigma$ - `contradiction`.⊠

In case $\lambda_e > \lambda_f$, note that $f - \epsilon e \in \Pi \ \forall \ \epsilon > 0$ small enough, whence (fixing such $\epsilon > 0$) $\frac{1}{\lambda_e^n} P^{\kappa n}(f - \epsilon e) \in \Pi \ \forall \ n \geq 1$; but

$$\frac{1}{\lambda_e^n} P^{\kappa n}(f - \epsilon e) = \frac{\lambda_f^n}{\lambda_e^n} f - \epsilon e - \epsilon \lambda_e^n e + o(1) \notin \Pi \text{ for } n \text{ large.} \quad \boxtimes \quad ☑$$

Write $\Sigma_0 = \{\sigma\}$, then $T^n x \to \sigma \ \forall \ x \in \Sigma$ and $T\sigma = \sigma$ whence $P\sigma = \lambda_+ \sigma$ where $\lambda_+ > 0$. This proves `(A)`.

¶7 $\nexists \ x \in \mathbb{R}^d, \ x \neq c\sigma$ (some $c \in \mathbb{R}$) such that $Px = \pm \lambda_+ x$.

Proof Otherwise (similar to the above) $\exists \ a \geq 0, \ b \in \mathbb{R}$ such that $g := a\sigma - bx \in \partial\Sigma$ whence $g = T^{2N} g \in T^{2N}\Sigma \subset \Sigma^o$ - `contradiction.`$\boxtimes$

Statement `(B)` follows from ¶7.

¶8 If $\mu \in \mathbb{R}$ is another e.v. of P, then $|\mu| < \lambda$.

Proof By ¶7, if not, then $|\mu| > \lambda$. Fix $Pe = \mu e$. For $\epsilon > 0$ sufficiently small, $\sigma \pm \epsilon e \in \Pi^o$ whence also $\{P^n(\sigma \pm \epsilon e)\} \subseteq \Pi^o$. However $\{P^n(\sigma \pm \epsilon e)\} = \{\pm \epsilon \mu^n e + o(\mu^n)\} \nsubseteq \Pi^o$ for large n. □

¶9 If $\mu \in \mathbb{C}, \ \mu \neq \lambda_+$ is an e.v. of P, then $|\mu| < \lambda$.

Proof Suppose that $\mu = \rho e^{i\theta} \notin \mathbb{R}$ and let $x \in \widetilde{E}_\mu \setminus \Pi$.
In case $\rho = |\mu| > \lambda$, note that for $\epsilon > 0$ sufficiently small, $\sigma \pm \epsilon x \in \Pi^o$, whence also $P^n(\sigma \pm \epsilon x) \in \Pi^o$. However, $\|P^n x\| \asymp \rho^n$ whence by (4),

$$P^n(\sigma \pm \epsilon x) = \pm \epsilon P^n x + \lambda^n \sigma = \pm \epsilon P^n x(1 + o(1))$$

are not both in Π^o.
In case $\rho = |\mu| = \lambda$, note that for appropriate $a, b \in \mathbb{R}$ and $x \in \widetilde{E}_\mu, \ a\sigma + bx \in \partial\Sigma$, whence as before, $T^n(a\sigma + bx) \to \sigma \in \Sigma^o$. However, $\exists \ n_k \to \infty$ such that $n_k\theta \mod 2\pi \to 0$ (i.e. $R_\theta^{n_k} \to \text{Id.}$), whence $\frac{1}{\lambda^{n_k}} P^{n_k}(a\sigma + bx) \to a\sigma + bx$ and $T^{n_k}(a\sigma + bx) \to a\sigma + bx \in \partial\Sigma$. ☑

Statement `(C)` follows from ¶8 & ¶9. The theorem is established. □

Corollary

$$p(\Sigma_A, T) = \log \lambda_+(A).$$

Proof

$$P_n(\Sigma_A, T) = \mathtt{Tr}(A^n) \propto \lambda_+(A)^n.$$

□

Exercise 9.3.

(i) Show that the TMS (Σ_A, T) is topologically mixing iff $\exists\ N > 1$ so that $A^N_{i,j} > 0\ \forall\ i,\ j \in S$.

(ii) Exhibit a TMS which is topologically transitive but not topologically mixing.

(iii) Show that if (X, S) is a topologically mixing topological dynamical system and (Y, T) is topologically transitive, then $(X \times Y, S \times T)$ is topologically transitive.

TOPOLOGICAL ENTROPY

Given a compact topological space X, and an open cover $\mathfrak{A}$ of X, define

$$\mathcal{N}(\mathfrak{A}) := \min\{|\mathcal{U}| :\ \mathcal{U} \subset \mathfrak{A} \text{ a subcover}\}.$$

The open cover $\mathfrak{A}$ *refines* the open cover $\mathfrak{B}$ (written $\mathfrak{A} \succ \mathfrak{B}$) if $\forall\ A \in \mathfrak{A}$, $\exists\ B \in \mathfrak{B}$ so that $A \subset B$.

Proposition E1 *If* $\mathfrak{B} \prec \mathfrak{A}$, *then* $\mathcal{N}(\mathfrak{B}) \le \mathcal{N}(\mathfrak{A})$.

Proof Suppose that $\mathfrak{A}' \subset \mathfrak{A}$ is a subcover, then since $\mathfrak{B} \prec \mathfrak{A}$, $\exists\ f : \mathfrak{A}' \to \mathfrak{B}$ so that $A \subset f(A)$. Evidently, $f(\mathfrak{A}') \subset \mathfrak{B}$ is a subcover and $|f(\mathfrak{A}')| \le |\mathfrak{A}'|$. ☐

Given open covers $\mathfrak{A}$ and $\mathfrak{B}$ let $\mathfrak{A} \vee \mathfrak{B} := \{A \cap B :\ A \in \mathfrak{A},\ B \in \mathfrak{B}\}$.

Evidently,

$$\mathcal{N}(\mathfrak{A} \vee \mathfrak{B}) \le \mathcal{N}(\mathfrak{A})\mathcal{N}(\mathfrak{B}). \tag{0}$$

Now let $T : X \to X$ be continuous.

For an open cover $\mathfrak{A}$ of X, set

$$a(n) := \log \mathcal{N}(\mathfrak{A}_0^{n-1}) \texttt{ where } \mathfrak{A}_0^{n-1} = \mathfrak{A}_0^{n-1}(T) := \bigvee_{k=0}^{n-1} T^{-k}\mathfrak{A}.$$

By (0), $a(m+n) \le a(m) + a(n)$ whence (!) $\frac{a(n)}{n} \xrightarrow[n\to\infty]{} \inf_\ell \frac{a(\ell}{\ell}$ and

$$\exists \lim_{n\to\infty} \tfrac{1}{n} \log \mathcal{N}(\mathfrak{A}_0^{n-1}(T)) =: h(T, \mathfrak{A}).$$

By proposition E1, if $\mathfrak{B} \prec \mathfrak{A}$, then $h(T, \mathfrak{B}) \le h(T, \mathfrak{A})$.

- $h(T, \mathfrak{A}_0^{K-1}) = h(T, \mathfrak{A})\ \forall\ K \ge 1$.

Proof

$$(\mathfrak{A}_0^{K-1})_0^{K-1} = \mathfrak{A}_0^{n+K-1}. \quad \square$$

Exercise 9.4. Show that if $T : X \to X$ is a homeomorphism, then $h(T, \mathfrak{A}_J^K) = h(T, \mathfrak{A})\ \forall\ \mathfrak{A}$ open cover, $J, K \in \mathbb{Z},\ J < K$ where $\mathfrak{A}_J^K := \bigvee_{\ell=J}^K T^{-\ell}\mathfrak{A}$.

Define the *topological entropy* of T by

$$h(T) := \sup_{\mathfrak{A}} h(T, \mathfrak{A}).$$

Proposition E2

If (Y, S) is a factor of (X, T) then

$$h(T) \ge h(S).$$

Proof

Suppose that $\pi : X \to Y$ is onto, continuous and $\pi \circ T = S \circ \pi$. If $\mathfrak{A}$ is an open cover of Y, then $\pi^{-1}\mathfrak{A}$ is an open cover of X and $\mathcal{N}(\pi^{-1}\mathfrak{A}) = \mathcal{N}(\mathfrak{A})$. Also $\pi^{-1}\bigvee_{k=0}^{K-1} S^{-k}\mathfrak{A} = \bigvee_{k=0}^{K-1} T^{-k}\pi^{-1}\mathfrak{A}$, whence

$$h(S, \mathfrak{A}) = h(T, \pi^{-1}\mathfrak{A}) \le h(T)$$

and $h(S) = \sup_{\mathfrak{A}} h(S, \mathfrak{A}) \le h(T)$. □

Calculation of $h(T)$ for T a subshift.

Let S be a finite set, let $X \subset S^{\mathbb{Z}}$ be a subshift and let T be the shift on X.

Consider the open cover $\alpha := \{[s]_0 \cap X :\ s \in S\}$.

¶1 $h(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} \log |\alpha_0^{n-1}|$.

Proof Since α_0^{n-1} is a partition of X, there are no (nontrivial) subcovers and $\mathcal{N}(\alpha_0^{n-1}) = |\alpha_0^{n-1}|$. ☑

For $n \ge 1$, consider

$$\alpha_{-n}^n = \bigvee_{k=-n}^{n} T^{-k}\alpha = \{[s_{-n}, \dots, s_n]_{-n} \cap X :\ s_{-n}, \dots, s_n \in S\}.$$

¶2 If $\mathfrak{B}$ is another open cover, then $\exists\ N \ge 1$ such that each $\alpha_{-N}^N \succ \mathfrak{B}$.

Proof Define $t(x, y) := \min\{|n| :\ x_n \ne y_n\} \le \infty$ (eq. iff $x = y$) and $d(x, y) := (\frac{1}{2})^{t(x,y)}$ then d is a metric generating the topology on X with $B_0(x, \frac{1}{2^{n+1}}) = [x_{-n}, \dots, x_n]_{-n}$.

Since $\mathfrak{B}$ is an open cover, $\forall\ x \in X,\ \exists\ B \in \mathfrak{B},\ N_x \ge 1$ such that $C_x := [x_{-N_x}, \dots, x_{N_x}]_{-N_x} \subset B$. The collection $\{C_x :\ x \in X\}$ is an open cover of X and by compactness $\exists\ F \subset X$ finite such that $X = \bigcup_{x\in F} C_x$. Let $N := \max_{x\in F} N_x$, then $\mathfrak{B} \prec \{C_x :\ x \in F\} \prec \alpha_{-N}^N$. ☑

¶3 $h(T) = h(T, \alpha)$.

Proof For any open cover $\mathfrak{B}$, by ¶2 $\exists\ N \geq 1$ such that $\alpha_{-N}^N \succ \mathfrak{B}$ whence

$$h(T,\mathfrak{B}) \leq h(T,\alpha_{-N}^N) \overset{\texttt{exercise}}{=} h(T,\alpha).\ \ V$$

Proposition E3

If $X = \Sigma_A$ is a topological Markov shift with transition matrix $A : S \times S \to \{0,1\}$, s.t. $\exists\ N \geq 1,\ A_{s,t}^N > 0\ \forall\ s,\ t \in S$; then

$$h(T) = \log \lambda_+(A) = P(T).$$

Proof By ¶1 & ¶3,

$$h(T) = h(T,\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \#\alpha_0^{n-1}.$$

By the Perron-Frobenius theorem,

$$\#\alpha_0^{n-1} = \sum_{s,t\in S} A_{s,t}^n \propto \lambda_+(A)^n,$$

whence

$$\frac{1}{n} \log \#\alpha_0^{n-1} \xrightarrow[n\to\infty]{} \log \lambda_+(A).\ \ ☑$$

Exercise 9.5: Conjugacy of TMS's.

Let S & S' be finite sets and let $\Sigma_A \subset S^{\mathbb{Z}}$ & $\Sigma_B \subset S'^{\mathbb{Z}}$ (where $A : S \times S \to \{0,1\},\ B : S' \times S' \to \{0,1\}$) be mixing TMS's and let T denote the shift map.

(a) Show that if (Σ_A, T) & (Σ_B, T) are topologically conjugate, then $\lambda_+(A) = \lambda_+(B)$.

(b) Show that for $k,\ \ell \geq 2,\ k \neq \ell$ that $(\{1,2,\dots,k\}^{\mathbb{Z}}, T)$ and $(\{1,2,\dots,\ell\}^{\mathbb{Z}}, T)$ are not topologically conjugate to (X_ℓ, T) for $k \neq \ell$.

Exercise 9.6: Frobenius theory of positive matrices ctd.

Suppose that $A \in M_{d\times d}$ ($:= d \times d$ matrices) is such that $A_{i,j} \geq 0\ \forall\ i,j$ and $\exists\ N \geq 1$ such that $A_{i,j}^{(N)} > 0\ \forall\ i,j$. Let $\lambda_{\max}(A)$ be the maximal eigenvalue of A.

(a) Show that $\lambda_{\max}(A^t) = \lambda_{\max}(A) =: \lambda_+$ where $A_{i,j}^t := A_{j,i}$.

Let $x,\ y \in \mathbb{R}_+^d$ be the positive eigenvectors $Ax = \lambda_+ x$ & $A^t y = \lambda_+ y$. Define $P \in M_{d\times d}$ by

$$p_{i,j} := \frac{A_{i,j} y_j}{\lambda_+ y_i}.$$

(b) Show that P is a *stochastic* matrix in the sense that $p_{i,j} \geq 0 \ \forall \ 1 \leq i,j \leq d$ and $\sum_{j=1}^{d} p_{i,j} = 1 \ \forall \ 1 \leq i \leq d$.

(c) Show that $\exists \ \pi \in \mathcal{P}(\{1,\ldots,d\})$ so that $\sum_{i=1}^{d} \pi_i p_{i,j} = \pi_j \ \forall \ 1 \leq j \leq d$.
`Hint` Normalize $x_i y_i$.

The probability vector π is aka the *invariant distribution* of P.

Exercise 9.7: Stochastic matrices.

Suppose that S is a finite set and $P : S \times S \to \mathbb{R}$ is a *stochastic* matrix in the sense that $p_{i,j} \geq 0 \ \forall \ i,j \in S$ and $\sum_{j\in S} p_{i,j} = 1 \ \forall \ i \in S$; and suppose that $\exists \ q \geq 1$ such that $p_{i,j}^{(q)} > 0 \ \forall \ i,j \in S$.

(a) Prove that $\exists \ 0 < \theta < 1 < M$ such that

$$|p_{i,j}^{(n)} - \pi_j| \leq M\theta^n \ \forall \ n \geq 1, \ \forall \ i,j \in S$$

where $\pi \in \mathcal{P}(S)$ is the invariant distribution of P.

b) Show that $\exists \ \mu \in \mathcal{P}(S^{\mathbb{Z}})$ such that

$$\mu([s_0, s_1, \ldots, s_N]) = \pi_{s_0} p_{s_0,s_1} \ldots p_{s_{N-1},s_N} \ \forall \ s_0, s_1, \ldots, s_N \in S.$$

The *closed support* of μ is $\text{Supp}\,(\mu) := \{x \in S^{\mathbb{Z}} : \ \mu(U) > 0 \ \forall \ x \in U \in \mathcal{T}\}$ (where $\mathcal{T}$ denotes the open sets in $S^{\mathbb{Z}}$).

c) Show that $\text{Supp}\,(\mu) = \Sigma_A$ where $A : S \times S \to \{0,1\}$ is defined by $A(s,t) = 1$ if $P(s,t) > 0$ and $A(s,t) = P(s,t) = 0$ otherwise.

d) Show that (Σ_A, T, μ) is a mixing probability preserving transformation.

d-ENTROPY

Separated sets.

Let Y be a set, and let ρ be a metric on Y. Recall that $F \subset Y$ is (ρ,ϵ)-*separated* if $\rho(x,y) \geq \epsilon \ \forall \ x,y \in F, \ x \neq y$; and that F is (ρ,ϵ)-*dense* in Y if $\forall \ y \in Y, \ \exists x \in F$ such that $\rho(x,y) < \epsilon$. Using Zorn's lemma it can be shown that $\exists$ maximal ϵ-separated sets.

Define

$$S(\rho,\epsilon) := \max\{|F| : \ F \subset Y \ (\rho,\epsilon) - \text{separated}\},$$

$$D(\rho,\epsilon) := \min\{|F| : \ F \subset Y \ (\rho,\epsilon) - \text{dense in } Y\},$$

and

$$N(\rho,\epsilon) :=$$

$$\min\{N \ge 1 :\ Y = \bigcup_{k=1}^{N} A_k,\ A_j \subset Y,\ \rho - \mathtt{diam}\,(A_j) < \epsilon\ \forall j\}.$$

Proposition E4

(i) $$D(\rho,\epsilon) \le S(\rho,\epsilon) \le D(\rho,\epsilon/2).$$

(ii) $$D(\rho,\epsilon) \le N(\rho,\epsilon) \le D(\rho,\epsilon/2).$$

Proof

(i) $S(\rho,\epsilon) \ge D(\rho,\epsilon)$ since a maximal (ρ,ϵ)-separated set is (ρ,ϵ)-dense.

To see $S(\rho,\epsilon) \le D(\rho,\epsilon/2)$ let F be (ρ,ϵ)-separated and let G be $(\rho,\epsilon/2)$-dense. $\exists\ f : F \to G$ such that $d(x, f(x)) < \epsilon/2\ \ \forall\ x \in F$. It follows that f is injective, since $f(x_1) = f(x_2) = y \implies d(x_1,x_2) \le d(x_1,y) + d(y,x_2) < \epsilon \implies x_1 = x_2$. Thus $|F| \le |G|$ whence $S(\rho,\epsilon) \le D(\rho,\epsilon/2)$.

(ii) Suppose that $Y = \bigcup_{k=1}^{N} A_k$ where $\rho - \mathrm{diam}\,(A_j) \le \epsilon$ and choose $x_i \in A_i\ \ (1 \le i \le N)$. Evidently $\{x_i :\ 1 \le i \le N\}$ is (ρ,ϵ)-dense whence $D(\rho,\epsilon) \le N(\rho,\epsilon)$. Now let F be $(\rho,\epsilon/2)$-dense, then $X = \bigcup_{y\in F} B(y,\epsilon/2)$ and $\rho - \mathrm{diam}\,(B(y,\epsilon/2)) \le \epsilon\ \forall y \in F$ thus $N(\rho,\epsilon) \le |F|$. ☐

Minkowski-Besicovitch Box dimension.

The *box dimension* of Y with respect to ρ is

$$\dim_b(Y,\rho) := \overline{\lim_{\epsilon\to 0}}\, \frac{\log D(\rho,\epsilon)}{\log 1/\epsilon}.$$

Exercise 9.8: `Box dimension.`

Show that:

a) if $X \subset \mathbb{R}^\kappa,\ X = \overline{X^o}$ and d is the Euclidean metric, then $\dim_b(X,d) = \kappa$;

b) $\dim_b(X, d^{(r)}) = \frac{\log 2}{\log 1/r}$ where $X := \{0,1\}^{\mathbb{N}},\ 0 < r < 1$ and $d^{(r)}(x,y) := r^{t(x,y)}$;

c) $\dim_b(C,d) = \frac{\log 2}{\log 3}$ where $C \subset [0,1]$ is the classical "middle third" Cantor set where $d(x,y) = |x - y|$.

d-entropy and separated sets.

For (X,d) a compact metric space, $T : X \to X$ a continuous map, define the sequence of dynamical metrics

$$d_n(x,y) = \max_{0\le k\le n-1} d(T^k x, T^k y).$$

Note that $D(d_n,\epsilon)$ is the minimum number of "initial conditions" which ensure ϵ-approximation up to time n of the dynamical system (under any initial condition).

The d- *entropy* of (X,T) is

$$h_d(T) := \lim_{\epsilon\to 0} \overline{\lim_{n\to\infty}} \tfrac{\log D(d_n,\epsilon)}{n}.$$

The d-entropy can be thought of as measuring the "degree of sensitivity of the dynamical system's dependence on initial conditions" (one of the components of so-called "chaos").

`Example.` Let $(X,T) = (\{0,1\}^{\mathbb{N}}, \text{shift})$ and define $d = d^{(r)}(x,y) := r^{t(x,y)}$ where $0 < r < 1$ and $\min\{n \ge 1 : x_n \ne y_n\} \le \infty$, then (X, d_r) is a compact metric space, the metrics $d^{(r)}$ $(0 < r < 1)$ are equivalent and (fixing $0 < r < 1$)

- $d_n(x,y) = \min\{\frac{d(x,y)}{r^{n-1}}, r\}$

Proof

$$d_n(x,y) = r^{\min_{0\le k\le n-1} t(T^k x, T^k y)} = r^{(t(x,y)-n+1)\vee 1} = \tfrac{d(x,y)}{r^{n-1}} \wedge r. \quad \square$$

- For $\epsilon \in [r^{K+1}, r^K)$,

$$B^{(d_n)}(x,\epsilon) = \{y \in X : \tfrac{d(x,y)}{r^{n-1}} \wedge r \le \epsilon\} = B^{(d)}(x, r^{n+K}) = [x_1,\ldots,x_{n+K-1}].$$

- For $\epsilon \in [r^{K+1}, r^K)$, $D(d_n,\epsilon) = 2^n$.
- $h_d(T) = \log 2.$

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Lemma E5

$$\exists \lim_{n\to\infty} \tfrac{\log N(d_n,\epsilon)}{n} \ \forall \ \epsilon > 0.$$

Proof

This is based on the (easy) observation that for $k, \ell \geq 1$,

$d_k - \text{diam}\,(A) < \epsilon,\ d_\ell - \text{diam}\,(B) < \epsilon \implies d_{k+\ell} - \text{diam}\,(A \cap T^{-k}B) < \epsilon.$

Thus $N(d_{k+\ell}, \epsilon) \leq N(d_k, \epsilon)N(d_\ell, \epsilon)$ and by subadditivity

$$\tfrac{\log N(d_n,\epsilon)}{n} \to \inf_{j\geq 1} \tfrac{\log N(d_j,\epsilon)}{j}.$$

□

Corollary E6

$$\begin{aligned} h_d(T) &= \lim_{\epsilon\to 0} \lim_{n\to\infty} \tfrac{\log N(d_n,\epsilon)}{n} \\ &= \lim_{\epsilon\to 0} \underline{\lim}_{n\to\infty} \tfrac{\log D(d_n,\epsilon)}{n} = \lim_{\epsilon\to 0} \overline{\lim}_{n\to\infty} \tfrac{\log D(d_n,\epsilon)}{n} \\ &= \lim_{\epsilon\to 0} \underline{\lim}_{n\to\infty} \tfrac{\log S(d_n,\epsilon)}{n} = \lim_{\epsilon\to 0} \overline{\lim}_{n\to\infty} \tfrac{\log S(d_n,\epsilon)}{n}. \end{aligned}$$

Proposition E7

If d' is another metric on X equivalent to d, then $h_{d'}(T) = h_d(T)$.

Proof

$\forall\ \epsilon > 0,\ \exists\ \delta(\epsilon) > 0$ such that $d'(x,y) < \delta(\epsilon) \implies d(x,y) < \epsilon$. It follows that for $n \geq 1$, $d'_n(x,y) < \delta(\epsilon) \implies d_n(x,y) < \epsilon$. Thus if $F \subset X$, then
F (d_n, ϵ)-separated $\implies$ F $(d'_n, \delta(\epsilon))$-separated,
whence

$$S(d_n, \epsilon) \leq S(d', n, \delta(\epsilon)),\ \&\ h_d(T) \leq h_{d'}(T).$$

□

Before proving that $h_d(T) = h(T)$, we need the concept of "`Lebesgue number`".

Definition Given an open cover Λ of a set subset K of a metric space (X, d), the *Lebesgue number* of Λ with respect to K is

$$\epsilon(\Lambda, K) := \sup\{\epsilon \geq 0 :\ \forall\ x \in K,\ \exists\ U \in \Lambda \ \text{ such that } \ B_0(x, \epsilon) \subset U\}.$$

Lebesgue's lemma says that if K is compact, then $\epsilon(\Lambda, K) > 0$.

Lebesgue's Lemma *Suppose that X is a metric space, and that $K \subset X$ is compact. If Λ is an open cover of K, then $\exists\ \epsilon = \epsilon(\Lambda, K) > 0$ such that*

$$\forall\ x \in K,\ \exists\ U \in \Lambda \ \ \textit{such that}\ \ B(x, \epsilon) \subset U.$$

Proof If not, then

$$\forall\ \epsilon > 0\ \exists\ x(\epsilon)\ \ \text{such that}\ \ B(x(\epsilon), \epsilon)) \not\subseteq U\ \ \forall\ U \in \Lambda.$$

In particular, $\exists\ x_n \in K$ and $\epsilon_n \to 0$ such that

$$B(x_n, \epsilon_n) \not\subseteq U\ \ \forall\ U \in \Lambda.$$

Passing to a subsequence, $\exists\ y \in K$ such that $x_n \to y$ and $\exists\ V \in \Lambda$ such that $y \in V$. Since V is open, $\exists\ d > 0$ such that $B(y, \delta) \subset V$. For $n \geq 1$ large enough, $\epsilon_n,\ d(y, x_n) < \delta/2$

$$z \in B(x_n, \delta/2)\ \Rightarrow\ d(y, z) \leq d(y, x_n) + d(x_n, z) < \delta$$

and $B(x_n, \epsilon_n) \subset B(x_n, \delta/2) \subset B(y, \delta) \subset V$ contradicting $B(x_n, \epsilon_n) \not\subseteq U\ \forall\ U \in \Lambda$.

☐

Theorem E8

$$h_d(T) = h(T).$$

Proof

$\leq$) If $\sup_{A \in \mathfrak{A}} \operatorname{diam}(A) \leq \epsilon$, then $d_n - \operatorname{diam.}(a) \leq \epsilon\ \forall\ a \in \mathfrak{A}_0^{n-1}$, whence $\mathcal{N}(\mathfrak{A}_0^{n-1}) \geq N(d_n, \epsilon)$. Thus, $h_d(T) \leq h(T)$. ☑

$\geq$) Let $\mathfrak{A}$ be an open cover of X, and suppose that $\eta > 0$ is smaller than its Lebesgue number (i.e. $\forall\ x \in X\ \exists\ A \in \mathfrak{A},\ B(x, \eta) \subset A$), then (!) $\forall\ x \in X\ \exists\ a \in \bigvee_{k=0}^{n-1} T^{-k}\mathfrak{A}$ such that $B_{d_n}(x, \eta) \subset a$.

Thus, for F (d_n, η)-dense, $\exists\ f : F \to \mathfrak{A}_0^{n-1}$ such that $B_{d_n}(x, \eta) \subset f(x)$, whence $f(F) \subset \mathfrak{A}_0^{n-1}$ is a subcover with $|f(F) \leq |F|$. This shows that $\mathcal{N}(\mathfrak{A}_0^{n-1}) \leq D(d_n, \eta)$, whence $h_d(T) \geq h(T)$. ☑

Exercise 10.1.

Let (X, T) be a continuous map of a compact metric space. Show that

(i) If $Y \subset X$ is closed and T-invariant, then $h(T|_Y) \leq h(T)$.

(ii) If $X = \bigcup_{i=1}^{L} Y_i$ where each Y_i is closed and T-invariant, then

$$h(T) = \max_{1 \leq i \leq L} h(T|_{Y_i}).$$

(iii) $h(T^n) = nh(T)\ \forall\ n \in \mathbb{N}$ and $h(T^{-1}) = h(T)$ if T is a homeomorphism.

(iv) $h(T \times S) = h(T) + h(S)$ whenever (Y, S) is also a continuous map of a compact metric space.

(iv) Is there a topological dynamical system (X, T) with $h(T) > 0$ but $\Pi(T) = \varnothing$?

More calculations of $h(T)$

¶1 If $T : X \to X$ is an isometry then $h(T) = 0$. To see this note that $d_n^{(T)} \equiv d$ and $D(d_n, \epsilon) \nrightarrow \infty$.

¶2 Lipschitz maps. The *box dimension* of the metric space (X, d) is

$$\dim_b(X) := \overline{\lim_{\epsilon \to 0}} \tfrac{\log D(d,\epsilon)}{\log 1/\epsilon}.$$

Lemma E9

Let (X, T) be a Lipschitz continuous map of a compact metric space, then

$$h(T) \le \dim_b(X) \max\{0, \log D_T\}$$

where $D_T := \sup_{x,y \in X} \frac{d(Tx,Ty)}{d(x,y)}$.

Proof

Let $L > 1 \vee D_T$, then given $\epsilon > 0,\ n \ge 1$,

$$\begin{aligned} &d(x, y) \le L^{-n}\epsilon \implies \\ &d(T^k x, T^k y) \le L^{k-n}\epsilon \le \epsilon\ \forall\ 0 \le k \le n \implies \\ &d_n(x, y) \le \epsilon, \end{aligned}$$

whence $S^{(T)}(d_n, \epsilon) \le S(d, L^{-n}\epsilon)$ and

$$\begin{aligned} \overline{\lim_{n\to\infty}} \tfrac{\log S^{(T)}(d_n,\epsilon)}{n} &\le \overline{\lim_{n\to\infty}} \tfrac{\log S(d,L^{-n}\epsilon)}{n} \le \overline{\lim_{n\to\infty}} \tfrac{\log D(d,L^{-n}\epsilon/2)}{n} \\ &\le \overline{\lim_{\delta\to 0}} \tfrac{\log D(d,\delta)}{\log 1/\delta} \log L \\ &\dim_b(X) \log L. \qquad \square \end{aligned}$$

¶3 Anzai skew products.

Consider $T = T_{\alpha,\psi} : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $T(x,y) = (x+\alpha, y+\psi(x))$ where $\psi : \mathbb{T} \to \mathbb{T}$ is C^1 (i.e. $\exists\ \Psi : \mathbb{R} \to \mathbb{R},\ C^1$ such that $\Psi(x) = \psi(x \mod 1) \mod 1$). We have that $DT(x) = \begin{pmatrix} 1 & 0 \\ \psi'(x) & 1 \end{pmatrix}$, whence (!) $\|DT(x)\| = O(|\psi'(x)|)$ and $D_T = O(\sup_x |\psi'(x)|)$.

Fixing ψ, we have that (for $n \ge 1$) $T^n(x,y) = (x+n\alpha, y+\psi_n(x))$ where $\psi_n(x) := \sum_{k=0}^{n-1} \psi(x+k\alpha))$, whence $D_{T^n} = O(n)$ as $n \to \infty$.

By lemma E9 and exercise 10.2 (iii), we have that

$$h(T) = \tfrac{h(T^n)}{n} = O(\tfrac{\log n}{n}) \text{ as } n \to \infty$$

whence $h(T) = 0$. ☐

¶4 "Hyperbolic" endomorphisms of $\mathbb{T}^2$.

Let $A \in \mathbb{G}_2(\mathbb{Z}) := \{A \in \mathrm{Gl}(2,\mathbb{R}) :\ a_{i,j} \in \mathbb{Z}\}$ with eigenvalues $\lambda_1,\ \lambda_2 \in \mathbb{R}$, $|\lambda_1| > |\lambda_2|$ and let $T = T_A : \mathbb{T}^2 \to \mathbb{T}^2,\ T_A(x+\mathbb{Z}^2) := Ax + \mathbb{Z}^2$. We show that

☺ $\quad h(T) = \sum_{i=1,2,\ |\lambda_i|>1} \log|\lambda_i|$.

Proof Set $\mu_i := |\lambda_i| \vee 1$, then $\mu_1 = |\lambda_1| > 1 \because |\det A| = |\lambda_1|\cdot|\lambda_2| \ge 1$.

It suffices to show that

☻ $\quad h(T) = \sum_{i=1,2} \log \mu_i$.

Let $u_i \in \mathbb{R}^2,\ Au_i = \lambda_i\ \ (i = 1,2)$ and consider $\mathbb{T}^2$ equipped with the metric d induced by the norm $\|a_1 u_1 + a_2 u_2\| := |a_1| \vee |a_2|$. Evidently for $x \in \mathbb{T}^2,\ n \ge 0,\ h,k \in \mathbb{R}$ small,

$$T^k(x + hu_1 + ku_2) = T^n x + h\lambda_1^n u_1 + k\lambda_2^n u_2 \quad (0 \le k < n)$$

whence

$$d_n(x, x+hu_1+ku_2) := \max_{0\le k<n} d(T^k x, T^k(x+hu_1+ku_2)) = \mu_1^n|h| \vee \mu_2^n|k|.$$

Thus

$$B^{(d_n)}(x,\epsilon) = \{x + hu_1 + ku_2 :\ |h| \le \tfrac{\epsilon}{\mu_1^n},\ |k| \le \tfrac{\epsilon}{\mu_2^n}\}.$$

If $F \subset \mathbb{T}^2$ is (d_n,ϵ)-dense, then $\mathbb{T}^2 \subset \bigcup_{x\in F} 1_{B^{(d_n)}(x,\epsilon)}$ whence

$$1 = m(\mathbb{T}^2) \le \sum_{x\in F} m(B^{(d_n)}(x,\epsilon)) = \tfrac{|F|\epsilon^2 \sin\theta}{\mu_1^n \mu_2^n}$$

where $\theta = \angle(0,u_1,u_2)$. Thus $D(d_n,\epsilon) \ge \frac{\mu_1^n\mu_2^n}{\epsilon^2 \sin\theta}$ and $h(T) \ge \log\mu_1 + \log\mu_2$.

To show $D(d_n,\epsilon) \ll \mu_1^n\mu_2^n$ (for fixed $\epsilon > 0$) choose $\Gamma \subset \mathbb{R}^2$ countable so that $\{\gamma + B^{(d_n)}(0,\epsilon) = B^{(d_n)}(\gamma,\epsilon) :\ \gamma \in \Gamma\}$ *tiles* $\mathbb{R}^2$ in the sense that

$$\bigcup_{\gamma\in\Gamma} B^{(d_n)}(\gamma,\epsilon) = \mathbb{R}^2 \ \ \&\ \ m(B_0^{(d_n)}(\gamma,\epsilon) \cap B_0^{(d_n)}(\gamma',\epsilon)) = 0 \quad (\gamma \ne \gamma' \in \Gamma).$$

Let $\Gamma_0 := \{\gamma \in \Gamma : \ B_0^{(d_n)}(\gamma, \epsilon) \cap [0,1]^2 \neq \varnothing$, then $F := \{\gamma + \mathbb{Z}^2 : \ \gamma \in \Gamma_0\}$ is (d_n, ϵ)-dense and $D(d_n, \epsilon) \le |F| \le |\Gamma_0|$. To estimate $|\Gamma_0|$, note that

$$B^{(d_n)}(\gamma, \epsilon) \subset [-\epsilon, 1+\epsilon]^2 \ \forall \ \gamma \in \Gamma_0$$

whence

$$|\Gamma_0| \le \tfrac{m([-\epsilon,1+\epsilon]^2)}{m(B^{(d_n)}(0,\epsilon)}$$

and (☻) follows. ☑

General endomorphisms of $\mathbb{T}^d$

Theorem 10.1 *Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a surjective endomorphism with $T(x + \mathbb{Z}^d) = A(x) + \mathbb{Z}^d$ with $A \in G_d(\mathbb{Z}) := \{A \in Gl(d, \mathbb{R}) : \ a_{i,j} \in \mathbb{Z}\}$. Let $\{\lambda_i : \ 1 \le i \le d\} \subset \mathbb{C}$ be the eigenvalues of A (counting multiplicity), then*

$$h(T) = \sum_{1 \le j \le d, \ |\lambda_j| > 1} \log |\lambda_j|.$$

Non-compact metric spaces.

Let (Y, ρ) be a metric space and let $K \subset Y$ be compact.

• $F \subset K$ is (ρ, ϵ)-*separated* if $\rho(x, y) \ge \epsilon \ \forall \ x, y \in F, \ x \neq y$; and that F is (ρ, ϵ)-*dense* in K if $\forall \ y \in K, \ \exists x \in F$ such that $\rho(x, y) < \epsilon$.

Define

$$S(K, \rho, \epsilon) := \max\{|F| : \ F \subset K \ (\rho, \epsilon) - \text{separated}\},$$

$$D(K, \rho, \epsilon) := \min\{|F| : \ F \subset Y \ (\rho, \epsilon) - \text{dense in } K\},$$

and

$$N(K, \rho, \epsilon) :=$$
$$\min\{N \ge 1 : \ K \subseteq \bigcup_{k=1}^{N} A_k, \ A_j \subset Y, \ \rho - \mathtt{diam}\,(A_j) < \epsilon \ \forall j\}.$$

Proposition 10.2

(i) $$D(K, \rho, \epsilon) \le S(K, \rho, \epsilon) \le D(K, \rho, \epsilon/2).$$

(ii) $$D(K, \rho, \epsilon) \le N(K, \rho, \epsilon) \le D(K, \rho, \epsilon/2).$$

Proof See propn. E4

d-entropy on non-compact spaces.

For (X,d) a metric space, $T:X\to X$ continuous define as before,

$$d_n(x,y)=\max_{0\le k\le n-1} d(T^kx,T^ky).$$

.

For $K\subset X$ compact, the d- *entropy of* (X,T) *on* K is

$$h_d(T,K):=\lim_{\epsilon\to 0}\overline{\lim_{n\to\infty}}\tfrac{\log D(d_n,\epsilon)}{n}.$$

The *d-entropy of* (X,T) is

$$h_d(T):=\sup\{h_d(T,K):\ K\subset X \text{ compact}\}.$$

Call a metric dynamical system (X,d,T) *uniformly continuous* (UCMDS) if T is uniformly continuous w.r.t. d (abbr. $T\in \mathtt{UC}(X,d)$. Note that if d' is a metric on X *uniformly equivalent* to d ($d\overset{\text{unif.}}{\cong}d'$) in the sense that

$$\mathtt{Id}:(X,d)\to(X,d')\ \&\ \mathtt{Id}:(X,d')\to(X,d)$$

are both uniformly continuous, then (X,d,T) is also a UCMDS.

10.2 Equivalence proposition *Let* (X,d,T) *be a* UCMDS *and let* $d\overset{\text{unif.}}{\cong}d'$, *then*

$$h_d(T)=h_{d'}(T).$$

Proof sketch It follows that $\forall\ \epsilon>0\ \exists\ 0<\delta(\epsilon)<\epsilon$ so that

$$d(x,y)<\delta(\epsilon)\implies d'(x,y)<\epsilon\ \ \&\ d'(x,y)<\delta(\epsilon)\implies d(x,y)<\epsilon.$$

It follows that for each $n\ge 1$,

$$d_n(x,y)<\delta(\epsilon)\implies d'_n(x,y)<\epsilon\ \ \&\ d'_n(x,y)<\delta(\epsilon)\implies d_n(x,y)<\epsilon$$

whence for each $K\subset X$ compact,

$$S(K,d_n,\epsilon)\le S(K,d'_n,\delta(\epsilon))\ \&\ \ S(K,d'_n,\epsilon)\le S(K,d_n,\delta(\epsilon)).\quad\dots\quad ☑$$

10.3 Localization proposition *Let* (X,d,T) *be a* UCMDS *and let* $K,\ K_1,\dots,K_N\subset X$ *be compact. If* $K\subset\bigcup_{j=1}^N K_j$, *then*

$$h_d(T,K)\le\max_{1\le j\le N}h_d(T,K_j).$$

Proof sketch For each $\epsilon>0\ \&\ n\ge 1$,

$$S(K,d_n,\epsilon)\le\sum_{j=1}^N S(K_j,d_n,\epsilon)\le N\max_{1\le j\le N}S(K_j,d_n,\epsilon).$$

$\exists\ n_t\to\infty$ so that

- $\frac{1}{n_t}\log S(K,d_{n_t},\epsilon) \xrightarrow[t\to\infty]{} \overline{\lim}_{n\to\infty}\frac{1}{n}\log S(K,d_n,\epsilon)$;
- $\exists\ J$ so that $\max_{1\le j\le N} S(K_j,d_{n_t},\epsilon) = S(K_J,d_{n_t},\epsilon)\ \forall\ t$.

It follows that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\frac{1}{n}\log S(K,d_n,\epsilon) &\xleftarrow[t\to\infty]{} \frac{1}{n_t}\log S(K,d_{n_t},\epsilon)\\ &\lesssim \frac{1}{n_t}\log S(K_J,d_{n_t},\epsilon)\\ &\lesssim \overline{\lim_{n\to\infty}}\frac{1}{n}\log S(KKJ,d_n,\epsilon).\end{aligned}$$

Thus $h_d(T,K)\le \max_{1\le j\le N} h_d(T,K_j)$. ☑

10.4 Small set corollary *For a* `UCMDS` (X,d,T)*, for each* $\epsilon>0$*,*

$$h_d(T)=\sup\{h_d(T,K):\ K\subset X\ \textit{compact}\ \&\ \textit{diam}\,K<\epsilon\}.$$

10.5 Entropy lifting proposition

Let (Z,ρ,R) *&* (X,d,T) *be a* `UCMDS`*s and suppose that* $\pi:Z\to X$ *is continuous, surjective & a* `uniform, local isometry`[8].

If $\pi\circ R=T\circ\pi$*, then*

$$h_d(T)=h_\rho(R).$$

Proof sketch Fix $0<\epsilon<\Delta$ so that $\rho(x,y)<\epsilon \implies \rho(Rx,Ry)<\Delta$ and $d(x,y)<\epsilon \implies d(Tx,Ty)<\Delta$.

Let $K\subset B_\rho(x,\Delta)$ be compact and let $F\subset K$ be (ρ_n,ϵ)-separated. It follows that $\pi(F)\subset\pi(K)$ is (d_n,ϵ)-separated since for $x\ne y\in F$, $\exists\ 0\le k<n$ so that $\epsilon\le\rho(R^kx,R^ky)<\Delta$ whence $d(T^k\pi(x),T^k\pi(y))=\rho(R^kx,R^ky)\ge\epsilon$. Thus $S(K,\rho_n,\epsilon)\le S(\pi(K),d_n,\epsilon)$.

The reverse inequality follows analogously, so $S(K,\rho_n,\epsilon)=S(\pi(K),d_n,\epsilon)$ ....☑

10.6 Lemma *Let* $T:\mathbb{R}^d\to\mathbb{R}^d$ *be a linear mapping. Suppose that* $\rho(x,y)=\|x-y\|$ *where* $\|\cdot\|$ *is a norm on* $\mathbb{R}^d$*. Let* m *be Lebesgue measure, then*

$$h_\rho(T)=\mathfrak{H}:=\overline{\lim_{n\to\infty}}\frac{1}{n}\log\frac{1}{m(B_{d_n}(0,1))}.$$

Proof sketch

Proof that $h_d(T,K)\ge\mathfrak{H}$ whenever $m(K)>0$:

[8] in the sense that $\exists\ \Delta>0$ so that $\pi:B_\rho(x,\Delta)\to B_d(\pi(x),\Delta)$ is an isometry $\forall\ x\in X$

Fix $K \subset \mathbb{R}^d$ compact with $m(K) > 0$. If F is (ρ_n, ϵ)-dense in K, then $K \subset \bigcup_{x\in F} B_{\rho_n}(x, 2\epsilon)$ whence

$$m(K) \le \sum_{x\in F} m(B_{\rho_n}(x, 2\epsilon)) = |F| m(m(B_{\rho_n}(0, 2\epsilon)) = |F|(2\epsilon)^d m(m(B_{\rho_n}(0, 1))$$

since

$$B_{\rho_n}(0, r) := \bigcap_{k=0}^{n-1} T^{-k} B(T^k x, r) = r B_{\rho_n}(0, 1),$$

whence

$$S(K, d_n, \epsilon) \ge \frac{m(K)}{(2\epsilon)^d m(m(B_{\rho_n}(0, 1))} \implies \frac{1}{n} S(K, d_n, \epsilon) \underset{n\to\infty}{\gtrsim} \frac{1}{n} \log \frac{1}{m(B_{d_n}(0, 1))}.$$

Proof that $h_d(T) \le \mathfrak{H}$:

Let $C_r = z + [-r, r]^d$ & $0 < \epsilon < r$. If $E \subset C_r$ is $(\rho_n, 2\epsilon)$-separated, then

$$C_{3r} \supset C_{r+2\epsilon} \supseteq \biguplus_{x\in E} B_{\rho_n}(x, \epsilon) \ \&$$

$$m(C_{3r}) \ge \sum_{x\in E} m(B_{\rho_n}(x, \epsilon)) = |E| \epsilon^d m(m(B_{\rho_n}(0, 1))$$

whence

$$S(C_r, \rho_n, \epsilon) \le \frac{m(C_{3r})}{\epsilon^d} \cdot \frac{1}{m(m(B_{\rho_n}(0, 1))}.$$

It follows from this that $h_d(T, C_r) \le \mathfrak{H}$. ☑

Week # 11, 1/1/2014.

11.1 Proposition (Entropy of a linear map) *Let $T:\mathbb{R}^d\to\mathbb{R}^d$ be a linear mapping. Suppose that $\rho(x,y)=\|x-y\|_2$, Let $\Lambda\subset\mathbb{C}$ be the collection of eigenvalues of T ocuuring with multiplicities $d_\lambda,\ \lambda\in\Lambda$, then*

$$h_\rho(T)=\sum_{\lambda\in\Lambda\ \ |\lambda|>1} d_\lambda\log|\lambda|.$$

Proof sketch

By Jordan's theorem

$$\mathbb{R}^d\cong\bigoplus_{\lambda\in\Lambda}V_\lambda$$

where $\dim V_\lambda=d_\lambda$ and $(T-\lambda\mathtt{Id})^{d_\lambda}|_{V_\lambda}\equiv 0$. In particular, $TE_\lambda\subset E_\lambda\ \forall\ \lambda\in\Lambda$. Let

$$W_+:=\bigoplus_{\lambda\in\Lambda,\ |\lambda|>1}V_\lambda\cong\mathbb{R}^{d_+}\ \ \&\ \ W_0:=\bigoplus_{\lambda\in\Lambda,\ |\lambda|\le 1}V_\lambda=V_+^\perp\cong\mathbb{R}^{d_0}$$

where $d_+=\sum_{\lambda\in\Lambda,\ |\lambda|>1}d_\lambda\ \&\ d_0=d-d_+$.

Let $\rho^j(x,y)=\|x-y\|_2,\ x,y\in W_j,\ j=+,0$ and set $\eta((x,y),(x',y')):=\rho^+(x,x')\vee\rho^0(y,y')\ \&\ m=m_+\times m_0$ where m_j is Lebesgue measure on $W_j,\ j=+,0$.

By lemma 10.6

$$\begin{aligned}h_\rho(T)&=h_\eta(T)\\&=\overline{\lim_{n\to\infty}}\frac{1}{n}\log\frac{1}{m(B_{\eta_n}(0,1))}\\&==\overline{\lim_{n\to\infty}}(\frac{1}{n}(\log\frac{1}{m_+(B_{\rho_n^+}(0,1))}+\log\frac{1}{m_0(B_{\rho_n^0}(0,1))}).\end{aligned}$$

Now

$$m_0(B_{\rho_n^0}(0,1))\le m(T^{-n}B_{\rho^0}(0,1))=\frac{m(B_{\rho^0}(0,1))}{|\det T|_{W_0}|^n}\ge m(B_{\rho^0}(0,1))$$

and

$$m_+(B_{\rho_n^+}(0,1))\le m(T^{-n}B_{\rho^+}(0,1))=\frac{m(B_{\rho^+}(0,1))}{|\det T|_{W_+}|^{n-1}}$$

whence

$$\begin{aligned} h_\rho(T) &= \varlimsup_{n\to\infty}(\frac{1}{n}(\log\frac{1}{m_+(B_{\rho_n^+}(0,1))}+\log\frac{1}{m_0(B_{\rho_n^0}(0,1))} \\ &\ge \varlimsup_{n\to\infty}\frac{1}{n}\log\frac{1}{m_+(B_{\rho_n^+}(0,1))} \\ &\ge \frac{n-1}{n}\log|\det T|_{W_+}| \approx \sum_{\lambda\in\Lambda\ |\lambda|>1} d_\lambda\log|\lambda|. \end{aligned}$$

Proof that $h_\rho(T)\le\sum_{\lambda\in\Lambda\ |\lambda|>1} d_\lambda\log|\lambda|$.

We have that $m=\prod_{\lambda\in\lambda} m_\lambda$ where m_λ is Lebesgue measure on V_λ. As above

$$h_\rho(T)=\varlimsup_{n\to\infty}\frac{1}{n}\sum_{\lambda\in\Lambda}\log\frac{1}{m_+(B_{\rho_n^\lambda}(0,1))}$$

where $\rho_\lambda(x,y)=\|x-y\|_2,\ \ x,y\in\ V_\lambda\ \&\ \ \rho(x,y)=\max_\lambda\rho_\lambda(x^\lambda,y^\lambda)$.

It thus suffices to show that

$$\varlimsup_{n\to\infty}\frac{1}{n}\log\frac{1}{m_\lambda(B_{\rho_n^\lambda}(0,1))}\le 0\vee d_\lambda\log|\lambda|\ \ \forall\ \lambda\in\Lambda.$$

Fix $\lambda\in\Lambda$ & $\mu>|\lambda|$ and for $x\in V_\lambda$, set

$$\|x\|_\mu:=\sum_{n\ge0}\frac{\|T^n x\|_2}{\mu^n}.$$

The series converges

$$\because\ \sqrt[n]{\frac{\|T^n x\|_2}{\mu^n}}\ =\ \frac{\sqrt[n]{\|T^n x\|_2}}{\mu}\ \xrightarrow[n\to\infty]{}\ \frac{|\lambda|}{\mu}<1$$

and thus defines a norm on V_λ. Moreover,

$$\|Tx\|_\mu=\sum_{n\ge0}\frac{\|T^{n+1}x\|_2}{\mu^n}=\mu\sum_{n\ge1}\frac{\|T^n x\|_2}{\mu^n}\le\mu\|x\|_\mu.$$

Writing $\Delta(x,y)=\|x-y\|_\mu$, we have $B_\rho(0,1))\supset B_\Delta(0,r)$ for some $r>0$, whence

$$\begin{aligned} B_{\rho_n^\lambda}(0,1))\ \supset\ B_{\Delta_n}(0,r)) &= \bigcup_{j=0}^{n-1} T^{-j}B_\Delta(0,r)) \\ &\supseteq \bigcup_{j=0}^{n-1} B_\Delta(0,\frac{r}{\mu^j}))=B_\Delta(0,\frac{r}{\mu^{n-1}})) \end{aligned}$$

and

$$m_\lambda(B_{\rho_n^\lambda}(0,1))\ge m_\lambda(B_\Delta(0,\frac{r}{\mu^{n-1}}))=m_\lambda(B_\Delta(0,1))\frac{r^{d_\lambda}}{\mu^{(n-1)d_\lambda}}.$$

Thus,

$$\frac{1}{n}\log\frac{1}{m_+(B_{\rho_n^\lambda}(0,1))} \le \frac{1}{n}\log\frac{1}{m_+(B_\Delta(0,1))} + \frac{n-1}{n}d_\lambda\log\mu - \frac{d_\lambda\log r}{n}$$

$$= d_\lambda\log\mu + O(\frac{1}{n}). \quad ☑$$

End of course "dynamical systems"