Lecture #25: 3/6/2010.

Multiplication of series

Motivation.

Suppose that $P(x) = \sum_{n=0}^{\infty} a_n x^n$ and $Q(x) = \sum_{n=0}^{\infty} b_n x^n$ are polynomials (ie $a_n = b_n = 0\ \forall$ large n), then $P(x)Q(x)$ is also a polynomial and

$$P(x)Q(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n.$$

Proof

$$P(x)Q(x) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} a_k b_\ell x^{k+\ell} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_k b_{n-k} x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n.$$

Merten's product theorem

Let $a_n,\ b_n \in \mathbb{R}\quad (n \ge 0)$ *and let* $c_n := \sum_{k=0}^{n} a_k b_{n-k}\quad (n \ge 0)$.

If $\sum_{n=0}^{\infty} a_n = A$ *converges absolutely and* $\sum_{n=0}^{\infty} b_n = B$ *converges, then* $\sum_{n=0}^{\infty} c_n = AB$ *converges.*

Here "$\sum_{n=0}^{\infty} b_n = B$ converges" means $\sum_{n=0}^{N} b_n \xrightarrow[N\to\infty]{} B \in \mathbb{R}$.

Proof Set

$$A_N := \sum_{n=0}^{N} a_n,\ B_n := \sum_{n=0}^{N} b_n,\ C_n := \sum_{n=0}^{N} c_n,$$

then

$$\begin{aligned} C_N &= \sum_{n=0}^{N} \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{N} a_k \sum_{n=k}^{N} b_{n-k} \\ &= \sum_{k=0}^{N} a_k \sum_{j=0}^{N-k} b_j = \sum_{k=0}^{N} a_k B_{N-k} \\ &= \sum_{k=0}^{N} a_k B + \sum_{k=0}^{N} a_k (B_{N-k} - B) \\ &=: A_N B + U_N. \end{aligned}$$

It suffices to show that $U_N \to 0$.

To this end, suppose that $|B_n|,\ |B| \le M$ and let $\epsilon > 0$, then $\exists\ N_0$ so that $\sum_{k>N_0} |a_k| < \epsilon$ and $\exists\ N_1$ so that $|B_n - B| < \epsilon\ \forall\ n > N_1$. For

$N > N_0 + N_1$,

$$\begin{aligned}|U_N| &\le \sum_{k=0}^{N} |a_k||B_{N-k} - B| \\ &= \sum_{k=0}^{N_0} |a_k||B_{N-k} - B| + \sum_{k=N_0+1}^{N} |a_k||B_{N-k} - B| \\ &< \epsilon \sum_{k=0}^{N_0} |a_k| + 2M \sum_{k=N_0+1}^{N} |a_k| \\ &< \epsilon \sum_{k=0}^{\infty} |a_k| + 2M\epsilon. \quad \square\end{aligned}$$

Examples. [14]

Fix $r,\ s > 0$ and set $a_n := \frac{(-1)^n}{(n+1)^r},\ b_n := \frac{(-1)^n}{(n+1)^s}$, then

$$c_n := \sum_{k=0}^{n} a_k b_{n-k} = (-)^n \sum_{k=0}^{n} \frac{1}{(k+1)^r (n-k+1)^s}.$$

We claim that

$$(\text{✗}) \qquad |c_n| \ge (n+1)^{1-r-s} \quad \forall\ n \ge 1.$$

Proof Set $x_n(k) := \frac{k}{n+1}$ for $0 \le k \le n$, then

$$\frac{(k+1)^r (n-k+1)^s}{(n+1)^{r+s}} = x_n(k+1)^r (1 - x_n(k))^s \le 1.$$

It follows that

$$\begin{aligned}|c_n| &= \sum_{k=0}^{n} \frac{1}{(k+1)^r (n-k+1)^s} \\ &\ge \sum_{k=0}^{n} \frac{1}{(n+1)^{r+s}} \\ &= (n+1)^{1-r-s}. \quad ☑(\text{✗})\end{aligned}$$

¶1 For eg $r = s = \frac{1}{2}$ that $\sum_{n=0}^{\infty} a_n,\ \sum_{n=0}^{\infty} b_n$ converge conditionally and $c_n \not\to 0$ whence $C_n \not\to$ and there's no product theorem.

¶2 For eg $r = \frac{5}{4},\ s = \frac{1}{4},\ \sum_{n=0}^{\infty} |a_n| < \infty,\ \ \sum_{n=0}^{\infty} b_n$ converges conditionally and $|c_n| \ge \frac{1}{\sqrt{n}}$ whence $\sum_{n=0}^{\infty} |c_n| = \infty$. This shows that the conditions of Merten's theorem do not guarantee absolute convergence of $\sum_{n=0}^{\infty} c_n$.

[14] Note that this bit got easier.

Abel's theorem on multiplication of series

Let $a_n,\ b_n \in \mathbb{R} \quad (n \ge 0)$ and let $c_n := \sum_{k=0}^n a_k b_{n-k} \quad (n \ge 0)$.

If $\sum_{n=0}^N a_n \to A,\ \sum_{n=0}^N b_n \to B$ and $\sum_{n=0}^\infty c_n$ converges, then $\sum_{n=0}^\infty c_n = AB$.

Proof The power series

$$A(x) := \sum_{n=0}^\infty a_n x^n,\ B(x) := \sum_{n=0}^\infty b_n x^n,\ C(x) := \sum_{n=0}^\infty c_n x^n$$

converge absolutely $\forall\ |x| < 1$ `(else these series would diverge for` $x = 1$`)`. By Mertens' theorem $C(x) = A(x)B(x) \forall\ |x| < 1$. Using Abel's continuity theorem,

$$AB \underset{x\to 1-}{\longleftarrow} A(x)B(x) = C(x) \underset{x\to 1-}{\longrightarrow} C := \sum_{k=0}^\infty c_k. \quad \square$$

Exercises.

For $a_n,\ b_n \in \mathbb{C}\ \ (n \ge 0)$ let $c_n := \sum_{k=0}^n a_k b_{n-k}$.

(i) State and prove Mertens product theorem for complex series.

(ii) Show that if $\sum_{n=0}^\infty |a_n| < \infty$ and $\sum_{n=0}^\infty |b_n| < \infty$, then so does $\sum_{n=0}^\infty |c_n| < \infty$.

(iii) Is it possible that $\sum_{n=0}^\infty |a_n| < \infty$ and $\sum_{n=0}^\infty |c_n| < \infty$ but $\sum_{n=0}^\infty |b_n| = \infty$?

(iv) Show that $\sum_{n=0}^\infty \frac{(z+w)^n}{n!} = \sum_{n=0}^\infty \frac{z^n}{n!} \sum_{n=0}^\infty \frac{w^n}{n!} \quad \forall\ z,\ w \in \mathbb{C}$.

(v) Show that $\exp[z] = \sum_{n=0}^\infty \frac{z^n}{n!} \quad \forall\ z \in \mathbb{C}$.

`Hint`: Use Euler's formula and (iv).

Converses to Abel's continuity theorem

The converse to Abel's continuity theorem is not true:

- $\sum_{n=0}^\infty (-1)^n x^n$ converges $\forall\ |x| < 1$ and
- $\sum_{n=0}^\infty (-1)^n x^n = \frac{1}{1+x} \underset{x\to 1-}{\longrightarrow} \frac{1}{2}$ but
- $\sum_{n=0}^N (-1)^n = \frac{1}{2}(1 + (-1)^N) \underset{N\to\infty}{\not\to}$

Tauber's theorem

Suppose $a_n \in \mathbb{R},\ na_n \underset{n\to\infty}{\longrightarrow} 0$ and

- *$\sum_{n=1}^\infty a_n x^n$ converges $\forall\ |x| < 1$; and*
- *$\sum_{n=1}^\infty a_n x^n \underset{x\to 1-}{\longrightarrow} s \in \mathbb{R}$, then*

$$\sum_{k=1}^n a_k \underset{n\to\infty}{\longrightarrow} s.$$

Proof By assumption, whenever $x_N \underset{N\to\infty}{\longrightarrow} 1-$, $\sum_{k=1}^{\infty} a_k x_N^k \underset{N\to\infty}{\longrightarrow} s$ and it suffices to show that

(⌧) $$\exists\ x_N \underset{N\to\infty}{\longrightarrow} 1- \text{ such that } \sum_{k=1}^{N} a_k - \sum_{n=1}^{\infty} a_n x_N^n \underset{N\to\infty}{\longrightarrow} 0.$$

We claim that (⌧) holds with $x_N := 1 - \frac{1}{N}$. To see this:

$$\sum_{k=1}^{N} a_k - \sum_{k=1}^{\infty} a_k x_N^k = \sum_{k=1}^{N} a_k(1 - x_N^k) - \sum_{k=N+1}^{\infty} a_k x_N^k$$
$$=: A_N + B_N.$$

Now

$$|A_N| \le \sum_{k=1}^{N} |a_k|(1 - x_N^k)$$
$$= (1 - x_N) \sum_{k=1}^{N} |a_k| \sum_{j=0}^{k-1} x_N^j$$
$$\le \tfrac{1}{N} \sum_{k=1}^{N} k|a_k|$$
$$\underset{N\to\infty}{\longrightarrow} 0;$$

and

$$|B_N| \le \sum_{k=N+1}^{\infty} |a_k| x_N^k = \sum_{k=N+1}^{\infty} k|a_k| \cdot \tfrac{1}{k} x_N^k$$
$$\le \tfrac{1}{N+1} \sup_{k\ge N+1} k|a_k| \sum_{k=N+1}^{\infty} x_N^k$$
$$= \sup_{k\ge N+1} k|a_k| \cdot \tfrac{1}{N+1} \cdot \tfrac{X_N^{N+1}}{1-x_N}$$
$$\le \sup_{k\ge N+1} k|a_k|$$
$$\underset{N\to\infty}{\longrightarrow} 0. \quad ☑$$

end of coursenotes