COURSE NOTES ON MEASURE THEORY

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Week # 1

§1 Classes of sets and set functions

Semi-rings. Let X be a set. A non-empty class $\mathcal{S} \subset 2^X$ is called a *semi-ring* if $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$ and $A \setminus B = \biguplus_{k=1}^N C_k$ where $N \in \mathbb{N}$ and $C_1, C_2, \dots, C_N \in \mathcal{S}$ (disjoint).

Example: boxes in $\mathbb{R}^d$. A (d-dimensional) *box* is a Cartesian product of finite intervals,

$$R = \prod_{k=1}^d I_k = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_k \in I_k \ \forall \ k\}$$

where for each $1 \le k \le d$, I_k is a finite interval. Let $\mathcal{S} = \mathcal{S}_d$ denote the collection of boxes in $\mathbb{R}^d$.

1.1 Proposition

$\mathcal{S}$ is a semi-ring.

Proof

Evidently, if R, R' are boxes then so is $R \cap R'$. We show that if R, R' are boxes then $R \setminus R'$ is a finite, disjoint union of (at most $2d$) boxes.

To see this write $R = \prod_{k=1}^d I_k$ and $R' = \prod_{k=1}^d J_k$. Note that if I, J are intervals, then $I \setminus J$ is the disjoint union of at most two intervals. We obtain that

$$R \setminus R' = \{x \in R : x_\ell \notin J_\ell \text{ for some } 1 \le \ell \le d\} = \bigcup_{\ell=1}^d A_\ell$$

where $A_\ell := \prod_{i=1}^{\ell-1} I_i \cap J_i \times I_\ell \setminus J_\ell \times \prod_{j=\ell+1}^d I_j$. Clearly, each A_k is the union of at most 2 disjoint boxes, and the $A_\ell \in \mathcal{A}$ $(1 \le \ell \le d)$ are disjoint.

These are notes of a course given in 2017.
1 week ~ 150 minutes.
Exercises are given at the end of each week's notes.
They are used in the notes without proof also to determine grades for the course.

Thus

$$R \setminus R' = \biguplus_{\ell=1}^{d} A_\ell = \biguplus_{j=1}^{2d} C_k$$

where $C_1, \dots, C_{2d} \in \mathcal{S}$ are disjoint. □

Semi-Algebras.

Let X be a set. A class $\mathcal{S} \subset 2^X$ is called a *semi-algebra* (of subsets of X) if

- $A, B \in \mathcal{S} \quad \Rightarrow \quad A \cap B \in \mathcal{S}$; and

- $C \in \mathcal{S} \quad \Rightarrow \quad X \setminus C = \biguplus_{k=1}^N C_k$ where $N \in \mathbb{N}$ and $C_1, C_2, \dots, C_N \in \mathcal{S}$ (disjoint).

Remark.

The semi-ring of sub-boxes of a finite box in $\mathbb{R}^d$ is a semi-algebra, but the semi-ring of all boxes in $\mathbb{R}^d$ is not.

Cylinder sets in shift spaces.

Let S be a finite set, and let $X := S^{\mathbb{N}}$. Given $N \geq 1$, $a_1, \dots, a_N \in S$, define the *cylinder*

$$[a_1, \dots, a_N] := \{x \in X : \ x_i = a_i \ \forall \ 1 \leq i \leq N\}.$$

The *length* of the cylinder $C = [a_1, \dots, a_N]$ is $\ell(C) = N$.

Proposition 1.2

$\mathcal{S} := \{\texttt{cylinders}\} \cup \{\varnothing\}$ *is a semi-algebra.*

`Proof`

Suppose that $a, b \in \mathcal{S}$. WLOG $\ell(a) = M \leq \ell(b) = N$ and

$$a = [a_1, a_2, \dots, a_M], \quad b = [b_1, b_2, \dots, b_N].$$

If $\exists \ 1 \leq k \leq N$ with $a_k \neq b_k$, then

$$a \cap b = \varnothing \in \mathcal{S}, \ a \setminus b = a, \ b \setminus a = b \in \mathcal{S}.$$

Otherwise

$$b = [a_1, \dots, a_M, b_{M+1}, \dots, b_N] \subset a$$

whence

$$a \cap b = b \in \mathcal{S}, \ b \setminus a = \varnothing \in \mathcal{S} \ \& \ a \setminus b = \varnothing \in \mathcal{S} \text{ in case } M = N$$

and in case $M < N$,

$$a \setminus b = \biguplus_{c_{M+1}, \dots, c_N \in S, \ (c_{M+1}, \dots, c_N) \neq (b_{M+1}, \dots, b_N)} [a_1, \dots, a_M, c_{M+1}, \dots, c_N]. \ \boxtimes$$

Rings and algebras. A non-empty class $\mathcal{R} \subset 2^X$ is called a *ring* if

$$A, B \in \mathcal{R} \;\Rightarrow\; A \cup B,\; A \cap B,\; A \setminus B \in \mathcal{R},$$

and an *algebra* (of subsets of X) if in addition, $X \in \mathcal{R}$. Note that any ring $\mathcal{R} \ni \varnothing$.

1.3 Intersection propositions

Let $\mathcal{C} \subset 2^X$ then:

$$\mathcal{R}(\mathcal{C}) := \bigcap_{2^X \supset \mathcal{R} \supset \mathcal{C} \text{ a ring}} \mathcal{R}$$

is a ring (known as the **ring generated by** $\mathfrak{R}$*);*

$$\mathcal{A}(\mathcal{C}) := \bigcap_{2^X \supset \mathcal{A} \supset \mathcal{C} \text{ an algebra}} \mathcal{A}$$

is an algebra (known as the **algebra generated by** $\mathcal{C}$*).*

Note that the ring generated by a semi-ring is an algebra iff the semi-ring is a semi-algebra.

- Is it true that if $\mathcal{R} \subset 2^X$ is a ring , then

$$\mathcal{A}(\mathcal{R}) = \mathcal{R} \cup \{X \setminus A :\; A \in \mathcal{R}\}?$$

1.4 Theorem (ring generated by a semi-ring)

Suppose that $\mathcal{S} \subset 2^X$ is a semi-ring, then:

$$\mathcal{R}(\mathcal{S}) = \{\bigcup_{k=1}^{n}{}^{\bullet}\, C_k :\; n \in \mathbb{N},\; C_1, \dots, C_n \in \mathcal{S} \;\text{ disjoint }\}.$$

Proof

Write

$$\mathcal{R}_0 := \{\bigcup_{k=1}^{n}{}^{\bullet}\, C_k :\; n \in \mathbb{N},\; C_1, \dots, C_n \in \mathcal{S} \;\text{ disjoint }\}.$$

Since $\mathcal{R}(\mathcal{S})$ is a ring containing $\mathcal{S}$, we have that $\mathcal{R}(\mathcal{S}) \supset \mathcal{R}_0$. To show equality, we prove that $\mathcal{R}_0$ is a ring. It is evident that $\mathcal{R}_0$ is closed under intersection and disjoint union.

¶1 If $A, B \in \mathcal{R}_0$, then $A \setminus B \in \mathcal{R}_0$.

It follows from the semi-ring property that $A \setminus C \in \mathcal{R}_0 \;\forall\; A \in \mathcal{R}_0$ and $C \in \mathcal{S}$. Suppose that $A, B \in \mathcal{R}_0$ and that $B = \bigcup_{k=1}^{n} C_k$ where $\{C_k\}_{k=1}^{n} \subset \mathcal{S}$. We have that

$$A \setminus C_1 \in \mathcal{R}_0 \implies A \setminus C_1 \setminus C_2 \in \mathcal{R}_0 \implies \dots \implies A \setminus C_1 \setminus \cdots \setminus C_n = A \setminus B \in \mathcal{R}_0.$$

¶2 If $A, B \in \mathcal{R}_0$, then $A \cup B \in \mathcal{R}_0$.

Here $A \cup B = (A \setminus B) \,\dot\cup\, B$ a disjoint union of sets in $\mathcal{R}_0$. □

Finite subcover property.

Let $\mathcal{S}$ be a semi-ring of subsets of X. We'll say that $\mathcal{S}$ has the *finite subcover property* (FSCP) if

$$(\text{🚘})\qquad A,\ A_1,\ A_2,\dots\in\mathcal{S},\ A\subset\bigcup_{n=1}^{\infty}A_n$$
$$\implies\ \exists\ N\in\mathbb{N}\text{ so that }\ A\subset\bigcup_{n=1}^{N}A_n.$$

Clopen set proposition *If X is a compact, topological space and $\mathcal{S}\subset$ `clopen sets`, then $\mathcal{S}$ has the finite subcover property.*

Proof This follows from the Heine-Borel theorem.

Example 1. Products of finite spaces.

Let $\Omega=\prod_{k=1}^{\infty}S_k$ where each S_k is finite.

As above, a *cylinder* is a set of form

$$[a_1,a_2,\dots,a_n]=\{x\in\Omega:\ x_k=a_k\ \forall\ 1\le k\le n\}$$

where $a_1\in S_1,\ a_2\in S_2,\dots,a_n\in S_n$. Also as above, the collection

$$\mathcal{S}:=\{\texttt{cylinders}\}$$

is a semi-algebra. We'll show

☺ $\mathcal{S}$ has the finite subcover property.

Proof of ☺

Define $d:\Omega\times\Omega\to[0,\infty)$ by

$$d(x,y):=\begin{cases}0 & x=y,\\ \frac{1}{2^{t(x,y)}} & x\ne y.\end{cases}$$

where $t(x,y):=\min\{n\ge 1:\ x_n\ne y_n\}$.

It is not hard to show that (Ω,d) is a metric space, and that the closed ball

$$B(x,\tfrac{1}{2^{n+1}})=[x_1,\dots,x_n].$$

Thus, if C is a cylinder then C is closed with

$$d-\texttt{diam}\ C=\frac{1}{2^{\ell(C)+1}}.$$

Note that cylinders are also open (*clopen*):

$$\Omega\setminus[c_1,\dots,c_N]=\bigcup_{a_1\in S_1,\dots,a_N\in S_N,\ (a_1,\dots,a_N)\ne(c_1,\dots,c_N)}[a_1,\dots,a_N]$$

is closed being a finite union of closed balls.

To complete the proof of ☺, it suffices to show that (Ω, d) is a compact metric space.

To see this, let $(x^{(k)})_{k\ge1}$ be a sequence in Ω. We'll show existence of a convergent subsequence.

There is a subsequence $n_1(1) < n_2(1) \uparrow \infty$ and $a_1 \in S_1$ so that $x_1^{(n_k(1))} = a_1 \ \forall\ k \le 1$.

There is also a subsequence $n_1(2) < n_2(2) \uparrow \infty$ of $n_1(1) < n_2(1) \uparrow \infty$ and $a_2 \in S_2$ so that $x_j^{(n_k(2))} = a_j \ \forall\ k \le 1,\ j = 1, 2$.

Continuing, we obtain a sequence of subsequences (each a subsequence of the previous) $n_1(k) < n_2(k) \uparrow \infty \quad (k \ge 1)$ and $a_1 \in S_1, a_2 \in S_2, \dots a_n \in S_n, \dots$ so that $x_j^{(n_k(\ell))} = a_j \ \forall\ k \le 1,\ 1 \le j \le \ell$.

Now diagonalize and set $N_\ell = n_\ell(\ell)$, then

$$x_j^{(N_\ell)} = a_j \ \forall \ \ 1 \le j \le \ell.$$

It follows that $t(x^{(N_\ell)}, a) \ge \ell + 1$ where $a := (a_1, a_2, \dots) \in S$ whence $d(x^{(N_\ell)}, a) \le \frac{1}{2^{\ell+1}} \xrightarrow[\ell\to\infty]{} 0$. ☑

Example 2. Products of countable spaces.

Here, we exhibit an algebra of subsets of $X := \mathbb{N}^{\mathbb{N}}$ with the finite subcover property.

Let

$$\mathcal{A}_{\mathbb{N}} := \{F, \mathbb{N} \setminus F :\ F \subset \mathbb{N} \setminus \{1\},\ \#(F) < \infty\},$$

then $\mathcal{A}_{\mathbb{N}}$ is an algebra.

An $\mathcal{A}_{\mathbb{N}}$-*cylinder* in $\mathbb{N}^{\mathbb{N}}$ is a set of form

$$[A_1, A_2, \dots, A_N] := \{x \in \mathbb{N}^{\mathbb{N}} :\ x_i \in A_i \ \forall\ 1 \le i \le N\}$$

where $A_1, A_2, \dots, A_N \in \mathcal{A}_{\mathbb{N}}$.

Denote

$$\mathcal{A} := \{\mathcal{A}_{\mathbb{N}}\texttt{- cylinders}\}.$$

As above, $\mathcal{A}$ is an algebra. We claim that

☻ $\mathcal{A}$ has the finite subcover property.

Proof of ☻ We construct a suitable compact topology on $\mathbb{N}^{\mathbb{N}}$.

Define $d : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ by

$$d(x, y) := \begin{cases} 0 & x = y; \\ 1 & x \ne y,\ x, y \ge 2; \\ \frac{1}{y} & x \ne y,\ x = 1; \end{cases}$$

then $(\mathbb{N}, d)$ is a compact, metric space.

Define the product metric $\rho : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to [0, \infty)$ by

$$\rho(x, y) := \sum_{k=1}^{\infty} \frac{d(x_k, y_k)}{2^k},$$

then $(\mathbb{N}^{\mathbb{N}}, \rho)$ is a compact, metric space and the collection $\mathcal{A}$ is an algebra of compect, open subsets of $\mathbb{N}^{\mathbb{N}}$. Thus $\mathcal{A}$ has the finite subcover property. ☑

§2 Additive set functions

Let $\mathcal{S}$ be a semi-ring. A set function $f : \mathcal{S} \to \mathbb{R}$ is *additive* if

$$C,\ C_1, C_2, \dots, C_N \in \mathcal{S},\ C = \biguplus_{k=1}^{N} C_k \ \Rightarrow\ f(C) = \sum_{k=1}^{N} f(C_k).$$

Example: Volume in $\mathbb{R}^d$.

Let $\mathcal{S}$ be the semi-ring of boxes in $\mathbb{R}^d$, and define $v : \mathcal{S} \to \mathbb{R}$ by $v(\prod_{k=1}^{d} I_k) := \prod_{k=1}^{d} |I_k|$ where $|I|$ denotes the length of I.

1.5 Proposition

The set function $v : \mathcal{S} \to \mathbb{R}$ is additive.

`Proof` For $A \subset \mathbb{R}^d$ and $\varepsilon > 0$ let $N_\varepsilon^{(d)}(A) := \#(\{\underline{n} \in \mathbb{Z}^d :\ \varepsilon\underline{n} \in A\})$.

Evidently, $N_\varepsilon^{(d)} : 2^{\mathbb{R}^d} \to [0, \infty]$ is additive:

(✉) $$N_\varepsilon^{(d)}(\biguplus_{k=1}^{N} A_k) = \sum_{k=1}^{N} N_\varepsilon^{(d)}(A_k).$$

Also, for $\prod_{k=1}^{d} I_k \subset \mathbb{R}^d$ a box,

(📱) $$N_\varepsilon^{(d)}(\prod_{k=1}^{d} I_k) = \prod_{k=1}^{d} N_\varepsilon^{(1)}(I_k).$$

`Proof of`(📱):

$$\{\underline{n} \in \mathbb{Z}^d :\ \varepsilon\underline{n} \in \prod_{k=1}^{d} I_k\} = \prod_{k=1}^{d} \{n \in \mathbb{Z} :\ \varepsilon n \in I_k\}$$

whence (📱). ☑

We claim next that

(☕) $$\varepsilon^d N_\varepsilon^{(d)}(R) \xrightarrow[\varepsilon \to 0+]{} v(R).$$

`Proof` By (📱) it suffices to prove (☕) for $d = 1$.

$N_\varepsilon^{(1)}([a,b]) = \lfloor \frac{b}{\varepsilon} \rfloor - \lceil \frac{a}{\varepsilon} \rceil + 1 \sim \frac{b-a}{\varepsilon}$ as $\varepsilon \to 0$ whence for $I \subset \mathbb{R}$, $\overline{I} = [a,b]$

$$\varepsilon N_\varepsilon^{(1)}(I) \le \varepsilon N_\varepsilon^{(1)}([a,b]) \underset{\varepsilon\to 0+}{\longrightarrow} b - a = |I|$$

and $\forall\ 0 < \delta < \frac{|I|}{2}$,

$$\varepsilon N_\varepsilon^{(1)}(I) \ge \varepsilon N_\varepsilon^{(1)}([a+\delta, b-\delta]) \underset{\varepsilon\to 0+}{\longrightarrow} b - a - 2\delta = |I| - 2\delta. \quad ☑(☕)$$

Now suppose that $R,\ R_1, R_2, \dots, R_N \in \mathcal{S},\ R = \biguplus_{k=1}^N R_k$, then by (✉) and (☕),

$$v(R) \underset{\varepsilon\to 0+}{\longleftarrow} \varepsilon^d N_\varepsilon(R) = \sum_{k=1}^N \varepsilon^d N_\varepsilon(R_k) \underset{\varepsilon\to 0+}{\longrightarrow} \sum_{k=1}^N v(R_k). ☑$$

Bernoulli set functions on shift space.

Let S be a finite set, $X = S^{\mathbb{N}}$ & $\mathcal{S} = \{\texttt{cylinders}\}$. Fix $p_j \in \mathcal{P}(S)$ $(j \ge 1)$ and define $\mu : \mathcal{S} \to [0,1]$ by

$$\mu([a_1, a_2, \dots, a_N]) := \prod_{j=1}^N p_j(a_j).$$

1.6 Proposition *$\mu : \mathcal{S} \to [0,1]$ is additive.*

Proof

Let $\mathcal{S}_n := \{C \in \mathcal{S} :\ \ell(C) = n\}$, then $\mathcal{A}_n := \mathcal{A}(\mathcal{S}_n) \uparrow\ \mathcal{A} := \mathcal{A}(\mathcal{S})$.

For $n \ge 1$,

$$\mathcal{A}_n = \{\biguplus_{j=1}^N C_j :\ N \in \mathbb{N},\ C_1, \dots, C_N \in \mathcal{S}_n \text{ are disjoint}\}$$

and we can define $\mu_n : \mathcal{A}_n \to [0,1]$ by

$$\mu_n(A) := \sum_{C \in \mathcal{S}_n,\ A \supset C} \mu(C).$$

Evidently $\mu_n : \mathcal{A}_n \to [0,1]$ is additive.

Next, for $k \le n$ we have that $\mathcal{S}_k \subset \mathcal{A}_n$ and we claim that $\mu_n|_{\mathcal{S}_k} \equiv \mu$.

To see this:

$$\begin{aligned}
\mu_n([c_1, \dots, c_k]) &= \sum_{a_{k+1}, \dots, a_n \in S} \mu([c_1, \dots, c_k, a_{k+1}, \dots, a_n]) \\
&= \sum_{a_{k+1}, \dots, a_n \in S} \mu([c_1, \dots, c_k]) \prod_{j=k+1}^n p_j(a_j) \\
&= \mu([c_1, \dots, c_k]).
\end{aligned}$$

To finish, suppose that $I = \biguplus_{C\in\mathcal{C}}^{N} C$ where $I \in \mathcal{S}$ & $\mathcal{C} \subset \mathcal{S}$ is a finite disjoint collection. We must show that

$$\sum_{C\in\mathcal{C}} \mu(C) = \mu(I).$$

Suppose that $n \ge 1$ is larger than the lengths of I and all the $C \in \mathcal{C}$, then $\mathcal{C} \subset \mathcal{A}_n$ and using $\mu_n|_{\mathcal{S}_k} \equiv \mu$ for $k \le n$:

$$\begin{aligned}\mu(I) &= \mu_n(I)\\ &= \sum_{C\in\mathcal{C}} \mu_n(C)\\ &= \sum_{C\in\mathcal{C}} \mu(C). \quad \boxtimes\end{aligned}$$

1.7 Ring extension proposition

Suppose that $\mathcal{S} \subset 2^X$ is a semi-ring and that $f : \mathcal{S} \to \mathbb{R}$ is additive, then there is an additive set function $F : \mathcal{R}(\mathcal{S}) \to \mathbb{R}$ such that $F|_{\mathcal{S}} \equiv f$.

Proof Using additivity of f:

if $A \in \mathcal{R}(\mathcal{S})$ and

$$A = \biguplus_{k=1}^{m} R_k = \biguplus_{k=1}^{n} R'_k$$

where $\{R_k\}_{k=1}^m$, $\{R'_k\}_{k=1}^n \subset \mathcal{S}$ are both disjoint collections, then

$$\sum_{k=1}^{m} f(R_k) = \sum_{k=1}^{m}\sum_{\ell=1}^{n} f(R_k \cap R'_\ell) = \sum_{\ell=1}^{n} f(R'_\ell).$$

Using the representation of $\mathcal{R}(\mathcal{S})$ in theorem 1.4, $\exists \ F : \mathcal{R}(\mathcal{S}) \to \mathbb{R}$ defined by

$$F(\biguplus_{k=1}^{m} R_k) := \sum_{k=1}^{m} f(R_k).$$

Evidently, F is additive and $F|_{\mathcal{S}} \equiv f$. $\boxtimes$

Total variation.

Let X be a set and let $\mathcal{A} \subset 2^X$ be an algebra. The *total variation* of the additive set function $\mu : \mathcal{A} \to \mathbb{R}$ is

$$\|\mu\| := \sup\{\sum_{k=1}^{N} |\mu(A_k)| : \ N \in \mathbb{N}, \ A_1, A_2, \dots, A_N \in \mathcal{A} \text{ disjoint}, \ X = \biguplus_{k=1}^{N} A_k\} \le \infty.$$

Example. Let $X = [0,1]$ and let $\mathcal{S}$ be the semi-algebra of subintervals. Given $f : X \to \mathbb{R}$ continuous and $I \in \mathcal{S}$, define $\mu(I) := f(b) - f(a)$ where $\overline{I} = [a,b]$, then $f : \mathcal{S} \to \mathbb{R}$ is additive and

$$\|\mu\| = \bigvee_0^1 f := \sup\{\sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)| : \ 0 = t_0 < t_1 < t_2 < \cdots < t_n = 1\}.$$

Jordan decomposition theorem *If $\mu : \mathcal{A} \to \mathbb{R}$ is additive and $\|\mu\| < \infty$, then $\exists$ positive, additive set functions $\mu_+, \ \mu_- : \mathcal{A} \to [0,\infty)$ so that $\mu = \mu_+ - \mu_-$.*

Proof

Define $m = m_\mu : \mathcal{A} \to [0,\infty]$ by

$$m(A) = \sup\{\sum_{n=1}^N |\mu(A_n)| : \ A_1, A_2, \ldots, A_n \in \mathcal{B} \text{ disjoint}, \ A = \biguplus_{n=1}^N A_n\},$$

then $m(A) \le m(X) = \|\mu\| < \infty \ \forall \ A \in \mathcal{A}$.

¶ We show that $m : \mathcal{A} \to [0,\infty)$ is additive.

To see this, let $A = \biguplus_{n=1}^N A_n$ where $A, \ A_1, A_2, \ldots, A_N \in \mathcal{A}$ and $A_n \ (1 \le n \le N)$ are disjoint.

Suppose $t_n < m(A_n)$, then, $\forall \ 1 \le n \le N$, $A_n = \biguplus_{k=1}^{J_n} A_{n,k}$ where $A_{n,k} \in \mathcal{B}$, $A_{n,k} \ (1 \le k \le J_n)$ are disjoint, and $\sum_{k=1}^{J_n} |\mu(A_{n,k})| > t_n$. It follows that

$$A = \biguplus_{1 \le n \le N, \ 1 \le k \le J_n} A_{n,k},$$

whence

$$\begin{aligned} m(A) &\ge \sum_{1 \le n \le N, \ 1 \le k \le J_n} |\mu(A_{n,k})| \\ &= \sum_{1 \le n \le N} \Big(\sum_{1 \le k \le J_n} |\mu(A_{n,k})| \Big) \\ &> \sum_{n=1}^N t_n. \end{aligned}$$

Therefore $m(A) \ge \sum_{n=1}^N m(A_n)$.

To obtain the reverse inequality, suppose $A = \bigcup_{k=1}^K E_k$ where $E_k \in \mathcal{A}$ and $E_k \ (1 \le n \le N)$ are disjoint. Then

$$\sum_{k=1}^K |\mu(E_k)| \le \sum_{n=1}^N \sum_{k=1}^K |\mu(E_k \cap A_N)| \le \sum_{n=1}^N m(A_n), \ \therefore m(A) \le \sum_{n=1}^N m(A_n). \ \boxtimes ¶$$

Now define $\mu_\pm := \frac{m \pm \mu}{2} : \mathcal{A} \to \mathbb{R}$. These are additive, and non-negative and

$$\mu = \mu_+ - \mu_-. \quad \boxtimes$$

§3 Countable Subadditivity

Let $\mathcal{D} \subset 2^X$. The set function $\mu : \mathcal{D} \to [0, \infty)$ is:

- *finitely subadditive* if

$$A,\ A_1, \dots, A_N \in \mathcal{D},\ A \subset \bigcup_{k=1}^N A_k \implies \mu(A) \le \sum_{k=1}^N \mu(A_k);$$

- *countable subadditive* (or *σ-subadditive*) if

$$A,\ A_1, A_2, \dots \in \mathcal{D},\ A \subset \bigcup_{k=1}^\infty A_k \implies \mu(A) \le \sum_{k=1}^\infty \mu(A_k);$$

1.8 Lemma `(addivity ⇒ finite subadditiivty)` *If $\mathcal{S} \subset 2^X$ is a semi-algebra, and $\mu : \mathcal{S} \to [0, \infty)$ is additive, then it is finitely subadditive.*

`Proof` Let $\mathcal{A} := \mathcal{A}(\mathcal{S})$ and $\widetilde{\mu} : \mathcal{A} \to [0, \infty)$ be additive so that $\widetilde{\mu}|_{\mathcal{S}} \equiv \mu$ (as in proposition 1.7). Additivity $\Rightarrow$ *monotonicity* i.e.

$$A,\ B \in \mathcal{A},\ A \subset B \implies \widetilde{\mu}(B) = \widetilde{\mu}(A) + \widetilde{\mu}(B \setminus A) \ge \widetilde{\mu}(A).$$

Now suppose that

$$A,\ A_1, \dots, A_N \in \mathcal{S} \ \ \&\ \ A \subset \bigcup_{k=1}^N A_k.$$

Let

$$B_1 = A_1,\ B_k := A_k \setminus \bigcup_{j=1}^{k-1} A_j \quad (2 \le k \le N),$$

then

$$B_1, \dots,\ B_N \in \mathcal{A} \ \ \&\ \ \bigcup_{k=1}^N A_k = \dot{\bigcup_{k=1}^N} B_k.$$

Thus

$$\begin{aligned}\mu(A) = \widetilde{\mu}(A) &\le \widetilde{\mu}(\bigcup_{k=1}^N A_k) = \widetilde{\mu}(\dot{\bigcup_{k=1}^N} B_k) \\ &= \sum_{k=1}^N \widetilde{\mu}(B_k) \le \sum_{k=1}^N \widetilde{\mu}(A_k) = \sum_{k=1}^N \mu(A_k). \quad ☑\end{aligned}$$

1.9 Proposition `finite subcover property &` σ`-subadditivity`
Let $\mathcal{C}$ be a semi-ring of subsets of X with the finite subcover property. Any additive $\mu : \mathcal{C} \to [0, \infty)$ is countable subadditive.

`Proof` Suppose that $C, C_1, C_2, \dots \in \mathcal{S}$ and $C \subseteqq \bigcup_{j\ge 1} C_j$. By assumption, $\exists\ N \ge 1$ so that $C \subseteqq \bigcup_{j=1}^N C_j$ and by finite subadditivity,

$$\mu(C) \le \sum_{j=1}^N \mu(C_j) \le \sum_{j=1}^\infty \mu(C_j). \quad \square$$

1.10 Example `additivity` $\not\Rightarrow$ `countable subadditivity.`

Let $X := \mathbb{Q} \cap [0,1])$, $\mathcal{Q} := \{R \cap \mathbb{Q} :\ R \in \mathcal{S}([0,1])\}$ and define $w : \mathcal{Q} \to [0,1]$ by $w(R \cap \mathbb{Q}) := |R|$ (the length of R), then $\mathcal{Q}$ is a semi-algebra and w is additive. But w is not countable subadditive since X is countable.

To see this, write $X = \{q_n :\ n \ge 1\}$ & $I_n := \{q_n\}$, then $I_n = [q_n, q_n] \in \mathcal{Q}$, $w(I_n) = 0$ and

$X = \bigcup_{n=1}^\infty I_n$ but $w(X) = 1$ & $\sum_{n=1}^\infty w(I_n) = 0$.

1.11 Proposition: `countable subadditivity of volume on` $\mathbb{R}^d$

Let B be a box and let $\mathcal{S} = \mathcal{S}(B)$ be the semi-algebra of sub-boxes of B, and let $\mu(R) = |R|$ as before, then $\mu : \mathcal{S} \to [0, \infty)$ is countable subadditive.

Remark. $\mathcal{S}(B)$ does not have the finite subcover property.

Proof

Suppose that $R, R_1, R_2, \dots \in \mathcal{S}$ and that $R \subset \bigcup_{n=1}^\infty R_n$. Fix $\varepsilon > 0$. For each $n \ge 1$, $\exists$ an open box $R_n' \supseteq R_n$ so that $\mu(R_n') < \mu(R_n) + \frac{\varepsilon}{2^n}$, and $\exists$ a compact box $R' \subseteq R$ so that $\mu(R') > \mu(R) - \varepsilon$. Evidently, $R' \subset \bigcup_{n=1}^\infty R_n'$ so that $\{R_n' :\ n \ge 1\}$ is an open cover of the compact R'. Thus $\exists\ N \ge 1$ so that $R' \subset \bigcup_{n=1}^N R_n'$. By finite subadditivity, $\mu(R') \le \sum_{n=1}^N \mu(R_n')$, whence

$(\ast)$

$$\mu(R) \le \mu(R') + \varepsilon \le \sum_{n=1}^N \mu(R_n') + \varepsilon \le \sum_{n=1}^N (\mu(R_n) + \frac{\varepsilon}{2^n}) + \varepsilon \le \sum_{n=1}^\infty \mu(R_n) + 2\varepsilon.$$

$\square$

Exercise N$^{\underline{\text{o}}}$1

1. Product spaces and cylinder sets. Let $\{X_\lambda :\ \lambda \in \Lambda\}$ be a collection of sets. Set

$$X = \prod_{\lambda \in \Lambda} X_\lambda := \{x : \Lambda \to \bigcup_{\lambda \in \Lambda} X_\lambda :\ x(\lambda) \in X_\lambda\ \forall\ \lambda \in \Lambda\}.$$

Given $N \in \mathbb{N}$, $\lambda_i \in \Lambda$, $A_i \subset X_{\lambda_i}$ $(1 \le i \le N)$ define the *cylinder set*

$$[A_1, \dots, A_N]_{\lambda_1, \dots, \lambda_N} := \{x \in X :\ x(\lambda_i) \in A_i\ \forall\ 1 \le i \le N\}.$$

Suppose that for each $\lambda \in \Lambda$, $\mathcal{S}_\lambda \subset 2^{X_\lambda}$ and let

$$\mathcal{S} := \{[A_1, \dots, A_N]_{\lambda_1,\dots,\lambda_N} : N \in \mathbb{N}, \ \lambda_i \in \Lambda, \ A_i \in \mathcal{S}_{\lambda_i} \ \ (1 \le i \le N)\} \subset 2^X.$$

Show that if for each $\lambda \in \Lambda$, $\mathcal{S}_\lambda$ is a semi-algebra, then $\mathcal{S}$ is a semi-algebra and that if in addition, for each $\lambda \in \Lambda$, $\mathcal{S}_\lambda$ has the finite subcover property, then so does $\mathcal{S}$.

2. Additive set functions on S^Λ.

Let S be a finite set, Λ be a set and let $X := S^\Lambda$.

A *cylinder set* is a set of form

$$\{x \in X : \ x(i) = a(i) \ \forall \ i \in F\} =: [a]_F$$

where $F \subset \Lambda$ finite and $a \in S^F$. Let $\mathcal{S} = \{\texttt{cylinders}\}$.

(i) Suppose that $[a_i]_{F_i} \in \mathcal{S}$ $(1 \le i \le N)$ are disjoint and that $C := \biguplus_{i=1}^N [a_i]_{F_i} \in \mathcal{S}$. Show that $\exists$ $G \subset S^F$ (where $F := \bigcup_{i=1}^N F_i$) so that $C = \biguplus_{g \in G} [g]_F$.

Now suppose that $\mu : \mathcal{S} \to [0, \infty)$ satisfies

$$(+) \qquad \sum_{s \in S} \mu([a_1, \dots, a_N, s]_{\lambda_1,\dots,\lambda_N,\lambda}) = \mu([a_1, \dots, a_N]_{\lambda_1,\dots,\lambda_N}).$$

(ii) Show that if $F \subset \Lambda$ is finite, $G \subset S^F$ and $C := \biguplus_{g \in G} [g]_F \in \mathcal{S}$, then $\mu(C) = \sum_{g \in G} \mu([g]_F)$.

(iii) Using (i) and (ii) (or otherwise) show that $\mu : \mathcal{S} \to [0, \infty)$ is additive.

3. Stieltjes set functions.

Let $R \subset \mathbb{R}$ be a closed, bounded interval and let $\mathcal{S} = \mathcal{S}(R)$ be the semi-algebra of sub-intervals of R.

(a) Given $F : R \to \mathbb{R}$ non-decreasing define $\nu_F : \mathcal{S} \to \mathbb{R}$ as follows. Given $I \in \mathcal{S}$, $\overline{I} = [a, b]$, set

$$\nu_F(I) = \begin{cases} F(b+) - F(a-) & I = [a, b], \\ F(b+) - F(a+) & I = (a, b], \\ F(b-) - F(a-) & I = [a, b), \\ F(b-) - F(a+) & I = (a, b) \end{cases}$$

where

$$F(x+) := \lim_{y \to x, \ y > x} F(y) \quad \& \quad F(x-) := \lim_{y \to x, \ y < x} F(y).$$

Show that $\nu_F : \mathcal{S} \to [0, 1]$ is additive and countable subadditive.

(b) Does $\mathcal{S}$ have the $\texttt{FSCP}$?

4. Finitely additive & infinite total variation?

Let $X := \mathbb{R}_+$ and let $\mathcal{A}$ be the algebra generated by finite intervals. Let $f : X \to \mathbb{R}$ be unformly continuous so that $\exists\ \lim_{R\to\infty} \int_0^R f(x)dx \in \mathbb{R}$. Here $\int$ denotes the Riemann integral.

(i) Show that for $A \in \mathcal{A}$, $\exists\ \lim_{R\to\infty} \int_{A\cap[0,R]} f(x)dx =: \mu(A)$ and that $\mu : \mathcal{A} \to \mathbb{R}$ is additive.

(ii) Show that $\|\mu\| = \lim_{R\to\infty} \int_0^R |f(x)|dx$.

5. Countably infinite σ-algebras?

Is there a set X and a σ-algebra $\mathcal{B} \subset 2^X$ which is countably infinite?

Week # 2

Monotone classes, σ-algebras and measures

Let X be a set. A class $\mathfrak{M}$ of subsets of X is called a *monotone class* if

$$A_n \in \mathfrak{M}\ (n \ge 1),\ A_n \xrightarrow{\text{mon}} A \quad \Rightarrow\ A \in \mathfrak{M}.$$

Here $A_n \xrightarrow{\text{mon}} A$ means
either $A_n \uparrow A$ i.e. $A_n \subset A_{n+1}$ & $A = \bigcup_{n=1}^\infty A_n$;
or $A_n \downarrow A$ i.e. $A_n \supset A_{n+1}$ & $A = \bigcap_{n=1}^\infty A_n$.

An algebra $\mathcal{B} \subset 2^X$ is called a *σ-algebra* if in addition

$$A_n \in \mathcal{B}\ (n \ge 1) \quad \Rightarrow \bigcup_{n=1}^\infty A_n \in \mathcal{B}.$$

Evidently a σ-algebra is also a monotone class, and an algebra which is a monotone class is also a σ-algebra.

2.1 Intersection proposition

Let $\mathfrak{R} \subset 2^X$ then:

$$\mathfrak{M}(\mathfrak{R}) := \bigcap_{\mathfrak{M}\supset\mathfrak{R}\ \text{a monotone class}} \mathfrak{M}$$

is a monotone class (known as the **monotone class generated by** $\mathfrak{R}$*);*

$$\sigma(\mathfrak{R}) := \bigcap_{\mathfrak{B}\supset\mathfrak{R}\ \text{a } \sigma\text{-algebra}} \mathfrak{B}$$

is a σ-algebra (known as the **σ-algebra generated by** $\mathfrak{R}$*);*

2.2 Monotone Class Theorem

Let $\mathcal{A}$ be an algebra of subsets of X, then

$$\mathfrak{M}(\mathcal{A}) = \sigma(\mathcal{A}).$$

Proof

Evidently $\mathfrak{M}(\mathcal{A}) \subset \sigma(\mathcal{A})$ as $\sigma(\mathcal{A})$ is a monotone class.

It suffices to show that $\mathfrak{M}(\mathcal{A})$ is an algebra, for then (being a monotone class) it is a σ-algebra and $\mathfrak{M}(\mathcal{A}) \supset \sigma(\mathcal{A})$.

To see this fix any $F \subset X$ set

$$\mathcal{L}(F) = \{E \subset X : E \cup F, E \cap F, E \cap F^c, E^c \cap F \in \mathfrak{M}(\mathcal{A})\}.$$

It follows that

- $E \in \mathcal{L}(F)$ iff $F \in \mathcal{L}(E)$;
- if $\mathcal{L}(F) \neq \varnothing$, then $\mathcal{L}(F)$ is a monotone class;
- $\mathcal{L}(A) \supset \mathfrak{M}(\mathcal{A}) \ \forall \ A \in \mathcal{A}$;
- $\mathcal{L}(A) \supset \mathfrak{M}(\mathcal{A}) \ \forall \ A \in \mathfrak{M}(\mathcal{A})$;
- $\mathfrak{M}(\mathcal{A})$ is an algebra. ☐

Let $\mathcal{A}$ be an algebra of subsets of X. The function $\nu : \sigma(\mathcal{A}) \to \mathbb{R}$ is *σ-additive* if it is additive and

$$A_n \in \sigma(A),\ A_n \uparrow A \implies \nu(A_n) \to \nu(A).$$

From this we obtain (!) that

$$A_n \in \sigma(A),\ A_n \downarrow A \implies \nu(A_n) \to \nu(A).$$

2.3 Proposition (Unicity of extension)

Let X be a set, let $\mathcal{A}$ be an algebra of subsets of X, and let $\mu_i : \sigma(\mathcal{A}) \to \mathbb{R}$ $(i = 1, 2)$ be σ-additive such that $\mu_1(A) = \mu_2(A) \ \forall \ A \in \mathcal{A}$, then $\mu_1 \equiv \mu_2$.

Proof

Set $\mathcal{C} := \{A \in \sigma(\mathcal{A}) : \ \mu_1(A) = \mu_2(A)\}$. Evidently $\mathcal{C} \supset \mathcal{A}$ and by σ-additivity, $\mathcal{C}$ is a monotone class, whence $\mathcal{C} \supset \mathfrak{M}(\mathcal{A}) = \sigma(\mathcal{A})$. ☐

2.4 Proposition (Approximation of a non-negative extension)

Let X be a set, let $\mathcal{A}$ be an algebra of subsets of X, and let $\mu : \sigma(\mathcal{A}) \to [0, \infty)$ be σ-additive, then $\forall \ A \in \sigma(\mathcal{A})$, $\varepsilon > 0$, $\exists \ A_0 \in \mathcal{A}$ such that $\mu(A \Delta A_0) < \varepsilon$.

Proof

Set $\mathcal{C} := \{A \in \sigma(\mathcal{A}) : \ \forall \ \exists \ A_0 \in \mathcal{A}, \ \mu(A\Delta A_0) < \varepsilon\}$, then $\mathcal{C} \supset \mathcal{A}$. We show directly that $\mathcal{C}$ is a σ-algebra.

If $A \in \sigma(\mathcal{A}), \ B \in \mathcal{A}$ then $B^c \in \mathcal{A}, \ A\Delta B = A^c \Delta B^c$ whence $A \in \mathcal{C} \ \Rightarrow A^c \in \mathcal{C}$.

Suppose $A, B \in \mathcal{C}$ and $\varepsilon > 0$, then $\exists \ A_0, B_0 \in \mathcal{A}$ with $\mu(A\Delta A_0) + \mu(B\Delta B_0) < \varepsilon$. Now $A_0 \cup B_0 \in \mathcal{A}$ and $(A \cup B)\Delta(A_0 \cup B_0) \subset (A\Delta A_0) \cup (B\Delta B_0)$ whence $\mu((A \cup B)\Delta(A_0 \cup B_0)) \le \mu(A\Delta A_0) + \mu(B\Delta B_0) < \varepsilon$. Thus $A \cup B \in \mathcal{C}$. This shows that $\mathcal{C}$ is an algebra.

Now let $C_n \in \mathcal{C} \ \ (n \ge 1)$. We show that $C := \bigcup_{n\ge 1} C_n \in \mathcal{C}$. To see this, set $A_1 = C_1, \ A_n := C_n \setminus \bigcup_{k=1}^{n-1} C_k \ \ (n \ge 2)$, then $A_1, A_2, \dots \in \mathcal{C}$ and $C = \biguplus_{n\ge 1} A_n$.

Fix $\varepsilon > 0$, then $\exists$:

- n_ε such that $\mu(\biguplus_{k>n_\varepsilon} A_k) < \frac{\varepsilon}{2}$,
- $D \in \mathcal{A}$ such that $m(D\Delta \biguplus_{k=1}^{n_\varepsilon} A_k) < \frac{\varepsilon}{2}$, then

$$C\Delta D \subset (\bigcup_{k=1}^{n_\varepsilon} A_k)\Delta D \uplus \biguplus_{j>n_\varepsilon} A_j,$$

whence

$$\mu(C\Delta D) \le \mu((\bigcup_{k=1}^{n_\varepsilon} A_k)\Delta D) + \mu(\biguplus_{j>n_\varepsilon} A_j) < \varepsilon.$$

Thus $\mathcal{C}$ is a σ-algebra and $\mathcal{C} \supset \sigma(\mathcal{A})$. ☐

Outer measures. Definition: Outer measure

An *outer measure* on X is a function $\overline{\mu} : 2^X \to [0, \infty]$ satisfying:

- $\overline{\mu}(\varnothing) = 0$;
- $\overline{\mu}(A) \le \overline{\mu}(B)$ whenever $A \subseteq B$ (*monotonicity*);
- $\overline{\mu}(\bigcup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty \overline{\mu}(A_n) \ \forall \ A_1, A_2, \dots \subset X$ (*sub-σ-additivity*).

2.5 Caratheodory's construction theorem

Let X be a set, and let $\overline{\mu}$ be an outer measure on X. Then

$$\mathcal{M} := \{E \subset X : \overline{\mu}(A) = \overline{\mu}(A \cap E) + \overline{\mu}(A \cap E^c) \ \ \forall \ A \subset X\}$$

is a σ-algebra, and $\overline{\mu}|_{\mathcal{M}}$ is a measure.

`Proof of Caratheodory's construction theorem`

Step 1 $\mathcal{M}$ *is an algebra.*

Proof Clearly $E \in \mathcal{M} \Leftrightarrow \ E^c \in \mathcal{M}$. Also $\varnothing \in \mathcal{M}$ since $\overline{\mu}(\varnothing) = 0$. We must show that $E, F \in \mathcal{M} \ \Rightarrow \ E \cup F \in \mathcal{M}$. Fix $A \subset X$. Since $E \in \mathcal{M}$,

$$\overline{\mu}(A) = \overline{\mu}(A \cap E) + \overline{\mu}(A \cap E^c),$$

and since $F \in \mathcal{M}$,

$$\overline{\mu}(A \cap E^c) = \overline{\mu}(A \cap E^c \cap F) + \overline{\mu}(A \cap E^c \cap F^c).$$

$$\begin{aligned}\therefore \overline{\mu}(A) &= \overline{\mu}(A \cap E) + \overline{\mu}(A \cap E^c) \\ &= \overline{\mu}(A \cap E) + \overline{\mu}(A \cap E^c \cap F) + \overline{\mu}(A \cap E^c \cap F^c) \\ &\geq \overline{\mu}(A \cap (E \cup F)) + \overline{\mu}(A \cap (E \cup F)^c)\end{aligned}$$

by subadditivity since $A \cap (E \cup F) = (A \cap E) \uplus (A \cap E^c \cap F)$. ☐

Step 2 *For $E_1, \dots, E_n \in \mathcal{M}$ disjoint:*

$$\overline{\mu}(A \cap \bigcup_{k=1}^{n}{}^{\!\!\!\cdot}\, E_k) = \sum_{k=1}^{n} \overline{\mu}(A \cap E_k) \;\; \forall \;\; A \subset X.$$

Proof Suppose $E, F \in \mathcal{M}$ are disjoint, and let $A \subset X$.

$$\begin{aligned}\overline{\mu}(A \cap (E \uplus F)) &= \overline{\mu}(A \cap (E \uplus F) \cap E) + \overline{\mu}(A \cap (E \uplus F) \cap E^c) \;\; \because \;\; E \in \mathcal{M} \\ &= \overline{\mu}(A \cap E) + \overline{\mu}(A \cap F) \;\; \because \;\; E \cap F = \varnothing.\end{aligned}$$

☐

Step 3 *If $E_1, E_2, \dots \in \mathcal{M}$ are disjoint, then*

$$\overline{\mu}(A \cap E) = \sum_{n=1}^{\infty} \overline{\mu}(A \cap E_n) \;\; \forall \;\; A \subset X, \;\; \textit{where } E := \bigcup_{n=1}^{\infty} E_n.$$

Note that it is not assumed that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$.

Proof Suppose $A \subset X$, then

$$\begin{aligned}\sum_{n=1}^{\infty} \overline{\mu}(A \cap E_n) &\geq \overline{\mu}(A \cap E) \geq \overline{\mu}(A \cap \bigcup_{n=1}^{N} E_n) \\ &= \sum_{n=1}^{N} \overline{\mu}(A \cap E_n) \to \sum_{n=1}^{\infty} \overline{\mu}(A \cap E_n) \geq \overline{\mu}(A \cap E)\end{aligned}$$

as $N \to \infty$. ☐

Step 4 *$\mathcal{M}$ is a σ-algebra.*

Proof Let $E_1, E_2, \dots \in \mathcal{M}$ and let

$$E := \bigcup_{n=1}^{\infty} E_n.$$

Set

$$F_1 = E_1, \quad F_{n+1} := E_{n+1} \setminus \bigcup_{k=1}^{n} E_k \;\; (n \geq 1),$$

then $F_1, F_2, \dots \in \mathcal{M}$, and are disjoint, whence for $A \subset X$,

$$\begin{aligned}\overline{\mu}(A) &= \overline{\mu}(A \cap \biguplus_{k=1}^{n} F_k) + \overline{\mu}(A \cap (\biguplus_{k=1}^{n} F_k)^c) \ (\forall\ n \geq 1)\\ &\geq \sum_{k=1}^{n} \overline{\mu}(A \cap F_k) + \overline{\mu}(A \cap E^c)\\ &\to \overline{\mu}(A \cap E) + \overline{\mu}(A \cap E^c) \text{ as } n \to \infty.\end{aligned}$$

☐

2.6 Caratheodory's extension theorem *Let X be a set, $\mathcal{S}$ be an semi-ring of subsets of X which σ-covers X.*

Suppose that $\mu : \mathcal{S} \to [0, \infty)$ is additive and countable subadditive, then there is a measure $\overline{\mu} : \sigma(\mathcal{S}) \to [0, \infty]$ such that $\overline{\mu}|_{\mathcal{S}} \equiv \mu$.

Proof Define, for $E \subset X$ (using the σ-covering property),

$$\overline{\mu}(E) := \inf \{\sum_{n=1}^{\infty} \mu(A_n) : \ A_1, A_2, \dots \in \mathcal{S},\ E \subset \bigcup_{n=1}^{\infty} A_n\}.$$

Let $\widehat{\mu} : \mathcal{R}(\mathcal{S}) \to [0, \infty)$ is the additive extension of μ obtained using 1.5.

- We claim first that $\overline{\mu} \equiv \widetilde{\mu}$ where

$$\widetilde{\mu}(E) := \inf \{\sum_{n=1}^{\infty} \widehat{\mu}(A_n) : \ A_1, A_2, \dots \in \mathcal{R}(\mathcal{S}),\ E \subset \bigcup_{n=1}^{\infty} A_n\}.$$

Since $\mathcal{R}(\mathcal{S}) \supset \mathcal{S}$ and $\widehat{\mu}|_{\mathcal{S}} \equiv \mu$, we have $\widetilde{\mu} \leq \overline{\mu}$.

For the reverse inequality, suppose that

$$E \subset X, \quad A_1, A_2, \dots \in \mathcal{R}(\mathcal{S}),\ E \subset \bigcup_{n=1}^{\infty} A_n.$$

By 1.4, for each $n \geq 1,\ \exists\ N_n \geq 1,\ C_{n,1}, \dots, C_{n,N_n} \in \mathcal{S}$ disjoint, such that $A_n = \biguplus_{k=1}^{N_n} C_{n,k}$. It follows that $E \subset \bigcup_{n=1}^{\infty} \biguplus_{k=1}^{N_n} C_{n,k}$, whence

$$\overline{\mu}(E) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{N_n} \mu(C_{n,k}) = \sum_{n=1}^{\infty} \widehat{\mu}(A_n).$$

- Next, we show that $\overline{\mu}$ is an outer measure.

Evidently $\overline{\mu}(\varnothing) \leq \mu(\varnothing) = 0$. Monotonicity is immediate. To see sub-σ-additivity, suppose that $E = \bigcup_{n \geq 1} E_n$ and let $\varepsilon > 0$. Choose $A_{n,1}, A_{n,2}, \dots \in \mathcal{S} \quad (n \geq 1)$ so that $E_n \subset \bigcup_{k \geq 1} A_{n,k}$ and $\sum_{k \geq 1} \mu(A_{n,k}) \leq \overline{\mu}(E_n) + \frac{\varepsilon}{2^n} \quad (n \geq 1)$. It follows that $E_n \subset \bigcup_{n,k \geq 1} A_{n,k}$, whence

$$\overline{\mu}(E) \leq \sum_{n,k \geq 1} \mu(A_{n,k}) \leq \sum_{n \geq 1} (\overline{\mu}(E_n) + \frac{\varepsilon}{2^n}) = \sum_{n \geq 1} \overline{\mu}(E_n) + \varepsilon.$$

• Continuing, we note that $\overline{\mu}(A) = \mu(A)\ \forall\ A \in \mathcal{S}$ by countable subadditivity.

• Finally, we show that $\mathcal{S} \subset \mathcal{M}$. This will establish the extension theorem by Caratheodory's construction theorem (2.1).

For this, it suffices to show that for $B \in \mathcal{S}$:

$$\overline{\mu}(F) \geq \overline{\mu}(F \cap B) + \overline{\mu}(F \cap B^c)\ \forall\ F \subset X.$$

To this end, fix $\varepsilon > 0$ and let $A_1, A_2, \cdots \in \mathcal{S}$ be such that

$$F \subset \bigcup_{n=1}^{\infty} A_n, \quad \overline{\mu}(F) + \varepsilon \geq \sum_{n=1}^{\infty} \mu(A_n).$$

For each $n \geq 1$, we have that

$$\mu(A_n) = \mu(A_n \cap B) + \widehat{\mu}(A_n \cap B^c)$$

where $\widehat{\mu} : \mathcal{R}(\mathcal{S}) \to [0, \infty)$ is the additive extension of μ obtained using 1.5.

It follows that

$$\begin{aligned}
\overline{\mu}(F) + \varepsilon &\geq \sum_{n=1}^{\infty} \mu(A_n) \\
&= \sum_{n=1}^{\infty} (\mu(A_n \cap B) + \widehat{\mu}(A_n \cap B^c)) \\
&= \sum_{n=1}^{\infty} \mu(A_n \cap B) + \sum_{n=1}^{\infty} \widehat{\mu}(A_n \cap B^c) \\
&\geq \overline{\mu}(F \cap B) + \widetilde{\mu}(F \cap B^c) \\
&= \overline{\mu}(F \cap B) + \overline{\mu}(F \cap B^c). \quad ☑
\end{aligned}$$

Lebesgue measure on $\mathbb{R}^d$.

Let $\mathcal{S}$ be the semi-ring of boxes in $\mathbb{R}^d$ which σ-covers $\mathbb{R}^d$, and let $v(R) = |R|$ as before. By 1.4 and 1.11, $v : \mathcal{S} \to [0, \infty)$ is additive and countable subadditive. By Caratheodory's extension theorem, there is a measure (aka *Lebesgue measure* on $\sigma(\mathcal{S})$) extending $v : \mathcal{S} \to [0, \infty)$.

Bernoulli measure with finite state space. Let Λ be a set, S be a finite set, and let $X = X_\Lambda := S^\Lambda$. As in exercise 1.1, and given $N \geq 1,\ \lambda_1, \ldots, \lambda_N \in \Lambda,\ a_1, \ldots, a_N \in S$, define the *cylinder*

$$[a_1, \ldots, a_N]_{\lambda_1, \ldots, \lambda_N} := \{x \in X :\ x(\lambda_i) = a_i\ \forall\ 1 \leq i \leq N\}.$$

Recall from your topology course that with respect to the product discrete topology, X is a compact Hausdorff space and cylinder sets are both open and compact.

Note that the topological space X_Λ is metrizable iff Λ is countable.

Let

$$\mathcal{S} := \{[a_1, \dots, a_N]_{\lambda_1,\dots,\lambda_N} : \ N \in \mathbb{N}, \ \lambda_i \in \Lambda, \ a_i \in S \ \ (1 \le i \le N)\} \cup \{\varnothing\},$$

then (ex. 1.1(a)) $\mathcal{S}$ is a semi-algebra.

Let $p : S \to (0,1)$ be a probability ($\sum_{s\in S} p(s) = 1$). Define $P : \mathcal{S} \to [0,\infty)$ by $P(\varnothing) := 0$ and $P([a_1, \dots, a_N]) := \prod_{k=1}^N p_{a_k}$. Fix $\lambda_1, \dots, \lambda_N, \lambda \in \Lambda$. Since

$$P([a_1, \dots, a_N, s]_{\lambda_1,\dots,\lambda_N,\lambda}) = P([a_1, \dots, a_N]_{\lambda_1,\dots,\lambda_N}) p_s \ \forall \ a_1, \dots, a_N, s \in S,$$

we have

$$(+) \qquad \sum_{s\in S} P([a_1, \dots, a_N, s]_{\lambda_1,\dots,\lambda_N,\lambda}) = \mu([a_1, \dots, a_N]_{\lambda_1,\dots,\lambda_N}).$$

and P is additive by exercise 1.2 and therefore finitely subadditive. Because each member of $\mathcal{S}$ is both compact and open, $\mathcal{S}$ has the `FSCP`, P is countable subadditive and $\exists$ a probability $P : \sigma(\mathcal{S}) \to [0,1]$ so that

$$P([a_1, \dots, a_N]_{\lambda_1,\dots,\lambda_N}) = \prod_{k=1}^N p_{a_k}.$$

Lebesgue measure and coin tossing.

Let $\Omega = \{0,1\}^{\mathbb{N}}$ and let P be as above with $p(0) = p(1) = \frac{1}{2}$. Define $\Phi : \Omega \to [0,1]$ by

$$\Phi(\omega_1, \omega_2, \dots) := \sum_{n\ge1} \frac{\omega_n}{2^n}.$$

This is a continuous map, therefore (!!) *Borel measurable* i.e.

$$A \in \mathcal{B}([0,1]) \implies \Phi^{-1} A \in \mathcal{B}(\Omega).$$

We claim

$$P(\Phi^{-1} A) = \mathtt{Leb}\,(A).$$

To see this, let A be a closed `dyadic interval`, i.e. of form

$$A = \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right] \ \text{ where } p, \ n \in \mathbb{N}, \ p < 2^n,$$

then (!)

$$\Phi^{-1}(A) = [a_1, a_2, \dots, a_n] \ \text{ where } a_1, a_2, \dots, a_n = 0, \ 1 \ \& \ \frac{p}{2^n} = \sum_{k=1}^n \frac{a_k}{2^k}.$$

Consequently

$$P(\Phi^{-1}(A)) = \frac{1}{2^n} = \mathtt{Leb}\,(A).$$

The collection of measurable sets A with this property is a σ-algebra (!!) and the claim is proved. ☑

Outer measures on metric spaces. Let (X, d) be a metric space. A *metric* outer measure on X is an outer measure $\overline{\mu} : 2^X \to [0, \infty]$ with the property that

$$A, B \subset X,\ d(A, B) := \inf_{x \in A, y \in B} d(x, y) > 0 \ \Rightarrow\ \overline{\mu}(A \cup B) = \overline{\mu}(A) + \overline{\mu}(B).$$

Example 1: `Metric outer measures from regular Borel measures.`

Let (X, d) be a metric space and let μ be a regular, finite Borel measure on X. Define $\overline{\mu} : 2^X \to [0, \infty)$ by $\overline{\mu}(A) := \inf\{\mu(U) :\ A \subset U \in \mathcal{B}(X)\}$.

Proposition 2.7 $\quad \overline{\mu} : 2^X \to [0, \infty]$ *is a metric outer measure.*

`Proof` It is easy to see that $\overline{\mu}$ is an outer measure.

Note also that

(☢) $\qquad \forall\ A \subset X,\ \exists\ B = B_A \in \mathcal{B}(X),\ B \supset A,\ \overline{\mu}(A) = \mu(B).$

In fact if $A \subset U_n$ open with $\mu(U_n) < \overline{\mu}(A) + \frac{1}{n}$ then one choice is the G_δ set $B_A = \bigcap_{n \geq 1} U_n$.

If $E,\ F \subset X$ and $d(E, F) > 0$, then the sets $B_E,\ B_F \in \mathcal{B}$ in (☢) can be chosen disjoint since if $d(E, F) > 3\varepsilon > 0$, then $E \subset B_o(E, \varepsilon)$ & $F \subset B_o(F, \varepsilon)$ (where $B_o(A, \varepsilon) := \bigcup_{x \in A} B_o(x, \varepsilon)$); the sets $B_o(E, \varepsilon)$ & $B_o(F, \varepsilon)$ being open and disjoint.

To see that $\overline{\mu}$ is a metric outer measure, fix $E,\ F \subset X,\ d(E, F) > 0$.

Let $B_E,\ B_F,\ B_{E \cup F} \in \mathcal{B}$ satisfy (☢) with $B_E \cap B_F = \varnothing$ and (WLOG) $B_E \uplus B_F \subset B_{E \cup F}$. It follows that

$$\begin{aligned}
\overline{\mu}(E \uplus F) &= \mu(B_{E \uplus F}) \\
&\geq \mu(B_E \uplus B_F) \\
&= \mu(B_E + \mu(B_F) \\
&= \overline{\mu}(E) + \overline{\mu}(F).\ \boxtimes
\end{aligned}$$

Pre-masses.

Let X be a set and let $\mathcal{C} \subset 2^X$ with $\varnothing \in \mathcal{C}$. A *pre-mass on* $\mathcal{C}$ is a function $\tau : \mathcal{C} \to [0, \infty]$ satisfying $\tau(\varnothing) = 0$.

Example: `Hausdorff-type pre-masses.`

For (X,d) be a metric space and $a:[0,\infty)\to[0,\infty]$ with $a(0)=0$, $\tau_a:2^X\to[0,\infty]$ defined by

$$\tau_a(A):=a(\mathtt{diam}\,A)$$

is a pre-mass.

Proposition 2.8 *Let $\tau:\mathcal{C}\to[0,\infty]$ be a pre-mass. The set function $\overline{\mu}=\overline{\mu}_{\tau,\mathcal{C}}:2^X\to[0,\infty]$ defined by*

$$\overline{\mu}_{\tau,\mathcal{C}}(A):=\inf\{\sum_{n\ge1}\tau(C_n):\ C_n\in\mathcal{C}\ \forall\ n\ \&\ A\subset\bigcup_{n\ge1}C_n\}$$

(with $\inf\varnothing:=\infty$*) is an outer measure.*

Proof (As in the proof of theorem 2.6).

Evidently $\overline{\mu}(\varnothing)\le\tau(\varnothing)=0$. Monotonicity is immediate. To see sub-σ-additivity, suppose that $E=\bigcup_{n\ge1}E_n$. and let $\varepsilon>0$. WLOG, $\overline{\mu}(E_n)<\infty\ \forall\ n$ and we may choose $A_{n,1},A_{n,2},\dots\in\mathcal{C}\ \ (n\ge1)$ so that $E_n\subset\bigcup_{k\ge1}A_{n,k}$ and $\sum_{k\ge1}\tau(A_{n,k})\le\overline{\mu}(E_n)+\frac{\varepsilon}{2^n}\ \ (n\ge1)$. It follows that $E_n\subset\bigcup_{n,k\ge1}A_{n,k}$, whence

$$\overline{\mu}(E)\le\sum_{n,k\ge1}\tau(A_{n,k})\le\sum_{n\ge1}(\overline{\mu}(E_n)+\frac{\varepsilon}{2^n})=\sum_{n\ge1}\overline{\mu}(E_n)+\varepsilon.\ \ ☑$$

Pre-masses on a metric space.

Let (X,d) be a metric space and let $\tau:\mathcal{C}\to[0,\infty]$ be a pre-mass.

For $r>0$, let

$$\mathcal{C}_r:=\{A\in\mathcal{C}:\ \mathtt{diam}\,A\ <\ r\}$$

and let $\mu_r=\overline{\mu}_{\tau,\mathcal{C}_r}$ be the outer measure as in proposition 1.

Proposition 2.9 *The set function $\mu:2^X\to[0,\infty]$ defined by*

$$\mu(A):=\lim_{r\to0}\mu_r(A)$$

is a metric outer measure.

Proof By proposition 1, each μ_r is an outer measure, whence so is $\mu=\sup_{r\to0}\mu_r$.

To check the metric property, fix $A,\ B\subset X,\ d(A,B)>0\ \&\ \mu(A),\ \mu(B)<\infty$ (the latter ensuring countable covers of A & B by elements of $\mathcal{C}_r\ \forall\ r>0$).

It suffices to show that

$$\mu_r(A\cup B)=\mu_r(A)+\mu_r(B)\ \ \forall\ r<\frac{d(A,B)}{3}$$

which in turn is true since if $A \cup B \subset \bigcup_{n\ge 1} C_n$ where $\mathtt{diam}\, C_n < \frac{d(A,B)}{3}$, then $\exists\ K \subset \mathbb{N}$ such that

$$A \subset \bigcup_{n\in K} C_n \ \ \&\ \ B \subset \bigcup_{n\notin K} C_n. \quad ☑$$

Theorem (Caratheodory) *If $\overline{\mu}$ is a metric outer measure on X, then $\mathcal{B}(X) \subset \mathcal{M}(\overline{\mu})$.*

Lemma *Let $\overline{\mu}$ be a metric outer measure on the metric space X. Then*

$$A_n \uparrow A,\ \overline{\mu}(A) < \infty,\ d(A_n, A \setminus A_{n+1}) > 0\ \forall\ n \ge 1\ \Rightarrow\ \overline{\mu}(A_n) \uparrow\ \overline{\mu}(A).$$

Proof Without loss of generality, $\overline{\mu}(A_n) \uparrow a < \infty$ as $n \to \infty$. We prove that $\overline{\mu}(A) \le a$.

Set $B_1 = A_1,\ B_n = A_n \setminus A_{n-1}\ (n \ge 2)$. We show first that

$$(☣) \qquad \sum_{k=1}^{\infty} \overline{\mu}(B_k) \ \le\ 2a.$$

`Proof` For $\ell \ge k+2,\ B_k \subset A_k,\ \&\ B_\ell \subset A \setminus A_{\ell-1} \subset A \setminus A_{k+1}$, and so

$$d(B_k, \bigcup_{\ell=k+2}^{\infty} B_\ell) > 0,$$

Therefore, for $\varepsilon = 0, 1$,

$$\begin{aligned}
a \ge \overline{\mu}(A_{2N}) &= \overline{\mu}\left(\bigcup_{k=1}^{2N} B_k\right) \\
&\ge \overline{\mu}\left(\bigcup_{k=1}^{N} B_{2k-\varepsilon}\right) \\
&= \overline{\mu}(B_{2-\varepsilon}) + \overline{\mu}\left(\bigcup_{k=2}^{N} B_{2k-\varepsilon}\right) \\
&\vdots \\
&= \sum_{k=1}^{N} \overline{\mu}(B_{2k-\varepsilon}) \\
&\xrightarrow[N\to\infty]{} \sum_{k=1}^{\infty} \overline{\mu}(B_{2k-\varepsilon}). \quad ☑\ (☣)
\end{aligned}$$

Using (☣),

$$\overline{\mu}(A) = \overline{\mu}\left(A_n \cup \bigcup_{k=n}^{\infty} B_k\right) \le \overline{\mu}(A_n) + \sum_{k=n}^{\infty} \overline{\mu}(B_k) \to a \text{ as } n \to \infty. \quad ☑$$

Proof of the theorem To prove the theorem, we show that for $B, E \subset X$, E closed,

$$\overline{\mu}(B) \ge \overline{\mu}(B \cap E) + \overline{\mu}(B \cap E^c)$$

Let $\varepsilon_j \downarrow 0$. Note that

$$B \supset (B \cap E) \cup (B \setminus B(E, \varepsilon_j)), \ \& \ d(B \cap E, B \setminus B(E, \varepsilon_j)) \ge \varepsilon_j$$

So

$$\overline{\mu}(B) \ge \overline{\mu}(B \cap E) + \lim_{j \to \infty} \overline{\mu}(B \setminus B(E, \varepsilon_j)).$$

Since E is closed, $B \setminus B(E, \varepsilon_j) \uparrow B \setminus E$.

We shall apply the lemma with $A_j = B \setminus B(E, \varepsilon_j)$ and $A = B \setminus E$. Evidently $A_k \subset B(E, \varepsilon_j)^c$ and $A \setminus A_{k+1} \subset B(E, \varepsilon_{k+1})$, whence $d(A_k, A \setminus A_{k+1}) \ge \varepsilon_k - \varepsilon_{k+1} > 0$. By the lemma, $\overline{\mu}(B \setminus B(E, \varepsilon_k)) = \overline{\mu}(A_k) \to \overline{\mu}(A) = \overline{\mu}(B \setminus E)$ whence

$$\overline{\mu}(B) \ge \overline{\mu}(B \cap E) + \overline{\mu}(B \setminus E).$$

□

N$^{\underline{\text{o}}}$Exercise 2

1. Outer measures.

Let $\overline{\mu} : 2^{\mathbb{R}} \to [0, \infty]$ be Lebesgue outer measure defined by

$$\overline{\mu}(E) := \inf\{\sum_{n=1}^{\infty} |I_n| : \ I_1, I_2, \dots \ \text{ intervals}, \ E \subseteqq \bigcup_{n=1}^{\infty} I_n\}.$$

a) Show that if $A \subset \mathbb{R}$, and $\overline{\mu}(A \cap J) \le \frac{1}{2}|J|$ for every interval J, then $\overline{\mu}(A) = 0$.

Let $A_n \subset [0, 1]$ $(n \ge 1)$.

b) Is it true that

$$A_1 \supset A_2 \supset \dots A_n \supset A_{n+1} \supset \dots \ \Rightarrow \ \overline{\mu}(A_n) \to \overline{\mu}(\bigcap_{n=1}^{\infty} A_n) \text{ as } n \to \infty?$$

2. An extended measure space. Let $(X, \mathcal{B}, m)$ be the unit interval equipped with Borel sets and Lebesgue measure.

a) Let $A \subset \mathbb{R}$, $\overline{\mu}(A) > 0$. Show that if $K \subset \mathbb{Q}$, $\overline{K} = \mathbb{R}$, then $\overline{\mu}(J \cap \bigcup_{k \in K}(k + A \mod 1)) = |J| \ \forall$ intervals J.

b) Show that $\exists$ a *partition* $\{E_n : \ n \ge 1\}$ of $[0, 1]$ (i.e. $E_n \subset [0, 1]$ disjoint & $\biguplus_{n \ge 1} E_n = [0, 1]$) with $\overline{\mu}(E_n)) = 1 \ \ \forall \ n \ge 1)$ where $\overline{\mu}$ denotes Lebesgue outer measure on $[0, 1]$.

`Hint` Take $E_n = \biguplus_{k \in K_n}(k + A \mod 1)$ for suitable $K_n \subset \mathbb{Q}$ and suitable (non-measurable) $A \subset X$.

Let $\{E_n : \ n \ge 1\}$ be a partition as in b), and let $\mathcal{B}_1$ be the σ-algebra generated by $\mathcal{B}$, and $\{E_n : \ n \ge 1\}$.

Show that

(c) $\mathcal{B}_1 = \{\biguplus_{n\ge 1} B_n \cap E_n : \ \ B_1, B_2, \dots \in \mathcal{B}\}$; that

(d) $$\biguplus_{n\ge 1} B_n \cap E_n = \biguplus_{n\ge 1} B_n' \cap E_n \Rightarrow$$
$$m(B_n \Delta B_n') = 0 \ \forall \ n \ge 1;$$

and that

(e) $$p((\biguplus_{n\ge 1} B_n \cap E_n) = \sum_{n\ge 1} \frac{m(B_n)}{2^n}$$

defines a probability $p : \mathcal{B}_1 \to [0,1]$ satisfying $p|_{\mathcal{B}} \equiv m$.

3 Bernoulli probabilities on countable shift spaces.

A (regular) *cylinder* in $X := \mathbb{N}^{\mathbb{N}}$ is a set of form

$$[n_1, n_2, \dots, n_k] := \{x \in X : \ x_j = n_j \ \forall \ 1 \le j \le k\}.$$

Let $\mathcal{C} := \{\texttt{cylinders}\}$.

In this exercise you construct, for any $p \in \mathcal{P}(\mathbb{N})$ with $p(s) > 0 \ \forall \ s \in \mathbb{N}$, a probability P on $(X, \sigma(\mathcal{C}))$ so that

(Å) $$P([a_1, a_2, \dots, a_N]) = \prod_{k=1}^{N} p(a_k) \quad \forall$$
$$[a_1, a_2, \dots, a_N] \in \mathcal{C}.$$

Let $\mathcal{A}_{\mathbb{N}}$ & $\mathcal{A}$ be as in example 2 on page5. Define $\mu : \mathcal{A}_{\mathbb{N}} = \{F, \ \mathbb{N} \setminus F : F \subset \mathbb{N} \setminus \{1\} \ \# F < \infty\}$ by $\mu(A) := \sum_{k \in A} p(a)$.

For $[A_1, A_2, \dots, A_N] \in \mathcal{A} := \{\mathcal{A}_{\mathbb{N}}\texttt{- cylinders}\}$, set

$$\widehat{\mu}([A_1, A_2, \dots, A_N]) := \prod_{k=1}^{M} \mu(A_k).$$

Show that

(i) $\widehat{\mu} : \mathcal{A} \to [0,1]$ is additive and countably subadditive;

(ii) there is a measure $P : \sigma(\mathcal{A})) \to [0,1]$ so that $P|_{\mathcal{A}} \equiv \widehat{\mu}$;

(iii) $\sigma(\mathcal{A}) = \sigma(\mathcal{C})$ and P satisfies (Å).

4. Regular Borel measures.

Let X be a topological space. The *Borel* σ-algebra is $\mathcal{B}(X) := \sigma(\{\texttt{open sets}\}$ and a *Borel probability* on X is a probability measure $P : \mathcal{B}(X) \to [0,1]$. In this exercise, you show that if X is a separable metric space, then Borel probabilities on X are *regular* in the sense that $P(A) = \inf \{P(U) : \ A \subset U, \ U \ \texttt{open}\}$.

Fix a Borel probability P on X and let

$$\mathcal{C} := \{A \in \mathcal{B}(X) : \ \forall\ \varepsilon > 0,\ \exists\ F \texttt{ closed}, U \texttt{ open},\ F \subset A \subset U,\ P(U \setminus F) < \varepsilon\}.$$

Show that:

(i) $\mathcal{C}$ is a σ-algebra;

(ii) $\{\texttt{open sets}\} \subseteq \mathcal{C}$;

(iii) P is regular.

5. Measurable union of a hereditary collection.

Let $(X, \mathcal{B}, m)$ be a finite measure space. Let $\mathfrak{H} \subset \mathcal{B}$ be a `hereditary collection` in the sense that

$$C \in \mathfrak{H},\ B \subset C,\ B \in \mathcal{B} \implies B \in \mathfrak{H}.$$

Show that

(i) $\exists\ A_1, A_2, \cdots \in \mathfrak{H}$ disjoint such that $U := \bigcupdot_{n=1}^{\infty} A_n$ `covers` $\mathfrak{H}$ in the sense that $A \subset U \bmod m\ \forall\ A \in \mathfrak{H}$;

`Hint` Take $\varepsilon_1 := \sup\{m(A) : \ A \in \mathfrak{H}\}$ & $A_1 \in \mathfrak{H},\ m(A_1) \geq \frac{\varepsilon_1}{2}$ and then take $\varepsilon_2 := \sup\{m(A) : A \in \mathfrak{H},\ A \cap A_1 = \varnothing\}$ & $A_2 \in \mathfrak{H},\ A_2 \cap A_1 = \varnothing,\ \ m(A_2) \geq \frac{\varepsilon_2}{2}\ \cdots$

(ii) if $V = \bigcupdot_{n=1}^{\infty} B_n \quad (B_n \in \mathfrak{H})$ also covers $\mathfrak{H}$, then $U = V \mod m$.

The set $U \in \mathcal{B}$ above is called the `measure theoretic union` of the hereditary collection $\mathfrak{H}$ and denoted $U(\mathfrak{H})$.

6. Non-atomicity.

Let $(X, \mathcal{B}, m)$ be a probability space. An *atom* of $(X, \mathcal{B}, m)$ is a set $A \in \mathcal{B}$ satisfying $m(A) > 0$ and $B \in \mathcal{B},\ B \subseteq A \Rightarrow m(B) = 0,\ m(A)$.

Show that if $(X, \mathcal{B}, m)$ is *non-atomic* (i.e. has no atoms), then $\forall\ p \in (0,1),\ \ \exists\ A \in \mathcal{B}$ with $m(A) = p$.

Hint: `Show first (using the previous exercise or not) that` $\forall\ \varepsilon > 0\ \exists$ `a finite partition` $\alpha \subset \mathcal{B}$ `of` X `with` $m(A) < \varepsilon\ \ \forall\ A \in \alpha$.

Week # 3

§4 Polish spaces

A *measurable space* $(X, \mathcal{B})$ is composed of a set X equipped with a σ-algebra $\mathcal{B} \subset 2^X$ of subsets of X. A common example of such is when X is a metric (or topological) space and

$$\mathcal{B} = \mathcal{B}(X) = \{\texttt{Borel sets in } X\} := \sigma(\{\texttt{open sets in } X\}).$$

Let $(X, \mathcal{B})$ & $(Y, \mathcal{C})$ be measurable spaces. A function $f : X \to Y$ is called *measurable* if $f^{-1}(C) \in \mathcal{B}\ \forall\ C \in \mathcal{C}$.

If Y is a metric space and $\mathcal{C} = \mathcal{B}(Y) := \sigma(\{\texttt{open sets}\})$, then $f : X \to Y$ is measurable iff $f^{-1}(U) \in \mathcal{B}\ \forall\ U \subset Y$ open.

In particular, if $(X,\mathcal{B})$ & $(Y,\mathcal{C})$ are both metric spaces eqipped with their Borel sets, then any continuous $f : X \to Y$ is measurable.

The measurable spaces $(X,\mathcal{B})$ and $(X',\mathcal{B}')$ are *isomorphic* ($(X,\mathcal{B}) \cong (X',\mathcal{B}')$) if $\exists$ a bimeasurable bijection (*isomorphism*) $\pi : X \to X'$.

Standard `(aka Polish )` **measurable spaces.** A *standard* measurable (or Borel) space is a measurable space $(X,\mathcal{B})$ where X is Polish space and $\mathcal{B} = \mathcal{B}(X)$ its Borel sets.

Uncountable examples.

$$X = \Omega := \{0,1\}^{\mathbb{N}},\ \mathbb{R}^d,\ C([0,1]),\ \cdots$$

It turns out that standard measurable spaces are isomorphic iff their cardinalities are the same, and that the possible cardinalities are:

$$(*)\qquad \mathbb{N} \cup \{\aleph_0, c\}.$$

Remark

Completeness (of the underlying Polish space) is necessary for $(*)$ to hold without the continuum hypothesis.

Kuratowski's isomorphism theorem

If X is an uncountable Polish space, then $X \cong \Omega := \{0,1\}^{\mathbb{N}}$.

Corollary

If X is an uncountable Polish space, then there is a countable, algebra $\mathcal{A} \subset \mathcal{B}(X)$ with `FSCP` *so that $\sigma(\mathcal{A}) = \mathcal{B}(X)$.*

Proof Let $\pi : X \to \Omega$ be an isomorphism of measurable spaces, then

$$\pi(\mathcal{A}(\{\texttt{cylinders}\})$$

is as required. ☑

Our proof of Kuratowski's theorem is a "measurable version" of the proof of the theorem of Cantor-Bernstein[1]

Let X be a separable metric space.

Consider the collection of `condensation points`

$$X_c := \{x \in X :\ |B(x,\varepsilon)| > \aleph_0\ \forall\ \varepsilon > 0\}.$$

Here (and throughout)

$$B(x,\varepsilon) := \{y \in X :\ d(x,y) \le \varepsilon\}\ \&\ B_o(x,\varepsilon) := \{y \in X :\ d(x,y) < \varepsilon\}.$$

By separability, $X \setminus X_c$ is countable and open, whence $|X| > \aleph_0$ iff $X_c \ne \varnothing$. A Polish space is called *perfect* if $X = X_c$.

[1] `Formulated by Yuval Peres while a student at TAU.`

Lemma 1 *If X is a Polish space with $|X| > \aleph_0$, then $\exists$ a closed set $K \subset X_c$ which is homeomorphic to $\{0,1\}^{\mathbb{N}}$.*

Proof

$\exists\, x(0), x(1) \in X_c$ and $\delta_1 > 0$ such that the closed balls $B(x(i), \delta_1)$ $(i = 0, 1)$ are disjoint, uncountable subsets of X. We can continue this to obtain $\delta_n > 0$, $\{x(\underline{i}) : i \in \{0,1\}^n\} \subset X_c$ $(n \geq 1)$ such that $\forall\ n \geq 1$, $\underline{i} \in \{0,1\}^n$, $B(x(\underline{i}, j), \delta_{n+1})$ $(j = 0, 1)$ are disjoint, uncountable subsets of $B(x(\underline{i}), \delta_n)$. By completeness, $\forall\ (i_1, i_2, \dots) \in \{0,1\}^{I\!N}$,

$$\bigcap_{n=1}^{\infty} B(x(i_1, \dots, i_n), \delta_n) \neq \varnothing.$$

The map $\pi : \Omega \to X$ defined by

$$\{\pi(i_1, i_2, \dots)\} := \bigcap_{n=1}^{\infty} B(x(i_1, \dots, i_n), \delta_n)$$

is continuous and injective. $K := \pi(\Omega)$ is the required set. ☑

Polish subsets. Let (X, d) be a Polish space.

A subset $Y \subset X$ is called a *Polish* (*subset*) if $\exists$ a metric ρ on Y equivalent to $d|_Y$ such that (Y, ρ) is a Polish space.

4.1 Proposition *Suppose that (X, d) is a Polish space, then $Y \subset X$ is Polish iff Y is a G_δ set.*

Proof WLOG $X = \overline{Y}$.

Polish $\implies$ G_δ :

Let ρ be a complete metric on Y generating the topology inherited from X. For $n \geq 1$, let

$$V_n = \{x \in X :\ \exists\ U \ni x \ \texttt{open},\ d - \texttt{diam}(U),\ \rho - \texttt{diam}(U \cap Y) \leq \frac{1}{n}\}.$$

It suffices to show

$$Y = \bigcap_{n=1}^{\infty} V_n.$$

Evidently $Y \subset \bigcap_{n=1}^{\infty} V_n$.

`Proof` that $Y \supset \bigcap_{n=1}^{\infty} V_n$.

If $x \in \bigcap_{n=1}^{\infty} V_n \cap \overline{Y}$, then $\exists\ W_1 \supset W_2 \supset \dots$, open in X such that

$$\forall\ n \geq 1 :\quad x \in W_n,\ \rho - \text{diam}\,(W_n \cap Y) \leq \frac{1}{n},\ \&\ d - \text{diam}\,(W_n) \leq \frac{1}{n}.$$

By completeness of (Y,ρ), $\bigcap_{n=1}^\infty \overline{W}_n \cap Y \ne \varnothing$. On the other hand, $\bigcap_{n=1}^\infty W_n = \{x\}$. Thus:

$$\{x\} = \bigcap_{n=1}^\infty \overline{W}_n \supseteq \bigcap_{n=1}^\infty \overline{W}_n \cap Y \ne \varnothing$$

and $x \in Y$. ☐

$G_\delta \implies$ **Polish**

Now let $Y = \bigcap_{n=1}^\infty U_n$ where U_n is open $(n \ge 1)$. For $n \ge 1$, define $f_n : U_n \to \mathbb{R}_+$ by $f_n(x) := \frac{1}{d(x,U_n^c)}$ (a d-continuous function on U_n) and define $\rho : Y \times Y \to [0,\infty)$ by

$$\rho(x,y) := d(x,y) + \sum_{n=1}^\infty \frac{1}{2^n} \cdot (|f_n(x) - f_n(y)| \wedge 1).$$

Then ρ is a metric on Y, and $\rho \ge d$.

To see equivalence of ρ and d on Y, suppose that $x_n, x \in Y$ and $x_n \xrightarrow{d} x$, then $f_k(x_n) \to f_k(x)\ \forall\ k \ge 1$ whence (!) $x_n \xrightarrow{\rho} x$.

To see completeness of (Y,ρ) let $(x_n)_{n\ge1}$ be a ρ-Cauchy sequence in Y, then $(x_n)_{n\ge1}$ is a d-Cauchy sequence in Y, and $(f_k(x_n))_{n\ge1}$ is a Cauchy sequence in $\mathbb{R}\ \forall\ k \ge 1$. It follows that $\exists\ x \in X$ such that $x_n \xrightarrow{d} x$ and $\exists\ \lim_{n\to\infty} f_k(x_n) \in \mathbb{R}\ \forall\ k \ge 1$. It follows that $x \in Y$ [2] and $x_n \xrightarrow{\rho} x$. ☐

Lemma 4.2 *If X is a Polish space, then $\exists$ a G_δ set $G \subset [0,1)^{\mathbb{N}}$ which is homeomorphic to X.*

Proof Choose an equivalent metric $d \le 3/4$, and a countable dense set $\mathcal{A} \subset X$. Define $f : X \to Z := [0,1)^{\mathcal{A}}$ by $f_a(x) := d(x,a)$. Clearly $f : X \to G := f(X)$ is bijective. We claim it is a homeomorphism.

Let $x_n, x \in X$.

If $x_n \to x$ as $n \to \infty$, then $d(x_n,a) \to d(x,a)$ as $n \to \infty\ \forall\ a \in \mathcal{A}$ and $f(x_n) \to f(x)$.

If $f(x_n) \to f(x)$, then $d(x_n,a) \to d(x,a)$ as $n \to \infty\ \forall\ a \in \mathcal{A}$ whence

$$d(x_n,x) \le d(x_n,a) + d(x,a) \underset{n\to\infty}{\longrightarrow} 2d(x,a)\ \forall\ a \in \mathcal{A}.$$

Let $\varepsilon > 0$. Since $\mathcal{A}$ is dense, $\exists\ a_\varepsilon \in \mathcal{A}$ such that $d(x,a_\varepsilon) < \varepsilon/2$ and

$$\varlimsup_{n\to\infty} d(x_n,x) \le 2d(x,a_\varepsilon) < \varepsilon.$$

---

[2] else $x \notin U_k$ and $f_k(x_n) \to \infty$ as $n \to \infty$ for some k

Thus $f : X \to G$ is a homeomorphism. The set G is now Polish (a complete equivalent metric being given by $\rho(f(x), f(y)) := d(x, y)$) and hence a G_δ. ☐

Lemma 4.3 *There is a Borel subset Z of $\Omega = \{0,1\}^{\mathbb{N}}$ which is Borel isomorphic to X.*

Proof Let $Y := \{\omega \in \{0,1\}^{\mathbb{N}} : \omega_n \nrightarrow 1\}$ then $\pi : Y \to [0,1)$ defined by $\pi(\omega) = \sum_{n=1}^{\infty} \frac{\omega_n}{2^n}$ is a Borel isomorphism.

The Borel isomorphism $\Psi = \phi \circ \psi : [0,1)^{\mathbb{N}} \to Z \in \mathcal{B}(\Omega)$ is as advertised, where:

$\phi : [0,1)^{\mathbb{N}} \to Y^{\mathbb{N}}$ is defined by $\phi(y_1, y_2, \dots)_k := \pi^{-1}(y_k)$,

$\psi : \Omega^{\mathbb{N}} = \{0,1\}^{\mathbb{N}^2} \to \Omega$ is defined by $\psi((\omega_u : \ u \in \mathbb{N}^2))_\ell = \omega_{\sigma(\ell)}$ where $\sigma : \mathbb{N} \to \mathbb{N}^2$ is bijective

and

$Z = \phi \circ \psi(G) \in \mathcal{B}(\Omega)$ where G is as in lemma 4.2. ☐

Proof of Kuratowski's theorem à la Cantor-Bernstein

We have bimeasurable maps $f : X \to f(X) \in \mathcal{B}(\Omega)$ and $g : \Omega \to g(\Omega) \in \mathcal{B}(X)$.

Define

$$X_{2n} = (g \circ f)^n (X \setminus g(\Omega)) \in \mathcal{B}(X), \ \Omega_{2n} = (f \circ g)^n (\Omega \setminus f(X)) \in \mathcal{B}(\Omega),$$

$$X_{2n+1} = g(\Omega_{2n}) \in \mathcal{B}(X), \quad \Omega_{2n+1} = f(X_{2n}) \in \mathcal{B}(\Omega),$$

and

$$X_\infty = X \setminus \bigcup_{n=0}^{\infty} X_n \in \mathcal{B}(X), \ \Omega_\infty = \Omega \setminus \bigcup_{n=0}^{\infty} \Omega_n \in \mathcal{B}(X).$$

Define $\alpha \subset \mathcal{B}(X)$ and $\beta \subset \mathcal{B}(\Omega)$ by

$$\alpha = \{A_0, A_1, X_\infty\}, \ \beta = \{B_0, B_1, \Omega_\infty\}$$

where

$$A_0 = \bigcup_{n=0}^{\infty} X_{2n}, \ A_1 = \bigcup_{n=0}^{\infty} X_{2n+1},$$

$$B_0 = \bigcup_{n=0}^{\infty} \Omega_{2n} = g^{-1}(A_1), \ B_1 = \bigcup_{n=0}^{\infty} \Omega_{2n+1} = f(A_0).$$

- We claim that α and β are partitions of X and Ω (respectively).

`Proof` For $x \in g(\Omega) \subset X$, call the point $g^{-1}(x) \in \Omega$ a *preimage* of x, and for $y \in f(X) \subset \Omega$, call the point $f^{-1}(y) \in X$ a *preimage* of y.

Define a map $N : X \cup \Omega \to \mathbb{Z}_+ \cup \{\infty\}$ by

$$N(z) = \max\{n \geq 0 : \ \exists\, (z_0, \dots, z_n), \ z_0 = z, \ z_{k+1} \text{ is a preimage of } z_k \ \forall\, 0 \leq k < n\},$$

then $X_n = X \cap [N = n]$ and $\Omega_n = \Omega \cap [N = n]$ for $0 \le n \le \infty$. $\square$

The required Borel isomorphism is $\pi : X \to \Omega$ defined by

$$\pi(x) = \begin{cases} f(x) & x \in A_0 \cup X_\infty, \\ g^{-1}(x) & x \in A_1. \quad \square \end{cases}$$

Universably nonmeasurable sets in Polish spaces.

We "construct" subsets of Polish spaces which are not `completion measurable` (in the sense of exercise 4.0 below) with respect to any non-atomic Borel probability on X.

We'll need:

Lemma 4.4

If X is a Polish space with $|X| = c$, then

$$|\{\texttt{uncountable closed subsets}\}| = c.$$

`Proof`

Let $\mathfrak{C} := \{\texttt{uncountable, closed subsets of } X\}$.

¶ To see that $|\mathfrak{C}| \le c$ let $\mathcal{U}$ be a countable base for the topology of X, then

$$|\mathfrak{C}| = |\{\texttt{open subsets of } X\}| \le |2^{\mathcal{U}}| = c.$$

¶ To show that $|\mathfrak{C}| \ge c$ we exhibit a continuum of disjoint, uncountable closed subsets of X. By lemma 1, it suffices to this in Ω. To this end fix $A \subset \mathbb{N}$ such that $|A| = |\mathbb{N} \setminus A| = \infty$. For $a \in \{0,1\}^A$, let $\Omega_a := \{\omega \in \Omega : \omega|_A = a\}$, then

- $\Omega_a \cap \Omega_{a'} \ne \varnothing \implies a = a'$;
- Ω_a is homeomorphic with $\Omega \ \forall \ a \in \{0,1\}^A$.

Thus $\mathcal{F} := \{\Omega_a : \ a \in \{0,1\}^A\}$ is a continuum of disjoint, uncountable closed subsets of Ω. ☑

Big sets & **Bernstein sets.**

Let X be an uncountable, Polish space. A *big set* in X is a set $B \subset X$ such that $F \cap B \ne \varnothing \ \forall \ F \subset X$ closed, uncountable,

$$F \cap B \ne \varnothing \ \& \ \ F \setminus B \ne \varnothing.$$

A *Bernstein set* in X is a big set whose complement is also big.

4.5 Bernstein's Theorem[3]

[3]Felix Bernstein, "Zur Theorie der trigonometrischen Reihen", Sitzungsber. Sachs. Akad. Wiss. Leipzig. Math.-Natur. Kl. 60 (1908), 325-338.

Bernstein sets exist in any uncountable Polish space.

`Proof` Let $\mathfrak{C} := \{\texttt{uncountable, closed subsets of } X\}$, then (as shown above) $|\mathfrak{C}| = c$.

The *initial segment* (רישא) of an ordinal β is

$$\Omega(\beta) := \{\alpha \ \texttt{ordinal}, \ \alpha \prec \beta\}.$$

Let ω_c be the minimal ordinal with $|\omega_c| := \texttt{card}\,\Omega(\omega_c) = c$. Write $\Omega_c := \Omega(\omega_c)$, then

$$|\alpha| < c \ \forall \ \alpha \in \Omega_c$$

and it follows that

$$\mathfrak{C} = \{F_\alpha : \ \alpha \in \Omega_c\}.$$

We'll also need the

`Transfinite recursion theorem.`

Suppose that Z is a set, $z \in Z$ and for $\alpha \in \Omega_c$, $f_\alpha : Z^{\Omega(\alpha)} \to Z$, then $\exists\, !\ f : \Omega_c \to Z$ such that $f(1) = z$ and

$$f(\alpha) = f_\alpha(f|_{\Omega(\alpha)}) \ \forall \ \alpha \in \Omega_c.$$

To use transfinite recursion, we claim

¶ $\forall \ \alpha \in \Omega_c, \ \exists \ \ \Psi_\alpha := (f_\alpha, g_\alpha) : (X \times X)^{\Omega(\alpha)} \to X \times X$ such that $\forall \ (u,v) \in (X \times X)^{\Omega(\alpha)}$,

(a) $f_\alpha(u,v) \ne g_\alpha(u,v)$;

(b) $f_\alpha(u,v), \ g_\alpha(u,v) \in F_\alpha$;

(c) $f_\alpha(u,v), \ g_\alpha(u,v) \notin u(\Omega(\alpha)) \cup v(\Omega(\alpha))$.

`Proof of` ¶:

Fix $(u,v) \in (X \times X)^{\Omega(\alpha)}$, then $|u(\Omega(\alpha)) \cup v(\Omega(\alpha))| < c$ whereas $|F_\alpha| = c$ whence $\#(F_\alpha \setminus u(\Omega(\alpha)) \setminus v(\Omega(\alpha))) > 2$ ensuring existence of such $f_\alpha(u,v), \ g_\alpha(u,v)$. ☑ ¶

By transfinite recursion $\exists \ \ \Psi = (p(\alpha), q(\alpha)) : \Omega_c \to X \times X$ such that $\Psi(\alpha) = \Psi_\alpha(\Psi|_{\Omega(\alpha)})$.

It follows that

(i) $p(\alpha) \ne q(\alpha)$;

(ii) $p(\alpha), \ q(\alpha) \in F_\alpha$;

(iii) $p(\alpha), \ q(\alpha) \notin p(\Omega(\alpha)) \cup q(\Omega(\alpha))$

whence $p(\Omega_c) \cap q(\Omega_c) = \varnothing$.

The Bernstein set constructed is $B := p(\Omega_c)$. Evidently $X \setminus B \supset q(\Omega_c)$. For each $\forall F \in \mathfrak{C},\ \exists\ \alpha \in \Omega_c,\ F = F_\alpha$ whence

$$p(\alpha) \in F \cap B \ \&\ q(\alpha) \in B^c \cap F. \ \boxed{\checkmark}$$

4.7 Proposition *If $B \subset X$ is a Bernstein set, and $\overline{\mu} : 2^X \to [0, \infty)$ is a metric outer measure on X with $\overline{\mu}(\{x\}) = 0\ \forall\ x \in X$, then $B \notin \mathcal{M}_{\overline{\mu}}$.*

`Proof`
Since $\overline{\mu}(\{x\}) = 0\ \forall\ x \in X$, any closed set of positive measure is uncountable. Thus, if $B \in \mathcal{M}$, then by regularity of $\overline{\mu}|_{\mathcal{M}}$, either

- $\overline{\mu}(B) > 0$ and $\exists\ C \in \mathfrak{C},\ C \subset B \implies \varnothing \ne C \cap B^c \subsetneqq B \cap B^c = \varnothing$; or
- $\overline{\mu}(B^c) > 0$ and $\exists\ C \in \mathfrak{C},\ C \subset B^c \implies \varnothing \ne C \cap B \subsetneqq B^c \cap B = \varnothing$. ⊠

See also exercise 4.3(a).

4.8 Ulam's theorem

Suppose that $|X| = \aleph_1$. If $\mu : 2^X \to [0, 1]$ is a measure, non-atomic in the sense that $\mu(\{x\}) = 0\ \forall\ x \in X$, then $\mu \equiv 0$.

`Proof` By the WOT, $\exists$ an ordering $\prec$ on X so that

- every subset has a minimal element; and
- $\{x \in X :\ x \prec y\}$ is at most countable $\forall\ y \in X$.

For $y \in X$ let $x \mapsto f_y(x) = f(x, y)$ be an injection of $\{x \in X :\ x \prec y\}$ into $\mathbb{N}$.

For $x \in X,\ n \in \mathbb{N}$ let $F_x^n := \{y \in X :\ x \prec y,\ f(x, y) = n\}$.

¶1 $F_x^n \cap F_{x'}^n = \varnothing\ \forall\ n \in \mathbb{N},\ x \ne x' \in X$.

`Proof` If $y \in F_x^n \cap F_{x'}^n$, then $f(x, y) = f(x', y) = n$ whence by construction of f, $x = x'$. $\boxed{\checkmark}$

¶2 $\bigcup_{n \in \mathbb{N}} F_x^n = \{y \in X :\ x \prec y\}$ since $\forall\ x \prec y,\ y \in F_x^{f(x,y)}$.

Now for fixed $n \in \mathbb{N}$,

$$|\{x \in X :\ \mu(F_x^n) > 0\}| \le \aleph_0.$$

Therefore

$$|\{x \in X :\ \exists\ n \in \mathbb{N},\ \mu(F_x^n) > 0\}| \le \aleph_0$$

and $\exists\ r \in X,\ \mu(F_r^n) = 0\ \forall\ n \in \mathbb{N}$.

Thus

$$(\text{☺})\qquad \begin{aligned}\mu(X) =&\mu(\{x\in X:\ x\le r\})+\mu(\bigcup_{n\in\mathbb{N}}F_r^n)\\ &=\mu(\{x\in X:\ x<r\})+\mu(\{r\})+\sum_{n\in\mathbb{N}}\mu(F_r^n)\\ &=0.\ \boxtimes\end{aligned}$$

Exercise №3, 29/3/2017 תרגיל פסח

1. Completions.

The measure space $(Z,\mathcal{C},m)$ is *complete* if arbitrary subsets of measureable null sets are measurable.

Let $(X,\mathcal{B},\mu)$ be a finite measure space.

The μ-*completion* of $\mathcal{B}$ is the σ-algebra

$$\overline{\mathcal{B}}_\mu := \sigma(\mathcal{B}\cup\bigcup_{N\in\mathcal{B},\ \mu(N)=0}2^N).$$

Here, you extend μ to $\overline{\mathcal{B}}_\mu$.

Show that

(i) $\overline{\mu}:2^X\to[0,\infty)$ defined by $\overline{\mu}(A):=\inf\{\mu(U):\ A\subset U\in\mathcal{B}\}$ is an outer measure;

(ii) $\mathcal{M}_{\overline{\mu}}=\overline{\mathcal{B}}_\mu$ and $\overline{\mu}|_{\mathcal{B}}=\mu$;

and that

(iii) a function $g:X\to\mathbb{R}$ is $\overline{\mathcal{B}}_\mu$–measurable iff $\exists\ f:X\to\mathbb{R}$ $\mathcal{B}$–measurable such that $f(x)=g(x)$ for $\overline{\mu}$-a.e. $x\in X$.

The measure space $(X,\overline{\mathcal{B}}_\mu,\overline{\mu})$ is called the μ-*completion* of $(X,\mathcal{B},\mu)$.

2. Measure algebra. Let $(X,\mathcal{B},m)$ be a finite measure space. Define a relation on $\mathcal{B}$ by $A\sim B$ if $m(A\Delta B)=0$.

a) Show that $\sim$ is an equivalence relation.

b) Let $\mathcal{B}^\sim=\{[A]:=\{A'\in\mathcal{B}:A'\sim A\}:A\in\mathcal{B}\}$ be the collection of equivalence classes.

Show that

$$d(a,b):=m(A\Delta B)\ \texttt{for}\ \ a,b\in\mathcal{B}^\sim,\quad A\in a,B\in b$$

defines a metric on $\mathcal{B}^\sim$, and that $(\mathcal{B}^\sim,d)$ is a complete metric space (called the *measure algebra* of $(X,\mathcal{B},m)$).

c) Show that the measure algebra of a measure space is isometric with the measure algebra of its completion.

d) Show that the following are equivalent:

(i) $\mathcal{B}^\sim$ is separable;

(ii) $\exists\ A_n \in \mathcal{B}\ \ (n \geq 1)$ such that $\forall\ A \in \mathcal{B}\ \exists\ A' \in \sigma(\{A_n\}_{n=1}^\infty)$ with $m(A\Delta A') = 0$;

e) Is there a probability space $(X, \mathcal{B}, \mu)$ equipped with sets $\{A_s : s \in (0,1)\} \subset \mathcal{B}$ such that $\mu(A_s \Delta A_t) \geq \frac{1}{4}\ \forall\ \ s \neq t$?

3. Polish probability spaces.

Let X be a Polish space, and let $p \in \mathcal{P}(X, \mathcal{B}(X))$ be non-atomic. Let $(I, \mathcal{B}(I), \lambda)$, the unit interval equipped with its Borel sets and Lebesgue measure. Show that $(X, \mathcal{B}(X), p)$ and $(I, \mathcal{B}(I), \lambda)$ are *isomorphic* in the sense that there are sets $X' \in \mathcal{B}(X),\ I' \in \mathcal{B}(I)$ such that $p(X \setminus X') = \lambda(I \setminus I') = 0$; and a bimeasurable bijection $\pi : X' \to I'$ satisfying $p \circ \pi^{-1} = \lambda$.

`Hint`: First use Kuratowski's theorem to show that $(X, , \mathcal{B}(X), p)$ is isomorphic with $(I, \mathcal{B}(I), q)$ for some nonatomic $q \in \mathcal{P}(I, \mathcal{B}(I))$.

4. Bernstein sets.

Let X be an uncountable polish space.

Show that there is a disjoint collection $\mathfrak{B}$ of Bernstein sets so that $|\mathfrak{B}| = c$ and $\dot\bigcup_{B \in \mathfrak{B}} B = X$.

5. Generalised Cantor sets. For $I = [a-c, a+c]$ a bounded closed interval, and $0 < h < 1$, define

$$I_0(h) := [a - c, a - (1-h)c],\ I_1(h) := [a + (1-h)c, a + c];$$

$$I'(h) := I_0(h) \cup I_1(h) = I \setminus (a - (1-h)c, a + (1-h)c).$$

For $A = \dot\bigcup_k I_k$ a disjoint union of closed intervals, define

$$A'(h) = \dot\bigcup_k I'_k(h).$$

Clearly $|A'(h)| = h|A|$. For $h_1, h_2, \dots \in (0,1)$ define

$$A_1 := [0,1],\quad A_{n+1} := A'_n(h_n),\quad A := \bigcap_{n=1}^\infty A_n.$$

a) Prove that A is closed and nowhere dense.

b) Prove that the Lebesgue measure of A is $m(A) = \lim_{n\to\infty} \prod_{k=1}^n h_n$.

6. Volterra's function (1881).

Here you construct this differentiable function $V:\mathbb{R}\to\mathbb{R}$ whose uniformly bounded derivative is not Riemann integrable on $[0,1]$.

a) For the construction, you need a closed, nowhere dense subset E of $[0,1]$, with positive Lebesgue measure. Construct such (e.g. using the previous exercise).

Set $f(x)=x^2\sin(\frac{1}{x})$ when $x\neq 0$, and $f(0)=0$ and recall that

- f is differentiable on $\mathbb{R}$;
- $f'(x)=2x\sin(\frac{1}{x})-\cos(\frac{1}{x})$ when $x\neq 0$, $f'(0)=0$; and
- there is a sequence $y_n\downarrow 0$ such that $f'(y_n)=0$.

For $r>0$, set $z=z_r:=\max\{z\in(0,r/2):f'(z)=0\}$, and define $f_r:\mathbb{R}\to\mathbb{R}$ by

$$f_r(x)=\begin{cases} f(x) & 0\le x\le z,\\ f(z) & z\le x\le r-z,\\ f(r-z) & r-z\le x\le r,\\ 0 & \text{else.}\end{cases}$$

b) For $I=(a,b)$, define $f_I:\mathbb{R}\to\mathbb{R}$ by $f_I(x)=f_{b-a}(x-a)$.

Show that $|f_I|\le\frac{|I|^2}{4}$; f_I is differentiable on $\mathbb{R}$ with $|f_I'|\le|I|+1$ and $\omega(f_I',x)=2\cdot 1_{\partial I}(x)$

where for $g:(x-\eta,x+\eta)\to\mathbb{R}$,

$$\omega(g,x):=\lim_{\varepsilon\to 0}\sup_{y,z\in(x-\varepsilon,x+\varepsilon)}|g(y)-g(z)|.$$

For $E\subset[0,1]$ a closed, nowhere dense set, write $\mathbb{R}\setminus E=\bigcup_n I_n$ where the I_n are disjoint, open intervals, and set

$$V(x):=\begin{cases} f_{I_n}(x) & x\in I_n\\ 0 & \text{else.}\end{cases}$$

c) Show that V is differentiable on $\mathbb{R}$, $\sup_{[0,1]}|V'|\le 2$, $V'|_E\equiv 0$ and that $\omega(V',x)=2\cdot 1_E(x)$.

Hint $|V(y)|\le|y-x|^2\ \forall\ x\in E,\ y\in\mathbb{R}$.

d) Show that if $m(E)>0$, then V' is not Riemann integrable on $[0,1]$.

Week # 4

MEASURABLE SETS IN POLISH SPACES: ANALYTIC SETS

Definition A subset of a Polish space is *analytic* if it is the continuous image of $\mathbb{N}^{\mathbb{N}}$ (i.e. the irrationals – see exercise 4.1(a)).

4.9 Proposition *Any Polish space is a continuous image of* $\mathbb{N}^{\mathbb{N}}$.

The proof is standard and uses the

4.10 Lemma *Let X be a separable metric space. Given $\varepsilon_k > 0 \ \ (k \geq 1)$, there is a collection of nonempty closed sets*

$$\{F(n_1, ..., n_k) : k, n_1, ..., n_k \in \mathbb{N}\}$$

such that $\overline{F(n_1, ..., n_k)^o} = F(n_1, ..., n_k) \ \forall = k, n_1, ..., n_k \in \mathbb{N}$,

$$\bigcup_{\nu=1}^{\infty} F(\nu) = X,$$

$$\bigcup_{\nu=1}^{\infty} F(n_1, ..., n_k, \nu) = F(n_1, ..., n_k))^o \forall \ k, n_1, ..., n_k \in \mathbb{N},$$

and

$$\sup\{\operatorname{diam} F(n_1, ..., n_k) : n_1, ..., n_k \in \mathbb{N}\} \leq \ \varepsilon_k \ \forall \ k \geq 1.$$

The lemma is proved using the Lindelöf property of separable metric spaces.

The following shows that Borel sets are analytic:

4.11 Theorem *Let X be Polish, then every non-empty $A \in \mathcal{B}(X)$ is an injective, continuous image of a Polish space.*

Proof Let

$$\mathfrak{P} := \{A \in \mathcal{B}(X) : \ \exists \ Z \text{ Polish \& a continuous bijection } \pi : Z \to A\};$$

$$\mathfrak{Q}_0 := \{A \in \mathfrak{P} : \ A^c \in \mathfrak{P}\} \ \& \ \ \mathfrak{Q} = \mathfrak{Q}_0 \cup \{\varnothing, X\}.$$

We show that $\mathfrak{Q} = \mathcal{B}(X)$.

If $\varnothing \neq U \subsetneqq X$ is open, then $U \in \mathfrak{P}$ (as a non-empty G_δ set), and $U^c \in \mathfrak{P}$ (as a non-empty closed set), hence $U \in \mathfrak{Q}$. We show that $\mathfrak{Q}$ is a σ-algebra.

First:

(🚲) $$A_n \in \mathfrak{P}, \ A := \bigcap_{n=1}^{\infty} A_n \neq \varnothing \implies A \in \mathfrak{P}.$$

To see this, let X_n be Polish, and $\pi_n : X_n \to X$ be a continuous injection with $\pi_n(X_n) = A_n$. Consider the product (Polish) space

$$\hat{X} := \prod_{n=1}^{\infty} X_n$$

and the continuous injection $\pi : \hat{X} \to X^{\mathbb{N}}$ defined by $(\pi(x_1, x_2, \dots))_n := \pi_n(x_n)$.

Now let

$$\begin{aligned}\tilde{X} &= \{x \in \hat{X} : \pi_n(x_n) = \pi_1(x_1)\ \forall\ n \ge 1\}\\ &= \pi^{-1}\{(x_1, x_2, \dots) \in A^{\mathbb{N}} :\ x_n = x_1\ \forall\ n \ge 1\},\end{aligned}$$

- a non-empty, closed subset of $\hat{X}$ and hence Polish, and consider $\tilde{\pi} : \tilde{X} \to X$ defined by $\tilde{\pi}(x_1, x_2, \dots) = \pi_1(x_1)$. Clearly this is continuous, and $\tilde{\pi}(\tilde{X}) = \bigcap_{n=1}^{\infty} A_n$. ☑(🚲)

Next:

(🚗) $$A_n \in \mathfrak{P} \text{ disjoint}, \ \Rightarrow\ \dot{\bigcup_{n=1}^{\infty}} A_n \in \mathfrak{P}.$$

`Proof` Let X_n be Polish, and $\pi_n : X_n \to X$ be a continuous injection with $\pi_n(X_n) = A_n$. Consider the union space $U := \{(k, y) : k \in \mathbb{N},\ x \in X_k\}$ which is Polish equipped with the metric

$$d((k,x),(k',y)) = \begin{cases} 1 & k \ne k', \\ d_k(x, x') & k = k'. \end{cases}$$

If $\pi((k,x)) := \pi_k(x)$, then π is continuous and injective and $\pi(U) = \bigcup_{n=1}^{\infty} A_n$. ☑(🚗)

Now, we show

(☞) $$A, B \in \mathfrak{Q}_0 \implies A \cup B \in \mathfrak{P}.$$

To see this

$$\begin{aligned} A, B \in \mathfrak{Q}_0 &\implies A, B, B^c \in \mathfrak{P} \\ &\overset{(🚲)}{\implies} A \cap B^c, B \in \mathfrak{P} \\ &\overset{(🚗)}{\implies} A \cup B = (A \cap B^c) \cup B \in \mathfrak{P}. \quad ☑(☞) \end{aligned}$$

To see that $\mathfrak{Q}$ is an algebra we must show that

$$A, B \in \mathfrak{Q} \implies A \cup B \in \mathfrak{Q}.$$

This true for $A, B \in \mathfrak{Q}_0$ by (🚲) & (☞) and the extension to $A, B \in \mathfrak{Q}$ is easy.

To see that $\mathfrak{Q}$ is a σ-algebra, it's enough to show that

$$A_n \in \mathfrak{Q} \ \Rightarrow\ A := \bigcup_{n=1}^{\infty} A_n \in \mathfrak{Q}.$$

If $\exists\ n,\ A_n = X$ then $A = X \in \mathfrak{Q}$.

So WLOG, we assume $A_n \in \mathfrak{Q}_0\ \forall\ n \ge 1$.

`Proof that` $A \in \mathfrak{P}$

Since $\mathfrak{Q}$ is an algebra, $\exists\ B_n \in \mathfrak{Q}$, disjoint, such that $A := \bigcupdot_{n=1}^\infty B_n$. For each $n \ge 1$, either $B_n \in \mathfrak{P}$ or $B_n = \varnothing$. WLOG $B_n \in \mathfrak{P}\ \forall\ n \ge 1$ & $A \in \mathfrak{P}$ by (🚗).

`Proof that` $A^c \in \mathfrak{P}$

For each $n \ge 1$, $A_n^c \in \mathfrak{P}$ whence

$$\left(\bigcup_{n=1}^\infty A_n\right)^c = \bigcap_{n=1}^\infty A_n^c \in \mathfrak{P}.$$

$\square$

Universally Measurable.

A subset of a Polish space $\mathfrak{a} \subset X$ is *universally measurable* if $\mathfrak{a} \in \overline{\mathcal{B}(X)}_p\ \forall\ \ p \in \mathcal{P}(X)$.

4.12 Luzin's Measurability Theorem

An analytic subset of a Polish space $\mathfrak{a} \subset X$ is universally measurable.

Proof We'll show that $\exists\ U \subset \mathfrak{a} \subset V,\ U, V \in \mathcal{B}(X)$ such that $p(U) = p(V)$.

Define $\overline{p} : 2^X \to [0,1]$ by

$$\overline{p}(A) := \inf\{p(B) :\ A \subset B \in \mathcal{B}(X)\}.$$

It follows from the basics that p is an outer measure.

If $A \subset X$ then $\exists\ U_A \supset A,\ U_A \in \mathcal{B}(X)$ with $\overline{p}(A) = p(U_A)$.

We'll show that $\forall\ \varepsilon > 0,\ \exists\ K_\varepsilon \subset \mathfrak{a}$ compact, such that $p(K_\varepsilon) \ge \overline{p}(\mathfrak{a}) - \varepsilon$ whence $\mathfrak{a} = \bigcup_{n=1}^\infty K_{\frac{1}{n}} \mod p$.

Suppose that $f : \mathbb{N}^{\mathbb{N}} \to X$ is continuous and $\mathfrak{a} = f(\mathbb{N}^{\mathbb{N}})$. For $k, n_1, \dots, n_k \ge 1$, let

$$L(n_1, \dots, n_k) = \{x \in \mathbb{N}^{\mathbb{N}} :\ x_j \le n_j \quad 1 \le j \le k\}.$$

By ex. 3.1(c), if $A_n \subset A_{n+1} \uparrow A = \bigcup_{n=1}^\infty A_n$, then $\overline{p}(A_n) \uparrow \overline{p}(A)$.

Thus, since $f(L(n)) \uparrow f(\mathbb{N}^{\mathbb{N}}) = \mathfrak{a}$, $\exists\ n_1$ such that

$$\overline{p}(f(L(n_1))) > \overline{p}(\mathfrak{a}) - \frac{\varepsilon}{2};$$

and since $f(L(n_1, n)) \uparrow f(L(n))$, $\exists\ n_2$ such that

$$\overline{p}(f(L(n_1, n_2))) > \overline{p}(f(L(n_1))) - \frac{\varepsilon}{2^2} = \overline{p}(\mathfrak{a}) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2^2}.$$

Continuing inductively, we obtain $n_k \ge 1\ \ (k \ge 1)$ such that

$$\overline{p}(f(L(n_1, \dots, n_k))) > \overline{p}(\mathfrak{a}) - \varepsilon \quad \forall\ k \ge 1.$$

Set

$$C_\varepsilon = \bigcap_{k=1}^{\infty} L(n_1, \dots, n_k)$$

then C_ε is compact in $\mathbb{N}^{\mathbb{N}}$, whence $K = K_\varepsilon := f(C_\varepsilon)$ is compact in X. We claim that $p(K_\varepsilon) > \overline{p}(\mathfrak{a}) - \varepsilon$.

To establish this, it is sufficient to show that

$$(☕) \qquad K = \bigcap_{k=1}^{\infty} \overline{f(L(n_1, \dots, n_k))}$$

for then,

$$p(K) \longleftarrow p(\overline{f(L(n_1, \dots, n_k))}) \ge \overline{p}(f(L(n_1, \dots, n_k))) > \overline{p}(\mathfrak{a}) - \varepsilon.$$

`Proof of (☕)`

Clearly $K \subset \bigcap_{k=1}^{\infty} \overline{f(L(n_1, \dots, n_k))}$.

Suppose that $x \in \bigcap_{k=1}^{\infty} \overline{f(L(n_1, \dots, n_k))}$. We'll show that $x \in K$. For each $k \ge 1$, $\exists\ y(k) \in \mathbb{N}^{\mathbb{N}}$ such that $d(x, f(y(k)) < \frac{1}{k}$ and $y_j(k) \le n_j$ $(1 \le j \le k)$. There is a subsequence $k_\ell \to \infty$ and $y \in \mathbb{N}^{\mathbb{N}}$ such that$y_j(k_\ell) = y_j \ \forall\ 1 \le j \le \ell$, whence $y \in C_\varepsilon$ and $y(k_\ell) \to y$. It follows that

$$x \leftarrow f(y(k_\ell)) \to f(y) \in K.$$

$\square$

Souslin Universality

Definition Let X be a set, and $\mathcal{F} \subset 2^X$. A subset A of $\mathbb{N}^{\mathbb{N}} \times X$ is *universal for* $\mathcal{F}$ if

$$\{A_y : \ y \in \mathbb{N}^{\mathbb{N}}\} = \mathcal{F}$$

where $A_y := \{x \in X : (y, x) \in A\}$.

Souslin's existence theorem *Let X be an uncountable Polish space. There is an analytic subset of X which is not Borel.*

Lemma *Suppose that X is a separable metric space. There is an open subset of $\mathbb{N}^{\mathbb{N}} \times X$ which is universal for the open subsets of X, and there is a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$ which is universal for the closed subsets of X.*

Proposition *Let X be Polish. There is an analytic subset of $\mathbb{N}^{\mathbb{N}} \times X$ which is universal for the analytic subsets of X.*

Proof Each analytic set in X is the projection of a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$. Choose a closed subset F of $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}} \times X)$ which is universal for the closed subsets of $\mathbb{N}^{\mathbb{N}} \times X$.

Set
$$G = \{(x,z) \in \mathbb{N}^{\mathbb{N}} \times X : \exists\, y \in \mathbb{N}^{\mathbb{N}},\ (x,y,z) \in F\}.$$
Evidently G is an analytic subset of $\mathbb{N}^{\mathbb{N}} \times X$. To see that G is universal for the analytic subsets of X, let $A \subset X$ be analytic. There is a closed $B \subset \mathbb{N}^{\mathbb{N}} \times X$ so that
$$A = \{y \in X : \exists\, x \in \mathbb{N}^{\mathbb{N}},\ (x,y) \in B\}.$$
By universality of F, $\exists\, u \in \mathbb{N}^{\mathbb{N}}$ such that
$$B = F_u.$$
Therefore,
$$\begin{aligned} A &= \{z \in X : \exists\, y \in \mathbb{N}^{\mathbb{N}},\ (y,z) \in F_u\} \\ &= \{z \in X : \exists\, y \in \mathbb{N}^{\mathbb{N}},\ (u,y,z) \in F\} \\ &= \{z \in X : (u,z) \in G\} \\ &= G_u. \end{aligned}$$
□

Proof of Souslin's existence theorem We first show that there is an analytic subset of $\mathbb{N}^{\mathbb{N}}$ which is not Borel.

To see this let $A \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be universal for the collection of analytic subsets of $\mathbb{N}^{\mathbb{N}}$, and let $B = \{x \in \mathbb{N}^{\mathbb{N}} : (x,x) \in A\}$, which is analytic being a continuous image of A.

We claim that $B \notin \mathcal{B}(\mathbb{N}^{\mathbb{N}})$. To see this, we show that B^c is not analytic.

If B^c is analytic, then $B^c = A_u$ for some $u \in A$. However,
$$u \in B^c \;\Leftrightarrow\; (u,u) \notin A \;\Leftrightarrow\; u \notin A_u = B^c \;\Leftrightarrow\; u \in B.$$

This contradiction blocks the analyticity of B^c, and shows that $B \notin \mathcal{B}(\mathbb{N}^{\mathbb{N}})$.

The general existence theorem follows from Kuratowski's isomorphism theorem. □

Definition [4]

Let X be Polish. Subsets $A_1, A_2, \cdots \subset X$ are (Borel) *separated* if $\exists\, B_1, B_2, \cdots \in \mathcal{B}(X)$ such that $B_i \cap B_j = \varnothing\ \forall\ i \neq j$ and $A_i \subset B_i\ \forall\ i \geq 1$.

[4] 23/11/95

Souslin's separation theorem *Disjoint analytic subsets of a Polish space are separated.*

Lemma 1 *Suppose $A_1, A_2, \dots, B_1, B_2, \dots \subset X$ and for each $m, n \ge 1$, A_m and B_n are separated, then $\bigcup_{m=1}^\infty A_m$ and $\bigcup_{n=1}^\infty B_n$ are separated.*

Proof We first fix $n \ge 1$ and show that $\bigcup_{m=1}^\infty A_m$ and B_n are separated.
To see this suppose that $A_m \subset F_m \in \mathcal{B}(X) \quad (m \ge 1)$ and $B_n \subset F_m^c \ \forall\ m \ge 1$; then

$$\bigcup_{m=1}^\infty A_m \subset F := \bigcup_{m=1}^\infty F_m \in \mathcal{B}(X),\ \&\ B_n \subset \bigcap_{m=1}^\infty F_m^c = F^c.$$

We now have that $\bigcup_{m=1}^\infty A_m$ and B_n are separated $\forall\ n \ge 1$. Let $B_n \subset G_n \in \mathcal{B}(X) \quad (n \ge 1)$ and $\bigcup_{m=1}^\infty A_m \subset G_n^c \ \forall\ n \ge 1$. It follows that

$$\bigcup_{n=1}^\infty B_n \subset G := \bigcup_{n=1}^\infty G_n \in \mathcal{B}(X),\ \&\ \bigcup_{m=1}^\infty A_m \subset \bigcap_{n=1}^\infty G_n^c := G^c.$$

☐

Lemma 2 *Suppose $A, B \subset X$ are disjoint and analytic, then A and B are separated.*

Proof There is no loss of generality in assuming $A, B \ne \varnothing$. Suppose $A = f(\mathbb{N}^\mathbb{N})$ and $B = g(\mathbb{N}^\mathbb{N})$ where $f, g : \mathbb{N}^\mathbb{N} \to X$ are continuous, and that A and B are not separated.
We claim that $\exists\ m_1, n_1 \ge 1$ such that $f([m_1])$ and $g([n_1])$ are not separated, else A and B are separated by lemma 1 since $A = \bigcup_{m=1}^\infty f([m])$ and $B = \bigcup_{n=1}^\infty g([n])$.
Continuing in this manner, we obtain $\underline{m}, \underline{n} \in \mathbb{N}^\mathbb{N}$ such that
$f([m_1, m_2, \dots, m_k])$ and $g([n_1, n_2, \dots, n_k])$ are not separated $\forall\ k \ge 1$.
However, since $f(\underline{m}) \ne g(\underline{n})$ (being contained in disjoint sets), we have by continuity of f and g that for some $k \ge 1$

$$\exists\ \varepsilon > 0 \text{ such that } B(f([m_1, m_2, \dots, m_k]), \varepsilon) \cap B(g([n_1, n_2, \dots, n_k]), \varepsilon) = \varnothing.$$

☐

Proof of the separation theorem Suppose that $A_1, A_2, \dots \subset X$ are disjoint and analytic. For each $n \ge 1$ $\bigcup_{k \ne n} A_n$ is analytic and disjoint from A_n; so

$$\forall\ n \ge 1\ \exists\ B_n \in \mathcal{B}(X),\ A_n \subset B_n,\ \&\ \bigcup_{k \ne n} A_n \subset B_n^c.$$

Set

$$C_n := B_n \setminus \bigcup_{k \ne n} B_k \in \mathcal{B}(X).$$

Clearly $C_i \cap C)j = \varnothing \ (i \neq j)$. To see that $A_n \subset C_n$, note that $A_n \subset B_n$ and

$$\forall\ k \neq n,\ A_n \subset \bigcup_{m \neq k} A_m \subset B_k^c,$$

whence $A_n \subset \bigcap_{k \neq n} B_k^c$ and $A_n \subset C_n$. □

Inverse function theorem *Suppose that X and Y are Polish spaces, and that $f : X \to Y$ is 1-1, and measurable, then $\exists\ g : Y \to X$ measurable suchj that $g \circ f$ =Id.*

Proof Let $\varepsilon_n \to 0$ as $n \to \infty$ and let for $n \geq 1$ $\alpha_n \subset \mathcal{B}(X)$ be a countable partition of X such that

$$\max_{a \in \alpha_n} \operatorname{diam}(a) \leq \varepsilon_n.$$

For each $n \geq 1$, $\{f(a) :\ a \in \alpha_n\}$ is a countable collection of analytic sets, ans by the separation theorem $\exists\ \{B_n(a) :\ a \in \alpha_n\} \subset \mathcal{B}(Y)$, disjoint such that $B_n(a) \supset a \ \ \forall\ a \in \alpha_n$.

Choose $x_a \in a \in \alpha_n$, fix $x_* \in X$, and define $g_n : Y \to X$ by

$$g_n(y) = \begin{cases} x_a & y \in B_n(a) \\ x_* & y \notin \bigcup_{a \in \alpha_n} B_n(a). \end{cases}$$

Clearly $g_n : Y \to X$ is measurable. Define $g : Y \to X$ by

$$g(y) = \begin{cases} \lim_{n\to\infty} g_n(y) & \exists\ lim_{n\to\infty} g_n(y), \\ x_* \quad \text{else}. \end{cases}$$

Clearly $g : Y \to X$ is measurable. Moreover for $x \in X$, $f(x) \in B_n(a_n(x)) \ \ \forall\ n \geq 1$ where $x \in a_n(x) \in \alpha_n$; whence $g_n(f(x)) = x_{a_n(x)} \to x$ as $n \to \infty$ $(\because x, x_{a_n(x)} \in a_n(x) \in \alpha_n)$, and $g(f(x)) = x$. □

Souslin's measurability theorem *Suppose that X and Y are Polish spaces, and that $f : X \to Y$ is 1-1, and measurable, then:*
$f(X) \in \mathcal{B}(X)$, and $f^{-1} : f(X) \to X$ is measurable.

Proof Let $g : Y \to X$ be as in the inverse function theorem. We have that :
$\{y \in Y :\ f \circ g(y) = y\} \in \mathcal{B}(Y) \ \because\ f \circ g : Y \to Y$ is measurable; and

$$f(X) = \{y \in Y :\ f \circ g(y) = y\}.$$

The measurability of $f^{-1} : f(X) \to X$ follows from $f^{-1} = g|_{f(X)}$. □

Cross sections. Consider the ordering on $\mathbb{N}^{\mathbb{N}}$ defined by $x < y$ if $\exists\ n \ge 1$ such that $x_n < y_n$ and $x_j = y_j\ \forall\ 1 \le j < n$ (if any). Every closed set has a mininum with respect to this ordering. Note that $\underline{1} \le \mathbb{N}^{\mathbb{N}}$ where $(\underline{a})_n := a\ \forall\ a, n \ge 1$. For $x < y \in \mathbb{N}^{\mathbb{N}}$, write

$$[x, y] := \{z \in \mathbb{N}^{\mathbb{N}} :\ x \le z \le y\},\ [x, y) = [x, y] \setminus \{y\} \dots.$$

Note that for each $x \in \mathbb{N}^{\mathbb{N}}$, $[\underline{1}, x)$ is open.

Exercise *Show that cylinder sets are generated by the countable collection*

$$\{[\underline{1}, x] :\ x \in \mathbb{N}^{\mathbb{N}},\ x_n \to a\}.$$

Cross section theorem *Let X and Y be Polish spaces, $\varnothing \ne \mathfrak{a} \subset X \times Y$ be analytic, and $\mathfrak{p} = \{x \in X :\ \exists\ y \in Y\ (x, y) \in \mathfrak{a}\}$, then $\exists\ f : \mathfrak{p} \to Y$ analytically measurable (i.e. $f^{-1}\mathcal{B}(Y) \subset \mathcal{A}(X)$) such that*

$$\{(x, f(x)) :\ x \in \mathfrak{p}\} \subset \mathfrak{a}.$$

Proof Let $T : \mathbb{N}^{\mathbb{N}} \to X \times Y$ be continuous with $\mathfrak{a} = T(\mathbb{N}^{\mathbb{N}})$.

Writing $\pi(x, y) = x,\ (\pi : X \times Y \to X)$ we have that $g := \pi \circ T : \mathbb{N}^{\mathbb{N}} \to \mathfrak{p}$ is continuous and onto. For each $x \in \mathfrak{p}$, $g^{-1}(\{x\})$ is closed in $\mathbb{N}^{\mathbb{N}}$. We define $h : \mathfrak{p} \to \mathbb{N}^{\mathbb{N}}$ by $h(x) := \min g^{-1}(\{x\})$.

Clearly $g \circ h$ =Id. To check analytic measurability of $h : \mathfrak{p} \to \mathbb{N}^{\mathbb{N}}$, note that

$$h^{-1}([\underline{1}, a)) = \{y \in \mathfrak{p} :\ h(y) < a\} = \{y \in \mathfrak{p} :\ \min g^{-1}(\{y\}) < a\} = g([\underline{1}, a))$$

is analytic $\forall\ a \in \mathbb{N}^{\mathbb{N}}$.

Writing $\psi(x, y) = yx,\ (\psi : X \times Y \to Y)$ we have that $f = \psi \circ T \circ h : \mathfrak{p} \to Y$ is analytically measurable, and

$$\{(x, f(x)) : x \in \mathfrak{p}\} = \{T \circ h(x) :\ x \in \mathfrak{p}\} \subset \mathfrak{a}.$$

$\square$

Remark $\exists\ \mathfrak{a} \in \mathcal{B}(X \times Y)$ such that $\nexists\ f : \mathfrak{p} \to Y$ Borel measurable.

Corollary: Section theorem[5] *Suppose that X and Y are Polish spaces, and that $T : X \to Y$ is measurable. There is an analytically measurable function $f : T(X) \to X$ such that*

$$T \circ f = Id_{T(X)}.$$

[5] 30/11/95

Proof Let $\mathfrak{a} \subset Y \times X$ be defined by

$$\mathfrak{a} := \{(T(x), x) : \ x \in X\} \overset{(!)}{\in} \mathcal{B}(Y \times X),$$

and let $\mathfrak{p}$, $f : \mathfrak{p} \to X$ be as in the cross section theorem; then $\mathfrak{p} = T(X)$ and

$$\{(x, f(x)) : x \in T(X)\} \subset \{(T(x), x) : \ x \in X\}$$

whence $T \circ f = \mathrm{Id}_{T(X)}$. □

Exercise *Let $(X, \mathcal{B}, m)$ be a standard probability space, and let $T : X \to X$ be measurable and measure preserving (i.e. $m(T^{-1}A) = m(A) \ \forall \ A \in \mathcal{B}$). Suppose that for m-a.e. $x \in X$, $|T^{-1}\{x\}| < \infty$. Prove that $\exists$ disjoint sets $A_1, A_2, \dots \in \mathcal{B}$ such that $\cup_{n=1}^{\infty} = X$ mod m, $TA_1, TA_2, \dots \in \mathcal{B}$, and $T : A_k \to TA_k$ is 1-1 and bimeasurable.*

EXERCISE N$^{\underline{\text{O}}}$4

1. Generalized Hausdorff-type measures.

Let (X, d) be a metric space, let $a : 2^X \to [0, \infty]$ with $a(\varnothing) = 0$ and define, for $\varepsilon > 0$, and $A \subset X$,

$$H^{(\varepsilon)}(A) := \inf\{\sum_{k=1}^{\infty} a(A_k) : A \subset \bigcup_{k=1}^{\infty} A_k, \ \mathtt{diam}\, A_k < \varepsilon \ \forall \ k \geq 1\};$$

where $\mathtt{diam}\, A := \sup_{x,y \in A} d(x, y)$; and let $H(A) := \lim_{\varepsilon \to 0} H^{(\varepsilon)}(A) \leq \infty$.

Show that $H : 2^X \to [0, \infty]$ is a metric outer measure.

3. Measure algebra. Let $(X, \mathcal{B}, m)$ be a finite measure space. Define a relation on $\mathcal{B}$ by $A \sim B$ if $m(A \Delta B) = 0$.

a) Show that $\sim$ is an equivalence relation.

b) Let $\mathcal{B}^{\sim} = \{[A] := \{A' \in \mathcal{B} : A' \sim A\} : A \in \mathcal{B}\}$ be the collection of equivalence classes.

Show that

$$d(a, b) := m(A \Delta B) \ \mathtt{for} \ \ a, b \in \mathcal{B}^{\sim}, \quad A \in a, B \in b$$

defines a metric on $\mathcal{B}^{\sim}$, and that $(\mathcal{B}^{\sim}, d)$ is a complete metric space (called the *measure algebra* of $(X, \mathcal{B}, m)$).

c) Show that the following are equivalent:

(i) $\mathcal{B}^{\sim}$ is separable;

(ii) $\exists \ A_n \in \mathcal{B} \ \ (n \geq 1)$ such that $\forall \ A \in \mathcal{B} \ \exists \ A' \in \sigma(\{A_n\}_{n=1}^{\infty})$ with $m(A \Delta A') = 0$;

d) Is there a probability space $(X, \mathcal{B}, \mu)$ equipped with sets $\{A_s : s \in (0,1)\} \subset \mathcal{B}$ such that $\mu(A_s \Delta A_t) \geq \frac{1}{4} \ \forall \ \ s \neq t$?

§5 Lebesgue integral on a finite measure space

5.1 Integral of a bounded, measurable function.

Countable additivity is not needed for the integration of bounded functions. This observation will be used when computing $L^{\infty *}$ (below).

A *finitely additive, finite measure space* is a triple $(X, \mathcal{B}, \mu)$ with $\mathcal{B}$ an algebra of subsets of X and $\mu : \mathcal{B} \to [0, \infty)$ finitely additive (e.g. a normal probability space). A function $f : X \to \mathbb{R}$ is *measurable* if $f^{-1}U \in \mathcal{B}$ for open sets $U \subset \mathbb{R}$.

Let be a finitely additive, finite measure space. A simple function is a finite linear combination of measurable indicators. The same simple function can be `represented` by many different finite linear combinations as above.

- A *representation* of the simple function f is a finite collection $\{(a_k, A_k)\}_{k=1}^N \subset \mathbb{R} \times \mathcal{B}$ satisfying `(and denoted by)` $f = \sum_{k=1}^N a_k 1_{A_k}$.
- The representation $f = \sum_{k=1}^N a_k 1_{A_k}$ is *disjoint* if the sets $\{A_k\}_{k=1}^N$ are disjoint.
- The *canonical representation* of the simple function $f : X \to \mathbb{R}$ is $f = \sum_{v \in f(X)} v 1_{[f=v]}$. This is disjoint.

The *integral* of the simple function $f : X \to \mathbb{R}$ is

$$\int_X f d\mu = \mu(f) := \sum_{v \in f(X)} v\mu([f = v]).$$

5.2 Proposition *Let $f : X \to \mathbb{R}$ be simple with representation $f = \sum_{k=1}^N a_k 1_{A_k}$ where $a_1, \dots, a_n \in \mathbb{R}$ and $A_1, \dots, A_n \in \mathcal{B}$, then*

$$\mu(f) = \sum_{k=1}^N a_k \mu(A_k).$$

`Proof` Suppose that $f = \sum_{k=1}^N a_k 1_{A_k}$ is a disjoint representation, then

$$[f = v] = \biguplus_{1 \le k \le N,\ a_k = v} A_k \ \forall\ v \in f(X)$$

and by additivity of μ,

$$\begin{aligned} \mu(f) &= \sum_{v \in f(X)} v\mu([f = v]) \\ &= \sum_{v \in f(X)} \sum_{1 \le k \le N,\ a_k = v} a_k \mu(A_k) \\ &= \sum_{k=1}^N a_k \mu(A_k). \quad ☑ \end{aligned}$$

For a general (non disjoint) representation, we need to consider the *partition generated by* $A_1, \dots, A_N \in \mathcal{B}$.

For $V \subset X$ define $V^1 := V$ and $V^0 := X \setminus V = V^c$.

For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \in \{0,1\}^N$ define $A_\varepsilon := \bigcap_{k=1}^N A_k^{\varepsilon_k}$.

Note that $X = \biguplus_{\varepsilon \in \{0,1\}^N} A_\varepsilon$ and $A_k = \biguplus_{\varepsilon \in \{0,1\}^N,\ \varepsilon_k = 1} A_\varepsilon$. We claim that

$$f = \sum_{\varepsilon \in \{0,1\}^N} \Big(\sum_{k=1}^N a_k \varepsilon_k\Big) 1_{A_\varepsilon}. \tag{$\boxed{\mathbf{i}}$}$$

`Proof of` $\boxed{\mathbf{i}}$:

$$\begin{aligned} f &= \sum_{k=1}^N a_k 1_{A_k} = \sum_{k=1}^N a_k \sum_{\varepsilon \in \{0,1\}^N,\ \varepsilon_k = 1} 1_{A_\varepsilon} \\ &= \sum_{\varepsilon \in \{0,1\}^N} \Big(\sum_{k=1}^N a_k \varepsilon_k\Big) 1_{A_\varepsilon}. \ \boxtimes \ (\boxed{\mathbf{i}}) \end{aligned}$$

This is a disjoint representation and so by the above,

$$\begin{aligned} \mu(f) &= \sum_{\varepsilon \in \{0,1\}^N} \Big(\sum_{k=1}^N a_k \varepsilon_k\Big) \mu(A_\varepsilon) \\ &= \sum_{k=1}^N a_k \sum_{\varepsilon \in \{0,1\}^N,\ \varepsilon_k = 1} \mu(A_\varepsilon) \\ &= \sum_{k=1}^N a_k \mu(A_k). \quad \boxtimes \end{aligned}$$

5.3 Proposition `(linearity and positivity)`

Let $f,\ g : X \to \mathbb{R}$ be simple functions, then

$$\int_X (af + bg) d\mu = a \int_X f d\mu + b \int_X g d\mu \quad \forall\ a, b \in \mathbb{R};$$

$$f \ge 0 \quad \Rightarrow \quad \int_X f d\mu \ge 0,$$

and

$$|\mu(f)| \le \int_X |f| d\mu \le \sup_X |f| \int_X 1 d\mu = \sup_X |f| \mu(X).$$

Integral of a bounded, measurable function.

Let $(X, \mathcal{B}, \mu)$ be a finite measure space.

5.4 Proposition

For $f : X \to \mathbb{R}$ bounded, measurable ,

$$\exists \lim_{g \texttt{ simple}, \ \sup_X |f-g| \to 0} \int_X g d\mu =: \int_X f d\mu.$$

`Proof` $\exists$ sequences of simple functions converging uniformly to f. By positivity, if g_n are simple and $g_n \to f$ uniformly, $\{\mu(g_n)\}_n$ is a Cauchy sequence. If $g_n,\ h_n$ are simple and $g_n,\ h_n \to f$ uniformly, then $\exists\ \lim_{n\to\infty} \mu(g_n) =: a,\ \lim_{n\to\infty} \mu(h_n) =: b$. To see that $a = b$, again by positivity,

$$|a - b| \leftarrow \ |\mu(g_n) - \mu(h_n)| \le \mu(X) \sup_X |g_n - h_n| \to 0. \quad \square$$

This number $\mu(f) = \int_X f d\mu$ is the *integral* (of f with respect to μ). The linearity and positivity are preserved under the limit.

5.5 Proposition

Suppose $f, g : X \to \mathbb{R}$ are bounded measurable functions, and $\alpha, \beta \in \mathbb{R}$.

(1) $$\int_A (\alpha f + \beta g) dm = \alpha \int_A f dm + \beta \int_A g dm.$$

(2) $$f \le g \quad \Rightarrow \quad \int_A f dm \le \int_A g dm.$$

Integral of unbounded, measurable functions.

For this, countable additivity is needed and henceforward, $(X, \mathcal{B}, m)$ is a finite (σ-additive) measure space.

Integral of a non-negative, measurable function.

For $f : X \to [0, \infty)$ measurable and $A \in \mathcal{B}$, define

$$\int_A f dm := \sup \{ \int_A g dm : g \le f, \ g \texttt{ simple } \} \le \infty.$$

- Note that if $f,\ g : X \to [0, \infty)$ are measurable,

(i) $f \le g$ on $A \in \mathcal{B} \implies \int_A f d\mu \le \int_A g d\mu$;

(ii)
$$\int_A f dm = \sup\{\int_A g dm : g \le f, \ \ g \texttt{ bounded, measurable }\}$$
$$= \sup\{\int_A g dm : g \le f, \ \ g \ge 0 \texttt{ measurable }\}.$$

Integrability.

The measurable function $f : X \to \mathbb{R}$ is called *integrable on* $A \in \mathcal{B}$ if $\int_A |f| dm < \infty$. As above, in this case $\int_A f_\pm dm < \infty$ where $f_\pm := \max\{\pm f, 0\}$ so that $f = f_+ - f_-$. We define

$$\int_A f dm := \int_A f_+ dm - \int_A f_- dm.$$

Before proving positivity and linearity for the integrals of integrable functions, we need some:

Basic convergence theory

Throughout this section, $(X, \mathcal{B}, m)$ is a probability space.

We'll need the

5.6 BC lemma[6]

Let $A_n \in \mathcal{B}$, $(n \ge 1)$.

If $\sum_{n\ge1} m(A_n) < \infty$, *then* $m([\sum_{n=1}^\infty 1_{A_n} = \infty]) = 0$.

Proof

$$m([\sum_{n=1}^\infty 1_{A_n} = \infty]) \xleftarrow[n\to\infty]{} m(\bigcup_{k=n}^\infty A_k)$$
$$\le \sum_{k=n}^\infty m(A_k)$$
$$\xrightarrow[n\to\infty]{} 0. \quad ☑$$

Let $f_n, f : X \to \mathbb{R}$ be measurable.

Convergence in measure. We say that f_n *converges in measure to* f written $f_n \xrightarrow[n\to\infty]{m} f$ if

$$m([|f_n - f| > \varepsilon]) \xrightarrow[n\to\infty]{} 0 \ \ \forall\ \varepsilon > 0.$$

[6] BC=Borel Cantelli

a.e. convergence. We say that f_n *converges a.e. to* f written $f_n \xrightarrow[n\to\infty]{\text{a.e.}} f$ if

$$m([|f_n \not\to f]) = 0.$$

5.7 Proposition

Suppose $f_n, f : X \to \mathbb{R}$ are measurable.

(i) $f_n \xrightarrow[n\to\infty]{\text{a.e.}} f \;\Rightarrow\; f_n \xrightarrow[n\to\infty]{m} f$;

(ii) $f_n \xrightarrow[n\to\infty]{m} f \;\not\Rightarrow\; f_n \xrightarrow[n\to\infty]{\text{a.e.}} f$;

(iii) $f_n \xrightarrow[n\to\infty]{m} f \;\Rightarrow\; \exists\ n_k \to \infty$ *such that* $f_{n_k} \xrightarrow[k\to\infty]{\text{a.e.}} f$.

Proof of (i)

For $\varepsilon > 0$,

$$\begin{aligned} m([|f_n - f| \ge \varepsilon]) &\le m(\bigcup_{k=n}^{\infty}[|f_k - f| \ge \varepsilon]) \\ &\xrightarrow[n\to\infty]{} m(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}[|f_k - f| \ge \varepsilon]) \\ &= m([\sum_{n=1}^{\infty} 1_{[|f_n - f| \ge \varepsilon]} = \infty]) \\ &= m([\varlimsup_{n\to\infty} |f_n - f| \ge \varepsilon]) \\ &= 0 \quad ☑ \end{aligned}$$

Proof of (ii) Exercise.

Proof of (iii) Fix $n_k \uparrow \infty$ so that

$$\sum_{k=1}^{\infty} m([|f_{n_k} - f| > \frac{1}{k}]) < \infty.$$

It is easy to see using the BC lemma that $f_{n_k} \xrightarrow[k\to\infty]{\text{a.e.}} f$. ☑

5.8 Egorov's Theorem

Suppose $f_n, f : X \to \mathbb{R}$ are measurable and that $f_n \to f$ a.e. as $n \to \infty$, then $\forall\ \varepsilon > 0\ \exists\ F \in \mathcal{B}$ such that $m(X \setminus F) < \varepsilon$ and $f_n \to f$ uniformly on F.

Proof Since $f_n \to f$ a.e., we have that

$$\forall\ \eta > 0,\ A(n, \eta) := \bigcap_{k \ge n}[|f_k - f| < \eta] \uparrow\ X \mod \mu$$

as $n \to \infty$. Thus, $\exists\ n_m\ \ (m \in \mathbb{N})$ so that $\mu(A(n_m, \frac{1}{m})^c) < \frac{\varepsilon}{2^m}$ and $F := \bigcap_{m=1}^{\infty} A(n_m, \frac{1}{m})$ is as advertised. □

5.9 Monotone convergence theorem (Beppo Levi)

Let $(X,\mathcal{B},m)$ be a finite measure space. Suppose $f_n, f : X \to [0,\infty)$ are measurable. If $f_n \uparrow f$ a.e., then

$$\int_A f_n dm \to \int_A f dm \ \text{as } n \to \infty \ \forall \ A \in \mathcal{B}.$$

`Proof`

It suffices, given $\varepsilon > 0$, $A \in \mathcal{B}$. $0 \le g \le f$, g simple to show that $\lim_{n\to\infty} \int_A f_n dm > \int_A g dm - \varepsilon$.

Let $K := \sup\ g$. By Egorov's theorem $\exists\ B \in \mathcal{B}$, $B \subset A$ such that $f_n \to f$ uniformly on B and $m(A \setminus B) < \frac{\varepsilon}{2K}$.

Choose n_0 such that

$$f_n(x) > g(x) - \frac{\varepsilon}{2m(B)} \ \forall \ n \ge n_0, \ x \in B,$$

then for $n \ge n_0$:

$$\begin{aligned}\int_A f_n dm &\ge \int_B f_n dm \\ &> \int_B g dm - \frac{\varepsilon}{2} \\ &= \int_A g dm - \int_{A\setminus B} g dm - \frac{\varepsilon}{2} \\ &> \int_A g dm - \varepsilon.\end{aligned}$$

□

5.10 Fatou's lemma

For $f_n \ge 0$ measurable,

$$\liminf_{n\to\infty} \int_A f_n dm \ge \int_A \liminf_{n\to\infty} f_n dm.$$

`Proof` Let $g_N(x) := \inf_{k\ge N} f_k(x)$, then $g_N \uparrow \liminf_{n\to\infty} f_n$, whence by the monotone convergence theorem, $\forall\ N \ge 1$

$$\int_A f_N dm \ge \int_A g_N dm \underset{N\to\infty}{\longrightarrow} \int_A \liminf_{n\to\infty} f_n dm. \quad \square$$

5.11 Proposition

Suppose $f, g : X \to [0, \infty)$ are measurable functions, and $\alpha, \beta \in \mathbb{R}_+$. Then

$$\int_A (\alpha f + \beta g) dm = \alpha \int_A f dm + \beta \int_A g dm. \tag{1}$$

$$f \le g \quad \Rightarrow \quad \int_A f dm \le \int_A g dm. \tag{2}$$

Proof Take limits of simple functions using the monotone convergence theorem. ☐

5.12 Lemma

If $f,\ g,\ h : X \to \mathbb{R}$ are integrable functions, $g,\ h \ge 0$ and $f = g - h$, then

$$\int_A f dm = \int_A g dm - \int_A h dm.$$

`Proof` Since $\int_A f dm := \int_A f_+ dm - \int_A f_- dm$, it suffices to show that if $g_i,\ h_i : X \to [0, \infty)$ $(i = 1, 2)$ are integrable functions, satisfying $g_1 - h_1 = g_2 - h_2$, then

$$\int_A g_1 dm - \int_A h_1 dm = \int_A g_2 dm - \int_A h_2 dm.$$

To see this, note that $g_1 + h_2 = g_2 + h_1$ whence

$$\int_A g_1 dm + \int_A h_2 dm = \int_A g_2 dm + \int_A h_1 dm. \quad ☑$$

5.13 Linearity and positivity theorem

Suppose $f, g : X \to \mathbb{R}$ are integrable functions, and $\alpha, \beta \in \mathbb{R}$. Then: $(\alpha f + \beta g)$ is integrable, and

$$\int_A (\alpha f + \beta g) dm = \alpha \int_A f dm + \beta \int_A g dm. \tag{1}$$

$$f \le g \quad \Rightarrow \quad \int_A f dm \le \int_A g dm. \tag{2}$$

`Proof` Recall $\int f := \int f_+ - \int f_-$.

- For $\alpha > 0$, $\alpha f_\pm = (\alpha f)_\pm$ and $(-f)_\pm = -f_\mp$ so (!) $\int (\alpha f) = \alpha \int f$.

• To see $\int(f+g)=\int f+\int g$ note that $f+g=(f_+ + g_+)-(f_- + g_-)$ and use the lemma.

• To see (2), let $f\le g$, then $g-f\ge 0$ whence

$$0\le\int(g-f)=\int g-\int f. \quad \boxtimes$$

Uniform integrability

Let $(X,\mathcal{B},m)$ be a finite measure space, and let

$$L(X,\mathcal{B},m):=\{f:X\to\mathbb{R}:\ \texttt{measurable}\}.$$

The family $\mathcal{F}\subset L(X,\mathcal{B},m)$ is called *uniformly integrable* (`UI`) if

$$\forall\ \varepsilon>0\ \exists\ M>0\ \ni\ \int_{[|f|\ge M]}|f|dm<\varepsilon\quad\forall\ f\in\mathcal{F}.$$

• If $\mathcal{F}\subset L(X,\mathcal{B},m)$ is uniformly integrable, then $\sup_{f\in\mathcal{F}}\|f\|_1<\infty$.

`Proof` Fix $M>0$ such that $\int_{[|f|\ge M]}|f|dm<1\ \ \forall\ f\in\mathcal{F}$, then $\sup_{f\in\mathcal{F}}\|f\|_1\le Mm(X)+1$. ☐

• If $\mathcal{F}\subset L(X,\mathcal{B},m)$, $p=1+\eta>1$ and $\sup_{f\in\mathcal{F}}\int|f|^p=:C<\infty$, then $\mathcal{F}$ is UI.

`Proof` For $f\in\mathcal{F}$,

$$\int_{[|f|\ge k]}|f|dm\le\frac{1}{k^\eta}\int|f|^p\le\frac{C}{k^\eta}. \quad \Box$$

• $\sup_{f\in\mathcal{F}}\|f\|_1<\infty\ \not\Rightarrow\ \mathcal{F}$ UI.

`Proof` Let $X=[0,1]$, $m=$Lebesgue, $f_n:=n1_{[0,\frac{1}{n}]}$, then $\|f_n\|_1=1$ but $\sup_n\int_{[f_n\ge k]}f_n=1\ \forall\ k>0$. ☐

5.14 Proposition (`dominated` $\Rightarrow$ `UI`) *If $\mathcal{F}\subset L(X,\mathcal{B},m)$ and $|f|\le g\ \forall\ f\in\mathcal{F}$ where g is integrable, then $\mathcal{F}$ is uniformly integrable.*

`Proof` We have $G_k:=g(1-1_{[g\ge k]})\uparrow\ g$ as $k\uparrow\infty$. By Beppo-Levi's theorem $\int G_k\uparrow\ \int g$. It follows that $\int_{[g\ge k]}gdm\xrightarrow[k\to\infty]{}0$. Thus

$$\sup_{f\in\mathcal{F}}\int_{[|f|\ge k]}|f|dm\le\int_{[g\ge k]}gdm\xrightarrow[k\to\infty]{}0. \quad \boxtimes$$

The converse implication is wrong. See exercise 5.2.

The idea of uniform integrability is to obtain the best possible convergence theorems.

5.15 Convergence theorem *Suppose that $f_n : X \to \mathbb{R}$ are measurable, $f_n \to 0$ a.e., and $\{f_n : n \ge 1\}$ is uniformly integrable, then*

$$\int_X f_n dm \to 0 \text{ as } n \to \infty.$$

`Proof` WLOG, $f_n \ge 0$.

$$\begin{aligned}\int_X f_n dm &\le \int_X 1_{[f_n<M]} f_n dm + \int_{[f_n \ge M]} f_n dm \\ &\le \int_X 1_{[f_n<M]} f_n dm + \sup_{n\ge 1} \int_{[f_n \ge M]} f_n dm.\end{aligned}$$

Given $\varepsilon > 0$ choose using UI $M = M_\varepsilon$ such that

$$\sup_n \int_{[f_n \ge M]} f_n dm < \frac{\varepsilon}{2}.$$

By Egorov's theorem, $\exists\ F \in \mathcal{B}$ with $m(X \setminus F) < \frac{\varepsilon}{2M}$ so that $f_n \to 0$ uniformly on F. Fix N_ε so that

$$f_n(x) < \frac{\varepsilon}{4}\ \forall\ x \in F,\ n > N_\varepsilon.$$

It follows that for $n > N_\varepsilon$,

$$\begin{aligned}\int_X 1_{[f_n<M]} f_n dm &= \int_F 1_{[f_n<M]} f_n dm + \int_{X\setminus F} 1_{[f_n<M]} f_n dm \\ &\le \frac{\varepsilon}{4} + Mm(X \setminus F) \\ &< \frac{\varepsilon}{2}. \quad \boxed{\checkmark}\end{aligned}$$

5.16 Corollary: `Dominated convergence theorem`

Suppose $f_n, f, g : X \to \mathbb{R}$ are measurable functions, and suppose that $f_n \to f$ a.e., $|f_n| \le g\ \forall\ n \ge 1$, and $\int_X g dm < \infty$, then f is integrable, and

$$\int_X f_n dm \to \int_X f dm \text{ as } n \to \infty.$$

Proof Exercise.

Exercise N$^{\underline{o}}$5

1. Tightness of probability on a polish space.

Let X be polish and let $p \in \mathcal{P}(X)$. Show that $\forall\ \varepsilon > 0,\ \exists\ K$ compact, such that $p(K) > 1 - \varepsilon$.

2. Uniformly integrable example.

Let $(X, \mathcal{B}, m) = ((0,1], \mathcal{B}((0,1]), \texttt{Lebesgue})$. For $x, y \in X$, define $r_y(x) := x \oplus y := x + y \mod 1$.

(i) Show that for each $a \in X$, $r_a : X \to X$ is a *measure preserving transformation* in the sense that $m(r_a A) = m(A)\ \forall\ A \in \mathcal{B}(X)$.

(ii) Show that if $g : [0,1) \to \mathbb{R}$ is integrable and $a \in X$, then so is $g \circ r_a$ with $\int_X g \circ r_a dm = \int_X g dm$.

(iii) Set $f(x) := \frac{1}{\sqrt{x}}$, $\mathcal{F} := \{f \circ r_a :\ a \in X\}$. Show that $\mathcal{F}$ is uniformly integrable and $\sup_{q \in X \cap \mathbb{Q}} f \circ r_q \equiv \infty$.

3. Convergence in measure.

Let $(X, \mathcal{B}, m)$ be a probability space, and suppose that $f_k : X \to \mathbb{R}$ $(k \geq 1)$ are measurable functions.

Is it true that

(i) $f_n \xrightarrow[n\to\infty]{m} 0 \Rightarrow f_n \xrightarrow[n\to\infty]{\texttt{a.e.}} 0$ **?**

(ii) $f_n \xrightarrow[n\to\infty]{m} 0 \quad \Rightarrow \quad \frac{1}{n}\sum_{k=1}^{n} f_k \xrightarrow[n\to\infty]{m} 0$ **?**

4. Baire space and the irrationals.

(a) Show (using continued fractions or otherwise) that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic with $(0,1) \setminus \mathbb{Q}$.

(b)☆ Show that in every perfect Polish space there is a dense G_δ set which is homeomorphic with $\mathbb{N}^{\mathbb{N}}$.

5. Analytic sets.

Show that countable unions and intersections of analytic sets are analytic if nonempty.

6 Non Borel analytic sets.

Let X, Y be Polish. A *projection* is a map of form $f : X \times Y \to X$, $f(x, y) = x$. The *projection of a set* is its image under a projection.

This problem is about Lebesgue's claim (1905) that the projection of a Borel set is Borel, and its refutation by Suslin (1917).

6.a Projections of closed sets.

Show that a subset of a Polish space X is analytic $\iff$ it is the projection of a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$.

`Hint` for $\Rightarrow$) Consider the graph of T where $T : \mathbb{N}^{\mathbb{N}} \to X$ continuous with $T(\mathbb{N}^{\mathbb{N}}) = A$.

`Definition of Universality` Let X be a set, and $\mathcal{F} \subset 2^X$. A subset A of $\mathbb{N}^{\mathbb{N}} \times X$ is *universal for* $\mathcal{F}$ if

$$\{A_y : \ y \in \mathbb{N}^{\mathbb{N}}\} = \mathcal{F}$$

where $A_y := \{x \in X : (y,x) \in A\}$.

6.b Universal open and closed sets.
Suppose that X is a separable metric space. Let $\mathfrak{u} := \bigcup_{k,n\in\mathbb{N}} [n]_k \times U_n$ where $[n]_k := \{x \in \mathbb{N}^{\mathbb{N}} : \ x_k = n\}$ and $\{U_n : \ n \ge 1\}$ is a base for the open sets in X. Show that $\mathfrak{u}$ is an open subset of $\mathbb{N}^{\mathbb{N}} \times X$ which is universal for the open subsets of X, and that there is a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$ which is universal for the closed subsets of X.

6.c Universal analytic sets.
Let X be Polish. Fix a closed subset F is of $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}} \times X)$ which is universal for the closed subsets of $\mathbb{N}^{\mathbb{N}} \times X$. Let $G := \{(x,z) \in \mathbb{N}^{\mathbb{N}} \times X : \exists \ y \in \mathbb{N}^{\mathbb{N}}, \ (x,y,z) \in F\}$. Show that G is an analytic subset of $\mathbb{N}^{\mathbb{N}} \times X$ which is universal for the analytic subsets of X.
`Hint` Use ex. 6.a repeatedly.

6.d Non-Borel, analytic subset of $\mathbb{N}^{\mathbb{N}}$*.*
Let

$$B := \{x \in \mathbb{N}^{\mathbb{N}} : \ (x,x) \in A\}$$

where $A \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is universal for the collection of analytic subsets of $\mathbb{N}^{\mathbb{N}}$. Show that

(i) B is analytic;

(ii) $\nexists \ u \in \mathbb{N}^{\mathbb{N}}$ with $B^c = A_u$;

(iii) B^c is not analytic;

(iv) B is not Borel.

6.e General non-Borel, analytic subsets. Show that in any uncountable polish space, there is a non-Borel, analytic subset.

6.f Lebesgue's claim (1905). Refute it.

Week # 5
§6 Approximation by continuous functions

6.1 Frechet's theorem *Let* X *be a polish space and let* $p \in \mathcal{P}(X)$ *and let* $f : X \to \mathbb{R}$ *be measurable, then*

(⚓) $\forall \ \Delta > 0 \ \exists \ h \in C_B(X)$ `with` $p([|f-h| \ge \Delta]) < \Delta.$

`Proof` `of (✠) for` $f = 1_F$ `where` F `is closed:`

Let $\varDelta > 0$ and let d be a Polish metric on X and for $n \ge 1$, define $h_n : X \to \mathbb{R}$ by $h_n(x) := (1 - nd(x,F)) \vee 0$. Evidently

$$h_n(x) = \begin{cases} \in [0,1] & \forall\ x \in X; \\ 1 & x \in F; \\ 0 & d(x,F) \ge \frac{1}{n}. \end{cases}$$

Thus $[h_n \ne 1_F] = B(x, \frac{1}{n}) \smallsetminus F \downarrow \varnothing$ whence $p([h_n \ne 1_F]) < \varDelta$ for n large. ☑

`Proof of (✠) for` $f = 1_A$ `where` $A \in \mathcal{B}(X)$:

Let $\varDelta > 0$, then $\exists\ F \subset A$ closed so that $p(A \smallsetminus F) < \frac{\varDelta}{2}$ and by (✠) for $f = 1_F$, $\exists\ h \in C_B(X)$ such that $p([|1_F - h| \ge \varDelta]) < \frac{\varDelta}{2}$. Thus

$$\begin{aligned} p([|1_A - h| \ge \varDelta]) &\le p([1_F \ne 1_A]) + p([|1_F - h| \ge \varDelta]) \\ &= p(A \smallsetminus F) + p([|1_F - h| \ge \varDelta]) \\ &< \varDelta. \ ☑ \end{aligned}$$

`Proof of (✠) for` s `simple`:

Suppose that $|s| \le M$, and let $\varDelta > 0$.

For each $y \in s(X)$, $\exists\ h_y \in C_B(X)$ such that

$$p([|1_{[s=y]} - h_y| \ge \frac{\varDelta}{M|s(X)|}]) < \frac{\varDelta}{|s(X)|}.$$

Define $h := \sum_{y \in s(X)} y h_y \in C_B(X)$, then

$$\begin{aligned} p([|s - h| \ge \varDelta]) &= p([|\sum_{y \in s(X)} y(1_{[s=y]} - h_y)| \ge \varDelta]) \\ &\le \sum_{y \in s(X)} p([|1_{[s=y]} - h_y| \ge \frac{\varDelta}{M|s(X)|}]) \\ &< \varDelta. \ ☑ \end{aligned}$$

`Proof of (✠) in general`:

Let $f : X \to \mathbb{R}$ be measurable and let $\varDelta > 0$.

- $b : X \to \mathbb{R}$ bounded, measurable with $p([f \ne b]) < \frac{\varDelta}{4}$ ($\because$ $p([|f| \ge n]) \xrightarrow[n \to \infty]{} 0$);
- $\exists\ s : X \to \mathbb{R}$ simple with $\sup |b - s| < \frac{\varDelta}{2}$;

- $\exists\ h \in C_B(X)$ such that $p([|s = h| \ge \frac{\Delta}{2}]) < \frac{\Delta}{4}$. It follows that

$$\begin{aligned} p([|h-f| \ge \Delta]) &\le p([f \ne b]) + p([|b-s| \ge \frac{\Delta}{2}]) + p([|s = h| \ge \frac{\Delta}{2}]) \\ &< \frac{\Delta}{4} + 0 + \frac{\Delta}{4} \\ &< \Delta. \ ☑\ (✠) \end{aligned}$$

6.2 Corollary: Luzin's theorem

Let X be a polish space and let $p \in \mathcal{P}(X)$ and let $f : X \to \mathbb{R}$ be measurable, then $\forall\ \varepsilon > 0\ \exists\ F \subset X$ compact with $p(F) > 1 - \varepsilon$ and $g \in C_B(X)$ so that $f|_F \equiv g|_F$.

Proof

By Frechet's theorem and proposition 5.7(iii), $\exists\ h_n \in C_B(X)$ such that $h_n \to f$ a.e. and by Egorov's theorem $\exists\ F \in \mathcal{B}$ such that $m(X \setminus F) < \varepsilon$ and such that this convergence is uniform on X. By tightness, F may be chosen to be compact. It follows that f is continuous on f. By Tietze's extension theorem $\exists\ g \in C_B(X)$ such that $g|_F \equiv f|_F$. ☑

§7 Product spaces and integration of measures

Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces. The *product measurable space* is $(X \times Y, \sigma(\mathcal{B} \times \mathcal{C}))$ where $\mathcal{B} \times \mathcal{C} := \{B \times C : \ B \in \mathcal{B},\ C \in \mathcal{C}\}$.

The *sections* of $A \subset X \times Y$ are defined by $A_x := \{y \in Y : (x, y) \in A\}$, and $A^y := \{x \in X : (x, y) \in A\}$. Evidently,

$$(A^c)_x = (A_x)^c, \quad (\bigcup_{t \in \Lambda} A(t))_x = \bigcup_{t \in \Lambda} (A(t))_x, \quad (A \times B)_x = \begin{cases} B & x \in A, \\ \varnothing & x \notin A. \end{cases}$$

The collection of σ-finite measures on the measurable space $(X, \mathcal{B})$ is denoted $\mathfrak{M}(X, \mathcal{B})$ and the collection of probabilities is

$$\mathcal{P}(X, \mathcal{B}) := \{\mu \in \mathfrak{M}(X, \mathcal{B}) : \ \mu(X) = 1\}.$$

7.3 Theorem (integration of probabilities) *Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces and let $\mu \in \mathcal{P}(X, \mathcal{B})$.*

Suppose that $\nu : X \to \mathcal{P}(Y, \mathcal{C})$ is measurable in the sense that $x \mapsto \nu_x(A)$ is measurable $\forall\ A \in \mathcal{C}$, then there is a unique

$$m \in \mathcal{P}(X \times Y, \sigma(\mathcal{B} \times \mathcal{C}))$$

such that

$$m(A \times B) = \int_A \nu_x(B) d\mu(x)\ \forall\ A \times B \in \mathcal{B} \times \mathcal{C}.$$

Proof

Since $\mathcal{B}\times\mathcal{C}$ is a semi-algebra, it suffices to show that $m_0 : \mathcal{B}\times\mathcal{C} \to [0,\infty)$ defined by

$$m_0(B\times C) = \int_B \nu_x(C)d\mu(x) \ \forall \ B\times C \in \mathcal{B}\times\mathcal{C}$$

is additive and countable subadditive.

To see additivity, suppose that

$$B\times C = \bigcupdot_{k=1}^N B_k\times C_k \text{ where } \ B,\ B_1,B_2,\dots B_N\in\mathcal{B},\ C,\ C_1,C_2,\dots C_N\in\mathcal{C},$$

then for $x\in X$,

$$(B\times C)_x = \bigcupdot_{k=1}^N (B_k\times C_k)_x,$$

whence

$$1_B(x)\nu_x(C) = \nu_x((B\times C)_x) = \sum_{k=1}^N \nu_x((B_k\times C_k)_x) = \sum_{k=1}^N 1_{B_k}(x)\nu_x(C_k),$$

and, integrating

$$m_0(B\times C) = \sum_{k=1}^N m_0(B_k\times C_k).$$

To prove that m_0 is countable subadditive, let

$$B,\ B_1,B_2,\dots\in\mathcal{B},\ C,\ C_1,C_2,\dots\in\mathcal{B}$$

and

$$B\times C\subset\bigcup_{j=1}^\infty B_j\times C_j,$$

then

$$\forall\ x\in X,\quad (B\times C)_x\subset\bigcup_{j=1}^\infty (B_j\times C_j)_x;$$

$$\therefore\ 1_B(x)\nu_x(C) = \nu_x((B\times C)_x) \le \sum_{j=1}^\infty \nu_x((B_j\times C_j)_x) = \sum_{j=1}^\infty 1_{B_j}(x)\nu_x(C_j),$$

and

$$\begin{aligned} m_0(B\times C) &= \int_B \nu_x(C)\mu(x) \\ &= \int_X (1_B(x)\nu_x(C))d\mu(x) \\ &\le \int_X \left(\sum_{j=1}^{\infty} 1_{B_j}(x)\nu_x(C_j)\right)d\mu(x) \\ &= \sum_{j=1}^{\infty} m_0(B_j\times C_j). \end{aligned}$$

The measure m exists by proposition 3.2 and is unique by proposition 2.3. ☐

7.4 Corollary (integration of measures) *Let $(X,\mathcal{B})$ and $(Y,\mathcal{C})$ be measurable spaces.*

Suppose that $\mu \in \mathfrak{M}(X,\mathcal{B})$ and that $\nu : X \to \mathfrak{M}(Y,\mathcal{C})$ is measurable and define $m_0 : \mathcal{B}\times\mathcal{C} \to [0,\infty]$ by $m_0(A\times B) = \int_A \nu_x(A)d\mu(x)$.

If $\exists\ A_n \in \mathcal{B},\ A_n \uparrow X$ and $B_n \in \mathcal{C},\ C_n \uparrow Y$ such that $m(A_n\times B_n) < \infty\ \forall\ n \ge 1$, then there is a unique $m \in \mathfrak{M}(X\times Y, \sigma(\mathcal{B}\times\mathcal{C}))$ such that $m|_{\mathcal{B}\times\mathcal{C}} \equiv m_0$.

In case $\nu_x \equiv \nu$ is constant, m is called *product* measure and denoted $m = \mu\times\nu$.

Remark. Suppose that $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ are $\mathbb{R}^d$, $\mathbb{R}^{d'}$ equipped with Lebesgue measure, then $\mu\times\nu$ is Lebesgue measure on $\mathbb{R}^{d+d'}$

Disintegration of sets, the Fubini-Tonelli theorems and generalizations

Let $(X,\mathcal{B})$ and $(Y,\mathcal{C})$ be measurable spaces.
The *sections* of $A \subset X\times Y$ are

$$A_u := \{y\in Y :\ (u,y)\in A\}\ \&\ A^v := \{x\in X :\ (x,v)\in A\} \quad (u\in X,\ v\in Y).$$

Theorem 7.5 *Let $\mu \in \mathfrak{M}(X,\mathcal{B})$ and let $\nu : X \to \mathfrak{M}(Y,\mathcal{C})$ be measurable, so that $\exists\ A_n \in \mathcal{B},\ A_n \uparrow X$ and $B_n \in \mathcal{C},\ C_n \uparrow Y$ with*

$$\int_{A_n} \nu_x(B_n)d\mu(x) < \infty\ \forall\ n.$$

Let $m \in \mathfrak{M}(X \times Y, \sigma(\mathcal{B} \times \mathcal{C}))$ *such that*

$$m(A \times B) = \int_A \nu_x(B) d\mu(x) \ \forall \ A \times B \in \mathcal{B} \times \mathcal{C}.$$

If $A \in \sigma(\mathcal{B} \times \mathcal{C})$, *then (i)* $A_x \in \mathcal{C} \ \forall \ x \in X$, *(ii) the function* $x \mapsto \nu_x(A_x)$ *is* $\mathcal{B}$*-measurable, and (iii)* $m(A) = \int_X \nu_x(A_x) d\mu(x)$.

Proof in case $m(X \times Y) < \infty$

The collection $\mathcal{D} \subset \sigma(\mathcal{B} \times \mathcal{C})$ of sets satisfying (i), (ii), and (iii) contains the algebra $\mathcal{A}$ generated by $\mathcal{B} \times \mathcal{C}$, and is a monotone class. By the monotone class theorem, $\mathcal{D} \supset \sigma(\mathcal{B} \times \mathcal{C})$. ☐

Theorem 7.6

If $h : X \times Y \to \mathbb{R}$ *is* $\sigma(\mathcal{B} \times \mathcal{C})$*-measurable, then* $y \mapsto h_x(y) := h(x, y)$ *is* $\mathcal{C}$*-measurable* $\forall \ x \in X$, *and*

(i)
$$h \geq 0 \ \Rightarrow x \mapsto \int_Y h_x d\nu_x \text{ is measurable on } X,$$
$$\& \int_{X \times Y} h dm = \int_X \left(\int_Y h_x d\nu_x \right) d\mu(x).$$

(ii)
$$h \in \mathcal{L}^1(m) \ \Rightarrow h_x \in \mathcal{L}^1(\nu_x) \text{ for a.e. } x \in X,$$
$$x \mapsto \int_Y h_x d\nu_x \text{ is integrable on } X,$$
$$\& \int_{X \times Y} h dm = \int_X \left(\int_Y h_x d\nu_x \right) d\mu(x).$$

(iii)
$$\int_X \left(\int_Y |h_x| d\nu_x \right) d\mu(x) < \infty \ \Rightarrow \ h \in \mathcal{L}^1(m).$$

Here, $\mathcal{L}^1(m)$ denotes the collection of m-integrable functions.

Proof In case h is an indicator function: $h = 1_A$, $A \in \sigma(\mathcal{B} \times \mathcal{C})$, this theorem follows from the previous theorem. Assume that $h : X \times Y \to \mathbb{R}$ is a simple function, $h = \sum_{k=1}^N a_k 1_{A_k}$ $A_k \in \sigma(\mathcal{B} \times \mathcal{C})$, then for $x \in X$, $h_x = \sum_{k=1}^N a_k 1_{(A_k)_x}$, $(A_k)_x \in \mathcal{C}$, whence h_x is $\mathcal{C}$-measurable. Moreover

$x \mapsto \int_Y h_x d\nu_x = \sum_{k=1}^N a_k \nu_x((A_k)_x)$ is $\mathcal{B}$-measurable, and

$$\int_X \left(\int_Y h_x d\nu_x \right) d\mu(x) = \int_X \left(\sum_{k=1}^N a_k \nu_x((A_k)_x) \right) d\mu(x)$$
$$= \sum_{k=1}^N a_k \int_X \nu_x((A_k)_x) d\mu(x) = \sum_{k=1}^N a_k m(A_k) = \int_{X \times Y} h dm.$$

If $h : X \times Y \to \mathbb{R}$ is $\sigma(\mathcal{B} \times \mathcal{C})$-measurable, then h is a pointwise limit of $\sigma(\mathcal{B} \times \mathcal{C})$-measurable simple functions, whence, $\forall\ x \in X$, h_x is a pointwise limit of $\mathcal{C}$-measurable simple functions, and hence is itself $\mathcal{C}$-measurable.

If $h \geq 0$ then h is a pointwise limit of an increasing sequence h_n of non-negative, $\sigma(\mathcal{B} \times \mathcal{C})$-measurable simple functions. By Lebesgue's monotone convergence theorem

$$\int_Y (h_n)_x d\nu_x \to \int_Y h_x d\nu_x \text{ as } n \to \infty \ \forall\ x \in X$$

whence $x \mapsto \int_Y h_x d\nu_x$ is $\mathcal{B}$-measurable on X. To complete the proof of (i), use monotone convergence again to show

$$\int_X \left(\int_Y h_x d\nu_x \right) d\mu(x) \ \leftarrow\ \int_X \left(\int_Y (h_n)_x d\nu_x \right) d\mu(x)$$
$$= \int_{X \times Y} h_n dm \ \to\ \int_{X \times Y} h dm \text{ as } n \to \infty.$$

Statement (iii) follows immediately from (i). To deduce (ii), note that by (i),

$$\int_X \left(\int_Y (h_x)_\pm d\nu_x \right) d\mu(x) = \int_{X \times Y} h_\pm dm$$

where $f_\pm := \max\{\pm f, 0\}$. ☐

7.7 Corollary (`Fubini-Tonelli theorem`)

If $h \in L^1(\mu \times \nu)$, then

$$\int_X \left(\int_Y h_x d\nu \right) d\mu(x) = \int_Y \left(\int_X h^y d\mu \right) d\nu(y) < \infty$$

where $h^y(x) = h(x, y)$.

Proof Apply 7.6 in case $\nu_x \equiv \nu$. ☐

Convolutions. Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable. For every $x \in \mathbb{R}$, the function $y \mapsto f(x-y)g(y)$ is measurable. In case this function is integrable, or non-negative, we can define the *convolution* of f and g at x by

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Let $m_{\mathbb{R}^d}$ denote Lebesgue measure on $\mathbb{R}^d$.

Note that $m_{\mathbb{R}^d} = \underbrace{m_{\mathbb{R}} \otimes \cdots \otimes m_{\mathbb{R}}}_{d\text{-times}}$.

Proposition 7.8 *If $f, g \in \mathcal{L}^1(m_{\mathbb{R}^d})$, then the function $h : \mathbb{R}^{2d} \to \mathbb{R}$ defined by $h(x,y) := f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$, and $f * g \in \mathcal{L}^1(m_{\mathbb{R}^d})$.*

Proof You establish the measurability of h in exercise 6.2.

For $y \in \mathbb{R}$, $\|h^y\|_1 = |g(y)|\|f\|_1$, whence, by Fubini's theorem, $h \in \mathcal{L}^1(m_{\mathbb{R}^{2d}})$.

Also by Fubini's theorem, we obtain that the convolution

$$f * g(x) := \int_{\mathbb{R}^d} h_x(y)dy$$

is defined at a.e. $x \in \mathbb{R}^d$, and is integrable:

$$\|f * g\|_1 \le \|h\|_1 = \int_{\mathbb{R}^d} \|h^y\|_1 dy = \|f\|_1 \|g\|_1.$$

$\square$

Sierpinski's counterexample sets

These show that for $A \subset \mathbb{R}^2$, the Borel measurability of the sections $A_x,\ A^x \subset \mathbb{R}\ \ (x \in \mathbb{R})$ does not imply Lebesgue measurability of A.

7.9 Sierpinski 1919

`Iterated integrals exist and differ.`

Let m be Lebesgue measure on $I := [0,1]$. Assuming the continuum hypothesis, $\exists\ M \subset I \times I$ such that for every $x, y \in I$,

$$M_x = \{t : (x,t) \in M\},\ M^y = \{t : (t,y) \in M\} \in \mathcal{B}(I) := \{\texttt{Borel sets}\}$$

but such that $m(M^y) = 0,\ m(M_x) = 1$.

It follows that $M \notin \mathcal{M}(I \times I) := \{\texttt{Lebesgue sets}\}$.

`Proof`

Assuming the continuum hypothesis, there is an order $\prec$ on $I = [0,1]$ such that $\{x \in I : x \prec y\}$ is countable for every $y \in I$. Set $M := \{(x,y) \in I^2 :\ x \prec y\}$. Evidently $M^y = \{t : (t,y) \in M\} \in \mathcal{B}(I)$ has zero measure being countable and $M_x = \{t : (x,t) \in M\} \in \mathcal{B}(I)$ has full measure being co-countable. ☑

7.10 Sierpinski 1920

`Section measures zero but set not measurable.`

$\exists\ E \subset \mathbb{R}^2$ *such that*

(a) $\#E \cap L \le 2\ \forall\ L$ *line;*

(b) $E \cap F \ne \varnothing\ \forall\ F \subset \mathbb{R}^2$ *closed, with positive measure.*

Note that (b) entails $\underline{m}(E] \cap A) = m(A)\ \forall\ A \in \mathcal{B}(\mathbb{R}^2)$ where m is Lebesgue measure on $\mathbb{R}^2$ and $\underline{m}(F) := \inf\{m(U) :\ F \subset\ U \text{ open}\}$.

`Proof` We first establish the

Lemma 7.11 *Suppose that* $A \in \mathcal{B}(\mathbb{R}^2)$, $m(A) > 0$ *and let* L_0 *be a line in* $\mathbb{R}^2$*, then* $\exists$ *a line* $L \parallel L_0$ *with* $|L \cap A| = c$.

`Proof of the Lemma` :

Let $T_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation in $\mathbb{R}^2$ about 0 so that $T_\theta L_0$ is a vertical line. By exercise 6.1, $m(T_\theta A) > 0$. By the Fubini-Tonelli theorem, $\exists$ a vertical line V with $m_V(V \cap T_\theta A) > 0$ (where m_V is linear measure on V), whence $|V \cap T_\theta A| = c$. It follows that $L := T_{-\theta}V \parallel L_0$ with $|L \cap A| = c$. ☑

Proof of 7.10 :

Let $\mathfrak{C}$ = {`closed subsets of positive measure in` $\mathbb{R}^2$}, then $\mathfrak{C} \cong \mathbb{R}^2 \cong c$.

As in Bernstein's theorem, we construct the set by transfinite recursion. Write $\mathfrak{C} = \{F_\alpha :\ \alpha \in \Omega_c\}$.

We claim that

¶ $\forall\ \alpha \in \Omega_c\ \exists\ f_\alpha : (\mathbb{R}^2)^{\Omega(\alpha)} \to \mathbb{R}^2$ such that $\forall\ u \in (\mathbb{R}^2)^{\Omega(\alpha)}$

(i) $f_\alpha(u) \in F_\alpha$;

(ii) $f_\alpha(u) \notin L(u(\beta), u(\gamma)) := \{tu(\beta)+(1-t)u(\gamma) :\ t \in \mathbb{R}\}\ \ \forall\ \beta,\ \gamma \in \Omega(\alpha)$.

`Proof of` ¶:

- The cardinality of the collection of directions of the lines

$$\{L(u(\beta), u(\gamma)) :\ \beta,\ \gamma \in \Omega(\alpha)\} < c$$

whence, using the lemma, $\exists$ a line L such that

- $L \nparallel L(u(\beta), u(\gamma)) \ \forall \ \beta, \ \gamma \in \Omega(\alpha)$;
- $|L \cap F_\alpha| = c$.

It follows that

$$|\bigcup_{\beta, \ \gamma \in \Omega(\alpha)} L \cap F_\alpha \cap L(u(\beta), u(\gamma))| < c,$$

whence

$$\exists \ f_\alpha(u) \in L \cap F_\theta \setminus \bigcup_{\beta, \ \gamma \in \Omega(\alpha)} L \cap F_\alpha \cap L(u(\beta), u(\gamma)). \quad ☑ \quad ¶$$

By transfinite recursion $\exists \ q : \Omega_c \to \mathbb{R}^2$ such that $q(\alpha) = f_\alpha(q|_{\Omega(\alpha)})$.

We claim that $E := q(\Omega_c)$ is as required. Evidently $q(\alpha) \in F_\alpha \ \forall \ \alpha \in \Omega_c$ and so E satisfies (b).

- To check (b) suppose otherwise that $\alpha, \ \beta, \ \gamma \in \Omega_c$ are distinct, and that L is a line with $q(\alpha), \ q(\beta), \ q(\gamma) \in L$. WLOG, $\alpha \prec \beta \prec \gamma$ and we have $q(\gamma) \in L(q(\alpha), q(\beta))$. However, by construction,

$$q(\gamma) = f_\gamma(q|_{\Omega(\gamma)}) \notin L(q(\alpha), q(\beta)). \quad ☑ \quad ¶$$

Such a set $E \subset \mathbb{R}^2$ is called a *Sierpinski set*. By regularity, a Sierpinski set intersects with every measurable set of positive measure.

¶ To see that a Sierpinski set E is not Lebesgue measurable, suppose otherwise, then by Fubini $m_{\mathbb{R}^2}(E) = 0$ whence $m_{\mathbb{R}^2}(E^c) > 0$ and $E \cap E^c \neq \varnothing$. ⊠

Exercise (Nº5), 26/4/2017

1. Baire's theorem. [7] Suppose that (X, d) is a complete metric space.

(i) Prove that if $U_n \subset X$ is open and dense in $X \ \forall \ n \geq 1$, then $G := \bigcap_{n \geq 1} U_n$ is dense in X.

(ii) Now suppose that $X = \bigcup_{n=1}^\infty F_n$ where each F_n is closed. Show that $\exists \ n \geq 1$ so that $F_n^o \neq \varnothing$.

(iii) Let $f_n \in C_B(X)$ and suppose that $f_n(x) \xrightarrow[n \to \infty]{} f(x) \ \ \forall \ x \in \mathbb{R}$. Show $\exists \ x \in X$ so that f is continuous at x.

`Hint` $[\omega_f < \varepsilon]$ is open and dense $\forall \ \varepsilon > 0$ where $\omega_f(x) := \lim_{r \to 0+} \mathtt{diam}\, f([x - r, x + r])$.

[7] Revision

2. Baire classes.

Let X be a polish space. The collection of *Baire class* 0 *functions* is $B_0 := C_B(X)$.

For $\alpha \in \Omega := \{\texttt{countable ordinals}\}$, define the collection of *Baire class* α *functions*

$$B_\alpha := \{f : X \to \mathbb{R} : \exists\ f_n \in B_{\beta_n},\ \beta_n \in \Omega,\ \beta_n < \alpha,\ f_n(x) \underset{n\to\infty}{\longrightarrow} f(x)\ \forall\ x \in X\}$$

`i.e.`

$B_1 = \{\texttt{limits of bounded continuous functions}\}$,

$B_2 := \{\texttt{limits of limits of bounded continuous functions}\}$,

$\vdots$

Show that

(i) $$B_\Omega := \bigcup_{\alpha\in\Omega} B_\alpha = \{\texttt{Borel measurable functions}\};$$

`Hint`: $\{A \in \mathcal{B}(X) :\ 1_A \in B_\Omega\}$ is a monotone class.

(ii) if $m \in \mathcal{P}(X)$ and $f : X \to \mathbb{R}$ is measurable then $\exists\ g \in B_2$ such that $f = g\ m$ - a.e..

Let $(X, \mathcal{B}, m) = ([0,1], \texttt{Borel}, \texttt{Leb})$. The rest of this exercise shows $\exists$ a measurable function $f : X \to \mathbb{R}$ for which $\nexists\ g \in B_1,\ f = g\ m$ - a.e..

Let $(\mathcal{B}^{\sim}, d)$ be the measure algebra of $(X, \mathcal{B}, m)$ as in exercise 3.2.

(iv) Show that

$$\mathfrak{d} := \{a \in \mathcal{B}^{\sim} :\ m(A \cap (s,t)) > 0\ \&\ m(A^c \cap (s,t)) > 0\ \forall\ A \in a,\ s,t \in \mathbb{Q},\ s < t\}$$

is dense in $\mathcal{B}^{\sim}$.

`Hint` Use Baire's theorem.

(v) Show that if $f = 1_A,\ A \in a \in \mathfrak{d}$, then $\nexists\ g \in B_1,\ f = g\ m$ - a.e..

3. Uniform integrability theorem.

Let $(X, \mathcal{B}, m)$ be a finite measure space. Show that the family $\mathcal{F} \subset L(X, \mathcal{B}, m)$ is UI $\Leftrightarrow$

$$\exists\ a : \mathbb{R}_+ \to \mathbb{R}_+ \ni \frac{a(x)}{x} \uparrow \infty \text{ as } x \uparrow \infty,\ \sup_{f\in\mathcal{F}} \int_X a(|f|)dm < \infty.$$

4. Linear change of variable.

Let

$$G := \{\texttt{nonsingular linear transformations of } \mathbb{R}^d\}$$

and let $m : \mathcal{B}(\mathbb{R}^d) \to [0,\infty]$ be Lebesgue measure.

(i) Show that if $T \in G$ then $m \circ T : \mathcal{B}(\mathbb{R}^d) \to [0,\infty]$ is a translation invariant, locally finite measure which is `homogeneous` in the sense that $m \circ T(aB) = a^d m \circ T(B) \ \forall \ a > 0,\ B \in \mathcal{B}(\mathbb{R}^d)$.

(ii) Show that $\exists$ a multiplicative homomorphism $\Delta : G \to (0,\infty)$ so that $m \circ T = \Delta(T)m$.

(iii) Show that $\Delta(T) = |\det T|$ for $T \in G$ orthogonal or diagonal.

`Hint` Choose suitable $A \in \mathcal{B}$.

Fix $T \in G$.

(iv) Show that for $T^tT = M^tM$ where $M = ŁU$ with U orthogonal & Ł diagonal; and that $|\det T| = \Delta(M)$.

(iii) Show that $\Delta(T) = \Delta(M)$.

`Hint` Choose suitable $A \in \mathcal{B}$.

5. Convolutions.

Let m denote Lebesgue measure on $\mathbb{R}^d$.

(i) Show that if $f,\ g : \mathbb{R}^d \to \mathbb{R}$ are Lebesgue measurable, then so is the function $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by $h(x,y) := f(x-y)g(y)$.

(ii) Prove that if $f, g : \mathbb{R}^d \to [0,\infty]$ are measurable, then $f * g = g * f$.

(iii) Show, using Hölder's inequality (or otherwise), that if $f \in L^1(\mathbb{R}^d)$, and $g \in L^p(\mathbb{R}^d)$ where $1 < p < \infty$, then $\|f * g\|_p \le \|f\|_1 \|g\|_p$. When is there equality?

(iv) Suppose that $h : \mathbb{R}^d \to \mathbb{R}_+$ is measurable, and suppose that $\int_{\mathbb{R}^d} h(x)dx = 1$. Define, for $t > 0$, $h_t : \mathbb{R}^d \to \mathbb{R}_+$ by $h_t(x) = \frac{1}{t}h(\frac{x}{t})$. Prove that for $1 \le p < \infty$,

$$\|f * h_t - f\|_p \xrightarrow[t\to 0]{} 0 \quad \forall \ f \in L^p(\mathbb{R}^d).$$

6. Fourier transform on $L^1(\mathbb{R}^d)$. For $f \in L^1(\mathbb{R}^d)$, define the *Fourier transform* of f by

$$\widehat{f}(t) := \int_{\mathbb{R}^d} f(x)e^{-i\langle x,t\rangle}dx \quad (t \in \mathbb{R}^d)$$

`where` $\langle x,t\rangle = \langle (x_1,\dots,x_d),(t_1,\dots,t_d)\rangle := \sum_{k=1}^d x_k t_k$.

a) Prove that $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ is bounded, and uniformly continuous $\forall \ f \in L^1(\mathbb{R}^d)$,

b) Show that $\widehat{f * g} \equiv \widehat{f}\,\widehat{g} \ \forall \ f,g \in L^1(\mathbb{R}^d)$.

c) Show that if $f : \mathbb{R}^d \to \mathbb{R}$ is C^1 and $\overline{[f \ne 0]}$ is compact, then $\widehat{(\nabla f)_k}(t) = it_k\widehat{f}(t)$ `where` $(\nabla f)_k(x) := \frac{\partial f}{\partial x_k}(x)$.

d) Prove the `Riemann-Lebesgue Lemma`, that if $f \in L^1(\mathbb{R}^d)$, then $\widehat{f}(t) \to 0$ as $t \to \infty$.

7. Measurability and Fubini.

(i) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be finite, non-atomic, Polish, probability spaces and let $m = \mu \times \nu : \mathcal{F} := \sigma(\mathcal{B} \times \mathcal{C}) \to [0,1]$ be product measure.

Show that $\exists\ E \subset X \times Y$ so that $\nu(E_x) = 0 = \mu(E^y)\ \forall\ x \in X,\ y \in Y$ and so that $E \notin \overline{\mathcal{F}}_m$.

`Hint`: Isomorphism.

(ii) Show that there is a continuum of disjoint Sierpinski sets in $\mathbb{R}^2$.

Week # 6

§8 Signed measures

A *signed measure* on the measurable space $(X, \mathcal{B})$ is a σ-additive set function $\mu : \mathcal{B} \to \mathbb{R}$. For example, if $\mu_\pm : \mathcal{B} \to [0, \infty)$ are measures, then $\mu = \mu_+ - \mu_-$ is a signed measure.

The *total variation* of a signed measure is a set function $m = m_\mu = |\mu| : \mathcal{B} \to [0, \infty]$ defined by

$$m(A) = \sup\{\sum_{n=1}^{\infty} |\mu(A_n)| : A_n \in \mathcal{B} \text{ disjoint},\ A = \bigcupdot_{n=1}^{\infty} A_n\}.$$

8.1 Theorem `(finite total variation)` *If $\mu : \mathcal{B} \to \mathbb{R}$ is a signed measure, then $m : \mathcal{B} \to [0, \infty)$ is a finite measure.*

Proof We must show that: (i) m is σ-additive, and that (ii) $m(X) < \infty$.

To prove (i), let $A = \bigcupdot_{n=1}^{\infty} A_n$ where $A,\ A_n \in \mathcal{B}$ and $A_n\ \ (n \ge 1)$ are disjoint. Suppose $t_n < m(A_n) (\le \infty)$, then, $\forall\ n \ge 1,\ A_n = \bigcupdot_{k=1}^{\infty} A_{n,k}$ where $A_{n,k} \in \mathcal{B},\ A_{n,k}\ \ (k \ge 1)$ are disjoint, and $\sum_{k=1}^{\infty} |\mu(A_{n,k})| > t_n$. It follows that

$$m(A) \ge \sum_{n,k=1}^{\infty} |\mu(A_{n,k})| > \sum_{n=1}^{\infty} t_n, \quad \therefore m(A) \ge \sum_{n=1}^{\infty} m(A_n).$$

To obtain the reverse inequality, suppose $A = \bigcupdot_{n=1}^{\infty} E_n$ where $E_n \in \mathcal{B}$ and $E_n\ \ (n \ge 1)$ are disjoint. Then

$$\sum_{n=1}^{\infty} |\mu(E_n)| \le \sum_{n,k=1}^{\infty} |\mu(E_n \cap A_k)| \le \sum_{n=1}^{\infty} m(A_n),\ \therefore m(A) \le \sum_{n=1}^{\infty} m(A_n).$$

To establish (ii), we use the

Lemma 8.2 *If $E \in \mathcal{B}$, $m(E) = \infty$, then $E = A \cup B$ where $A, B \in \mathcal{B}$ are disjoint, $m(B) = \infty$ and $|\mu(A)|, |\mu(B)| > 1$.*

Using the lemma, if $m(X) = \infty$, then there are disjoint sets $A_n \in \mathcal{B}$ $(n \ge 1)$ such that $|\mu(A_n)| > 1 \ \forall \ n \ge 1$, contradicting the countable additivity of μ. ☐

Proof of lemma 8.2 Let $t = 2(1 + |\mu(E)|)$, then $\exists \ E_n \in \mathcal{B}$ $(n \ge 1)$ disjoint sets, such that $E = \bigcup_{n=1}^{\infty} E_n$ and $\sum_{n=1}^{\infty} |\mu(E_n)| > t$. It follows that $\exists \ S \subset \mathbb{N} \ \ni \ |\sum_{n \in S}^{\infty} \mu(E_n)| > \frac{t}{2}$. Set $A = \bigcup_{n \in S} E_n$ and $B = E \setminus A$, then

$$|\mu(A)| > 1 + |\mu(E)|, \ \& \ |\mu(B)| \ge |\mu(A)| - |\mu(E)| > 1.$$

By additivity of m, one of $m(A)$, $m(B)$ is infinite. ☐

8.3 Corollary (`σ-additive Jordan decomposition`) *If $\mu : \mathcal{B} \to \mathbb{R}$ is a signed measure, then $\mu = \mu_+ - \mu_-$ where $\mu_\pm : \mathcal{B} \to [0, \infty)$ are finite measures.*

`Proof` $\mu_\pm := \frac{m \pm \mu}{2}$.

Hahn decomposition of signed measures.

Let X be a set, let $\mathcal{A} \subset 2^X$ be an algebra and let $\mu : \mathcal{A} \to \mathbb{R}$ be additive. Define $m = m_\mu = |\mu| : \mathcal{A} \to [0, \infty]$ by

$$m(A) = \sup\{\sum_{n=1}^{N} |\mu(A_n)| : \ A_1, A_2, \dots, A_n \in \mathcal{B} \text{ disjoint}, \ A \supset \dot{\bigcup_{n=1}^{N}} A_n\}.$$

As on page 9, m is additive, and, in case $\|\mu\| := |\mu|(X) < \infty$, we have the `Jordan decomposition`:

$$\mu = \mu_+ - \mu_- \ \text{ where } \ \mu_\pm = \frac{m \pm \mu}{2} : \mathcal{A} \to [0, \infty).$$

In case $\mathcal{A}$ is a σ-algebra and μ is a signed measure, then $|\mu|(X) < \infty$ and $|\mu|$ is σ-additive.

8.4 Theorem (`Hahn decomposition`)

There are disjoint sets $A_\pm \in \mathcal{B}$ such that $X = A_+ \cup A_-$, and

$$\mu_\pm(B) = |\mu|(B \cap A_\pm) = \pm\mu(B \cap A_\pm) \ \forall \ B \in \mathcal{B}.$$

Positive sets.

Let $\mu : \mathcal{B} \to \mathbb{R}$ be a signed measure on the measurable space $(X, \mathcal{B})$. Call $A \in \mathcal{B}$ a *positive set* for μ if $\mu(B) \ge 0 \ \ \forall \ B \in \mathcal{B}, \ B \subset A$. Let

$$\mathcal{P}_\mu := \{\texttt{positive sets for } \mu \ \}.$$

8.5 Proposition

(1) *$\mathcal{P}_\mu$ is a heriditary σ-ring.*
(2) $\mu(A) = |\mu|(A) \ \forall \ A \in \mathcal{P}_\mu$.
(3) *If $A \in \mathcal{B}, \ \mu(A) > 0$, then $\exists \ P \subset A, \ \mu(P) > 0, \ P \in \mathcal{P}_\mu$.*

Proof of (3) by exhaustion

Let $\varepsilon_1 := \sup\{(-\mu(B)) \vee 0 : \ B \in \mathcal{B}, \ B \subset A\}$ and choose $B_1 \in \mathcal{B}, \ B_1 \subset A, \ -\mu(B_1) \ge \frac{\varepsilon_1}{2}$. Note that $A \in \mathcal{P}_\mu$ iff $\varepsilon_1 = 0$ in which case $B_1 = \varnothing$ is a possible choice.

Let $\varepsilon_2 := \sup\{(-\mu(B)) \vee 0 : \ B \in \mathcal{B}, \ B \subset A \setminus B_1\}$ and choose $B_2 \in \mathcal{B}, \ B_2 \subset A \setminus B_1, \ -\mu(B_2) \ge \frac{\varepsilon_2}{2}$.

Continue to obtain:

- $\varepsilon_1 \ \ge \ \varepsilon_2 \ \ge \ \dots \ \ge 0$,
- disjoint sets $B_n \in \mathcal{B} \ (n \ge 1)$ so that

$$-\mu(B_n) \vee 0 \ge \frac{\varepsilon_n}{2} \ \ \forall \ n \ge 1 \ \ \&$$

$$\varepsilon_{n+1} = \sup\{(-\mu(B)) \vee 0 : \ B \in \mathcal{B}, \ B \subset A \setminus \bigcup_{j=1}^{n}{}^{\!\!\!\!\!\cdot} \ B_j\}.$$

Let $B := \dot\bigcup_{n\ge1} B_n$, then

$$\mu(B) = \sum_{n\ge1} \mu(B_n) \le 0, \text{ whence } \sum_{n\ge1} \varepsilon_n < \infty.$$

Set $P := A \setminus B$, then

$$\mu(P) = \mu(A) - \mu(B) > 0.$$

To see that $P \in \mathcal{P}_\mu$, suppose not; then

$$\exists \ \varepsilon > 0, \ B \in \mathcal{B}, \ B \subset P, \ -\mu(B) > \varepsilon.$$

However, $B \subset A \setminus \dot\bigcup_{j=1}^n B_j \ \forall n \ge 1$ whence $-\mu(B) \le \varepsilon_n \to 0.$ ⊠

Proof of Hahn decomposition

We show that there are disjoint sets $A_\pm(\mu) \in \mathcal{B}$ such that $X = A_+ \dot\cup A_-$, and

$$|\mu|(B \cap A_\pm(\mu)) = \pm\mu(B \cap A_\pm(\mu)) \ \forall \ B \in \mathcal{B}.$$

Let $A_\pm = A_\pm(\mu) \in \mathcal{B}$ be the measure theoretic union of $\mathcal{P}_{\pm\mu}$, then by (1) in the lemma, $A_\pm(\mu) \in \mathcal{P}_{\pm\mu}$.

It remains to show that $|\mu|(X \setminus (A_+ \dot\cup A_-)) = 0$.

If not $\exists\ G \in \mathcal{B},\ G \subset X \setminus A_+ \cup A_-,\ \varepsilon = \pm 1$ with $\varepsilon\mu(G) > 0$.

By (3) in the proposition $\exists\ P \subset G,\ |\mu|(P) > 0,\ P \in \mathcal{P}_{\varepsilon\mu}$. But then $P \overset{|\mu|}{\subset} A_\varepsilon$ and

$$0 < |\mu|(P) = |\mu|(P \cap (A_+ \cup A_-)) + |\mu|(P \setminus A_+ \cup A_-) = |\mu|(P \cap (A_+ \cup A_-)) = 0. \ \boxtimes$$

Absolute continuity and singularity

Let $(X, \mathcal{B})$ be a measurable space, and let $\mu, \nu : \mathcal{B} \to [0, \infty)$ be measures. The measure ν is *absolutely continuous* with respect to μ ($\nu \ll \mu$) if $A \in \mathcal{B},\ \mu(A) = 0 \ \Rightarrow\ \nu(A) = 0$. The measures μ and ν are *singular* ($\mu \perp \nu$) if $\exists\ A \in \mathcal{B} \ni \mu(A) = 0,\ \&\ \nu(A^c) = 0$.

8.6 Theorem (Radon-Nikodym for finite measures) *Let μ, m be finite measures on the measurable space $(X, \mathcal{B})$.*

If $\mu \ll m$ then $\exists\ f \in L^1(m),\ f \geq 0$ such that

$$(*) \qquad \mu(A) = \int_A f dm \quad \forall\ A \in \mathcal{B}.$$

The function f (determined up to equality a.e.) is known as the *Radon-Nikodym derivative* and denoted $f = \frac{d\mu}{dm}$.

Proof

For $q \in \mathbb{Q}_0 := \mathbb{Q} \cap [0, \infty)$, set $B_q := A_-(\mu - qm)$ where $\{A_\varepsilon(\mu - qm) : \varepsilon = \pm\}$ is the Hahn decomposition of $\mu - qm$. Note that $B_0 = \varnothing$.

We have

$$B_a \setminus B_b = A_-(\mu - am) \cap A_+(\mu - bm) \in \mathcal{P}_{am-\mu} \cap \mathcal{P}_{\mu - bm} \quad \forall\ a, b \in \mathbb{Q}_0.$$

For $b > a$,

$$0 \leq (\mu - bm)(B_a \setminus B_b) \leq (\mu - am)(B_a \setminus B_b) \leq 0$$

whence

$$m(B_a \setminus B_b) = 0.$$

Define the sets $\{C_q : \ a \in \mathbb{Q}_0\} \subset \mathcal{B}$ by

$$C_q := B_q \setminus \bigcup_{t \in \mathbb{Q}_0,\ t > q} B_t;$$

then

$$m(B_q \Delta C_q) = 0 \ \forall\ q \in \mathbb{Q}_0 \ \ \&\ \ C_a \subseteq C_b \ \forall\ a,\ b \in \mathbb{Q}_0,\ a < b.$$

Define $f : X \to \mathbb{R}$ by

$$f(x) := \inf\{q \in \mathbb{Q}_0 : \ x \in C_q\}.$$

It follows that $[f \leq q] = C_q$ whence f is measurable.

Moreover, by the above if $a < b \in \mathbb{Q}_0$, then

$$[a < f \le b] = C_b \setminus C_a \in \mathcal{P}_{am-\mu} \cap \mathcal{P}_{\mu-bm},$$

whence for $Z \in \mathcal{B}$,

$$am(Z \cap [a < f \le b]) \le \mu(Z \cap [a < f \le b]) \le bm(Z \cap [a < f \le b])$$

with the consequence that (!)

$$\mu(Z) = \int_Z f dm \;\; \forall \; Z \in \mathcal{B}. \;\; ☑$$

8.7 Lebesgue Decomposition Theorem

Let $\lambda, \mu : \mathcal{B} \to [0, \infty)$ be finite measures, then $\exists$ and finite measures $\lambda_{ac}, \; \lambda_s : \mathcal{B} \to [0, \infty), \; \lambda_{ac} \ll \mu$ & $\lambda_s \perp \mu$ such that

$$\lambda(A) = \int_A g d\mu + \lambda_s(A) \;\; \forall \;\; A \in \mathcal{B}.$$

Proof

Let $\rho = \lambda + \mu$. By the Radon-Nikodym theorem $\exists \; h \in L^1(\rho)$ so that

$$\lambda(A) = \int_A h d\rho \;\; \forall \;\; A \in \mathcal{B},$$

whence

(∗)
$$\int_A h d\mu = \int_A (1-h) d\lambda \;\; \forall \;\; A \in \mathcal{B}.$$

It follows that:

- $h \ge 0 \;\; \rho$–a.e. $(\texttt{else} \;\; 0 \le \lambda([h < 0]) = \int_{[h<0]} h d\rho < 0)$;
- $h \le 1 \;\; \rho$–a.e. $(\texttt{else} \;\; \rho([h > 1]) \ge \lambda([h > 1]) = \int_{[h>1]} h d\rho > \rho([h > 1]))$.

Now define $\lambda_{ac}, \; \lambda_s : \mathcal{B} \to [0, \infty)$ by

$$\lambda_{ac}(A) := \int_A g d\mu \text{ where } g := \frac{h}{1-h} 1_{[h<1]}; \;\; \& \quad \lambda_s(A) = \lambda(A \cap [h = 1]).$$

Evidently $\lambda_{ac} \ll \mu$ and $\lambda_{ac} + \lambda_s = \lambda$.

Moreover, $\lambda_s \perp \mu$ because $\lambda_s([h \ne 1]) = 0$ and

$$\mu([h = 1]) = \int_X h 1_{[h=1]} d\mu = \int_X 1_{[h=1]} (1-h) d\lambda = 0. \;\; ☑$$

8.8 Corollary (R-N for signed measures)

Let μ be a signed measure, and m be a measure on the measurable space $(X,\mathcal{B})$.

If $\mu \ll m$ then $\exists\ f \in L^1(m)$ such that

$$\mu(A) = \int_A f dm \quad \forall\ A \in \mathcal{B}.$$

8.9 Corollary (R-N for infinite measures)

Let μ, m be σ-finite measures on the measurable space $(X,\mathcal{B})$.

If $\mu \ll m$ then $\exists\ f : X \to [0,\infty)$, $\mathcal{B}$-measurable, such that

$$\mu(A) = \int_A f dm \quad \forall\ A \in \mathcal{B}.$$

§9 Conditional Expectations, Conditional Probabilities and disintegrations

The following is the converse to the theorem 7.3 on "`integration of probabilities`" for Polish spaces.

9.1 Theorem (disintegration of probabilities)

Suppose that X, Y are Polish spaces and that $m \in \mathcal{P}(X \times Y)$.

Let $\mu \in \mathcal{P}(X)$ be the `marginal` *of m defined by $\mu(A) = m(A \times Y)$, then there is a set $X_0 \in \mathcal{B}(X)$, $X_0 = X \mod \mu$, and a measurable map $x \mapsto \nu_y \ \ (X_0 \to \mathcal{P}(Y))$ such that*

$$m(A \times B) = \int_A \nu_x(B) d\mu(x) \quad \forall\ A \in \mathcal{B}(X),\ B \in \mathcal{B}(Y).$$

`Proof` Let $\mathcal{A} \subset \mathcal{B}(Y)$ be a countable, generating algebra with the `FSCP`.

For $B \in \mathcal{B}(Y)$, define the measure $\nu_B : \mathcal{B}(X) \to [0,\infty)$ by $\nu_B(A) := m(A \times B)$, then $\nu_B \ll \mu \equiv \nu_\Omega$ and so by the RN theorem, $\exists$ a $\mathcal{B}(X)$-measurable function $x \mapsto v_x(B) = \frac{d\nu_B}{d\mu}(x)$ so that

$$\int_A v_x(B) d\mu(x) = m(A \times B) \quad \forall\ A \in \mathcal{B}(X)$$

whence $v_x(Y) = 1$ for μ-a.e. $x \in X$.

Also, if $A_1, \dots \in \mathcal{B}(Y)$ are disjoint, then

$$u_x(\dot{\bigcup_{k=1}^{\infty}} A_k) = \sum_{k=1}^{\infty} u_x(A_k),$$

for μ-a.e. $x \in X$.

Since $\mathcal{A}$ is countable, there is a set $X_0 \in \mathcal{B}(X)$, $X_0 = X \bmod \mu$ such that

$$u_x(\dot{\bigcup_{k=1}^{n}} A_k) = \sum_{k=1}^{n} u_x(A_k) \quad \forall x \in X_0$$

whenever $A_1, ..., A_n \in \mathcal{A}$ are disjoint.

Since $\mathcal{A}$ has the `FSCP`,

$$\exists \ \{\nu_x : x \in X_0\} \subset \mathcal{P}(Y)$$

such that

$$\nu_x(A) = u_x(A) \ \ \forall \ A \in \mathcal{A}, \ x \in X_0.$$

In order to complete the proof of the theorem, we note that the collections

$$\mathcal{D} := \{A \in \mathcal{B}(Y) : x \mapsto \nu_x(A) \text{ measurable}\},$$

and

$$\mathcal{E} := \{B \in \mathcal{B}(Y) : \int_A \nu_x(B) d\mu(x) = m(A \times B) \ \forall \ A \in \mathcal{B}(X)\}$$

are both monotone classes containing $\mathcal{A}$. ☑

9.2 Theorem (Existence of conditional expectations) *Suppose that $(X, \mathcal{B}, m)$ is a probability space and that $\mathcal{C} \subset \mathcal{B}$ is a sub-σ-algebra, then for every $f \in L^1(\mathcal{B})$, $\exists!$ $g \in L^1(\mathcal{C})$ such that*

$$\int_C f dm = \int_C g dm \ \forall \ C \in \mathcal{C}.$$

This is proved using the Radon-Nikodym theorem. The function $g \in L^1(\mathcal{C})$ is clearly unique mod m. It is called the *conditional expectation* of f with respect to $\mathcal{C}$, and denoted

$$g = E(f|\mathcal{C}).$$

9.3 Proposition (Properties of conditional expectations)

a) If $A_n \in \mathcal{B}$ $(n \geq 1)$ are disjoint, and $A = \bigcup_{n=1}^{\infty} A_n$, then

$$E(1_A|\mathcal{C}) = \sum_{n=1}^{\infty} E(1_{A_n}|\mathcal{C}) \ \ a.e.$$

b) If $\mathcal{C} = \sigma(\alpha)$ where $\alpha \subset \mathcal{B}$ is a countable partition of X, then

$$E(f|\mathcal{C}) = \sum_{A \in \alpha} 1_A E(f|A) \ \ a.e. \ \forall \ f : X \to [0, \infty)$$

measurable, where $E(f|A) := \frac{1}{m(A)} \int_A f dfm$.

c) If $f : X \to \mathbb{R}$ is bounded, $\mathcal{B}$-measurable, and $g : X \to \mathbb{R}$ is bounded, $\mathcal{C}$-measurable, then $E(fg|\mathcal{C}) = gE(f|\mathcal{C})$.

d) If $\mathcal{C}_1 \subset \mathcal{C}_2$ then $E(E(f|\mathcal{C}_1)|\mathcal{C}_2) = E(E(f|\mathcal{C}_2)|\mathcal{C}_1) = E(f|\mathcal{C}_1)$.

e) $\|E(f|\mathcal{C})\|_p \le \|f\|_p \ \forall \ 1 \le p \le \infty$.
f) $\int_X (f - E(f|\mathcal{C}))g dm = 0 \ \forall \ f \in L^2(X), \ g \in L^2(\mathcal{C})$.

Remark

$E(\cdot|\mathcal{C})$ is the orthogonal projection $P : L^2(\mathcal{B}) \to L^2(\mathcal{C})$, whence

$$\|f - E(f|\mathcal{C})\|_2 \le \|f - g\|_2 \ \forall \ f \in L^2(\mathcal{B}), \ g \in L^2(\mathcal{C})$$

with equality iff $g = E(f|\mathcal{C})$.

Regular conditional probabilities.

Suppose that $(X, \mathcal{B}, m)$ is a probability space and that $\mathcal{C} \subset \mathcal{B}$ is a sub-σ-algebra.

A *regular conditional probability on* $(X, \mathcal{B}, m)$ *given* $\mathcal{C}$ is a function $p : X_0 \times \mathcal{B} \to [0, 1]$ where $X_0 \in \mathcal{C}, \ X_0 = X$ mod m, such that

(a) for every $x \in X_0, \ m_x \in \mathcal{P}(X, \mathcal{B})$ where $m_x(A) := p(x, A)$;

(b) for every $A \in \mathcal{B}$, the $\mathbb{R}$-valued function $x \mapsto m_x(A) = p(x, A)$ is $\mathcal{C}$-measurable, and

(c) $\int_C m_x(A) dm(x) = m(A \cap C) \quad \forall \ A \in \mathcal{B}, \ C \in \mathcal{C}$ (i.e. $p(\cdot, A) = E(1_A|\mathcal{C})$ a.e.).

9.4 Theorem (Existence of regular, conditional probabilities)

Suppose that $(X, \mathcal{B}, m)$ *is a Polish probability space and that* $\mathcal{C} \subset \mathcal{B}$ *is a sub-*σ*-algebra, then there is a regular conditional probability on* $(X, \mathcal{B}, m)$ *given* $\mathcal{C}$.

Proof

By Kuratowski's isomorphism theorem, we may assume (!) that $X = \Omega$.

Let $\mathcal{A}$ denote the algebra of finite unions of cylinder sets in X, then each set in $\mathcal{A}$ is both open, and compact.

Consequently any non-negative, finitely additive set function $\mu : \mathcal{A} \to \mathbb{R}_+$ satisfies Caratheodory's condition (*) (as on p.11) and extends to a measure ν on $(X, \sigma(\mathcal{A}))$ defined by

$$\nu(E) = \inf\{\sum_{n=1}^{\infty} \mu(A_n) : \bigcup_{n=1}^{\infty} A_n \supseteq E, \ A_n \in \mathcal{A}\}.$$

Choose $\mathcal{C}$-measurable functions

$$u_x(A) = E(1_A|\mathcal{C})(x) \text{ a.e. } (A \in \mathcal{B}),$$

then

$$\int_C u_x(A) dm(x) = m(A \cap C) \quad \forall \ A \in \mathcal{B}, \ C \in \mathcal{C}.$$

Also, if $A_1, \dots \in \mathcal{B}$ are disjoint, then

$$u_x(\bigcup_{k=1}^{\infty}{}^{\!\!\cdot}\, A_k) = \sum_{k=1}^{\infty} u_x(A_k),$$

for m-a.e. $x \in X$. Since $\mathcal{A}$ is countable, there is a set $X_0 \in \mathcal{C},\ X_0 = X$ mod m such that

$$u_x(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} u_x(A_k) \quad \forall x \in X_0$$

whenever $A_1, \dots, A_n \in \mathcal{A}$ are disjoint.

It follows from the remarks above that there are measures

$$\{m_x : x \in X_0\}$$

on $(X, \mathcal{B})$ such that

$$m_x(A) = u_x(A) \quad \forall\ A \in \mathcal{A},\ x \in X_0.$$

In order to complete the proof of the theorem, we note that the collections

$$\mathcal{D} := \{A \in \mathcal{B} : x \mapsto m_x(A)\ \mathcal{C} - \text{measurable}\},$$

and

$$\mathcal{E} := \{A \in \mathcal{B} : \int_C m_x(A) dm(x) = m(A \cap C)\ \forall\ C \in \mathcal{C}\}$$

are both monotone classes containing $\mathcal{A}$. ☐

Exercise N$^{\underline{\text{o}}}$6

1. Measure space isomorphism.

Measure spaces $(X, \mathcal{B}, p)$ and $(Y, \mathcal{C}, q)$ are *isomorphic* if there are sets $X' \in \mathcal{B},\ Y' \in \mathcal{C}$ such that $p(X \setminus X') = q(Y \setminus Y') = 0$; and a bijection $\pi : X' \to Y'$ satisfying $\pi^{-1}(\mathcal{C} \cap Y') = \mathcal{B} \cap X'$ and $p \circ \pi^{-1} = q$. The map π is called a *measure space isomorphism* and denoted $\pi : (X, \mathcal{B}, p) \to (Y, \mathcal{C}, q)$.

A *Polish* measure space is a σ-finite measure space $(X, \mathcal{B}, m)$ where X is a Polish space, $\mathcal{B} = \mathcal{B}(X)$ and $m \in \mathfrak{M}(X, \mathcal{B})$. In this exercise, you prove that

a non-atomic, Polish probability space is isomorphic to the unit interval equipped with Borel sets and Lebesgue measure

(i) Let X be a metric space, and suppose that $p : \mathcal{B}(X) \to [0,1]$ is a probability. The *support* of p is defined by

$$S_p := \{x \in X : p(B(x, \varepsilon)) > 0\ \forall\ \varepsilon > 0\}.$$

Show that S_p is a closed subset of X, and that if X is separable, then $p(S_p^c) = 0$.

(ii) Let $I = [0,1]$ & $p \in \mathcal{P}(I)$ be a non-atomic probability. Define $\pi : I \to I$ by $\pi(x) := p([0,x])$. Show that $\pi : (I, \mathcal{B}(I), p) \to (I, \mathcal{B}, \texttt{Leb})$ is a measure space isomorphism.

(iii) Prove that a non-atomic, Polish probability space is isomorphic to the unit interval equipped with Borel sets and Lebesgue measure.

2. Lebesgue measure spaces and conditional probabilities.

A (nonatomic) *Lebesgue* measure space is a measure space which is isomorphic to a subinterval of $\mathbb{R}$ equipped with Lebesgue subsets, and Lebesgue measure.

(a) Show that the completion of a non-atomic, Polish probability space is Lebesgue.

(b) Let $(X, \mathcal{B}_1, p)$ be an `extended measure space` where

$X = [0,1],\ \mathcal{B}_1 := \{E(B_1, B_2) := (B_1 \cap E) \cup (B_2 \cap E^c) : \quad B_1, B_2 \in \mathcal{B}(X)\}$

where $E \subset [0,1]$ satisfies $\overline{\mu}(E) = \overline{\mu}([0,1] \setminus E) = 1$, $\overline{\mu}$ denoting Lebesgue outer measure on $[0,1]$ (e.g. E a Bernstein set) and

$$p(E(B_1, B_2)) = \frac{1}{2}(\texttt{Leb}\,(B_1) + \texttt{Leb}\,(B_2)) \quad (B_1, B_2 \in \mathcal{B}(X)).$$

Construction as in exercise 2.2.

Show that

(i) $E(1_{E(B_1,B_2)} \| \mathcal{B}(X)) = \frac{1}{2}(1_{B_1} + 1_{B_2})$ a.e.;

(ii) $\nexists$ "`regular conditional probabilities given` $\mathcal{B}(X)$" at any point satisfying (i).

(iii) $(X, \overline{(\mathcal{B}_1)}_p, p)$ is not a Lebesgue measure space where $\overline{(\mathcal{B}_1)}_p$ is the p-completion of $\mathcal{B}_1$.

3. A measure on product space.

Let $I = [0,1]$ and $\Omega = \{0,1\}^{\mathbb{N}}$ be equipped with their natural topologies.

a) Show that there is a probability $m : \mathcal{B}(I \times \Omega) \to [0,1]$ such that $\forall\ n \geq 1,\ a_1, \dots, a_n \in \{0,1\}$ & $J \in \mathcal{B}(I)$,

$$m(J \times [a_1, \dots, a_n]) = \int_J t^{s_n}(1-t)^{n-s_n} dt$$

where

$$[a_1, \dots, a_n] = \{(x_1, x_2, \dots) \in \Omega : x_k = a_k\ 1 \leq k \leq n\}$$

and $s_n = a_1 + \cdots + a_n$.

b) Show that

$$\int_{I\times\Omega}(A_n - T)^2 dm = \frac{1}{6n}$$

where $T(t,x) := t$ and $A_n(t,x) := \frac{1}{n}\sum_{k=1}^n x_k$.

c) Define $\pi : \Omega \to \{0,1\}^{\mathbb{N}}$ by $\pi(\omega,t) := \omega$.

By identifying a regular, conditional probability on $(\Omega, \mathcal{B}(\Omega), m)$ given $\mathcal{A} = I \times \mathcal{B}(\Omega)$ (or otherwise), show that

$$\pi : (\Omega, \mathcal{B}(\Omega), m) \to (\{0,1\}^{\mathbb{N}}, \mathcal{B}(\{0,1\}^{\mathbb{N}}, \mu)$$

is a measure space isomorphism where $\mu(A) := m(I \times A)$.

4. Non-singular bijections.

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and let $A, B \in \mathcal{B}$. A *non-singular bijection* of A and B is a $T : A \to B$, such that both T and T^{-1} are measurable, and such that $m \circ T,\ m \circ T^{-1} \ll m$. Show that

(i) if $T : A \to B$ is a non-singular bijection, then $\exists\ T' : A \to \mathbb{R}$, positive on A, such that $m(TC) = \int_C T' dm\ \forall\ C \in \mathcal{B} \cap A$;

(ii) (`Chain rule for R-N derivatives`) if $S : A \to B$, and $T : B \to C$ are non-singular bijections, then so is $T \circ S : A \to C$, and $(T \circ S)' = T' \circ S \cdot S'$ a.e..

(iii) Let $T : X \to X$ be a non-singular bijection of X. For $1 \le p < \infty$ and $f : X \to \mathbb{R}$ measurable, define $V_p(f) := (T')^{\frac{1}{p}} f \circ T$. Show that $V_p : L^p(\mu) \to L^p(\mu)$ is an invertible isometry.

(iv) Let $(X, \mathcal{B}, \mu) = ([0,1], \texttt{Borel}, \texttt{Leb.})$ and let $T : X \to X$ be C^1 and strictly increasing. Is $T : X \to X$ necessarily a non-singular bijection?

5. Invariant measures on groups.

Let $G = \{\left(\begin{smallmatrix} x & y \\ 0 & 1\end{smallmatrix}\right) :\ (x,y) \in \mathbb{R}^2,\ x > 0\}$ equipped with the topology inherited from $\mathbb{R}^2$, and matrix multiplication.

Find measures $m_L,\ m_R : \mathcal{B}(G) \to [0,\infty]$ such that $\forall\ f : G \to \mathbb{R}_+$ measurable,

$$\int_G f(gh)dm_L(h) = \int_G f(h)dm_L(h),\ \int_G f(gh)dm_R(g) = \int_G f(h)dm_R(h).$$

Hint: Try $dm_J(\left(\begin{smallmatrix} x & y \\ 0 & 1\end{smallmatrix}\right)) = p_J(x,y)dxdy \quad (J = L, R)$.

6. Fourier transform on $L^2(\mathbb{R})$.

Let $\varphi(t) := e^{-\frac{t^2}{2}} \quad (t \in \mathbb{R})$.

a) Show that $\widehat{\varphi} = \sqrt{2\pi}\varphi$.

`Hint` $\frac{d}{dx}\widehat{\varphi}(x) = -x\widehat{\varphi}(x)$??

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

b) Show that

$$\int_{\mathbb{R}} \varphi\left(\frac{x}{\sqrt{n}}\right)|\widehat{f}(x)|^2 dx = \sqrt{2n\pi}\int_{\mathbb{R}} \varphi(x\sqrt{n})g(x)dx$$

where $g(x) = \int_{\mathbb{R}} f(x+y)\overline{f}(y)dy$.

c) Deduce that

$$\|\widehat{f}\|_2^2 \leftarrow \sqrt{2n\pi}\int_{\mathbb{R}} \varphi(x\sqrt{n})g(x)dx \;\to\; 2\pi g(0) = 2\pi\|f\|_2^2.$$

Week # 7

§10 Banach spaces and linear functionals

Let $(X, \mathcal{B}, m)$ be a σ-finite measure space. Denote the collection of $f : X \to \mathbb{R}$ measurable by

$$\mathcal{L}(X, \mathcal{B}, m) = \{f : X \to \mathbb{R} : \; f^{-1}\mathcal{B}(\mathbb{R}) \subset \mathcal{B}\}.$$

For $p > 0$ and $f \in \mathcal{L}$, let

$$\|f\|_p = \left(\int_X |f|^p dm\right)^{\frac{1}{p}},$$

$$\|f\|_\infty := \inf\{K > 0 : \; m([|f| > K]) = 0\} \le \infty,$$

and

$$\mathcal{L}^p = \mathcal{L}^p(X, \mathcal{B}, m) = \{f \in \mathcal{L} : \; \|f\|_p < \infty.\}$$

It is not hard to show that if $m(X) < \infty$, then

$$\|f\|_p \xrightarrow[p\to\infty]{} \|f\|_\infty.$$

Also, for $f \in L^1$, $g \in L^\infty$ non-we have $fg \in L^1$ and

$$|\int_X fgdm| \le \|f\|_1\|g\|_\infty.$$

More generally,

10.1 Hölder's inequality *For* $1 < p < \infty$, $q = \frac{p}{p-1}$,

$$\left|\int_X fgdm\right| \le \|f\|_p\|g\|_q$$

with equality iff $fg \geq 0$ *a.e. and there is a constant* $c > 0$ *so that* $|f|^p = c|g|^q$.

Proof See exercises.

10.2 Minkowski's inequality *For*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \quad \forall\ f, g \in \mathcal{L}^p,\ 1 \leq p \leq \infty$$

with equality when $p \in (1, \infty)$ *iff* f *&* g *are linearly dependent.*

Proof See exercises.

From Minkowski's inequality, it follows that if $L^p = \mathcal{L}^p/\sim$ where $\sim$ means "a.e. equality", then for $1 \leq p < \infty$, $(L^p, \|\cdot\|_p)$ is a normed linear space.

10.3 Theorem $(L^p, \|\cdot\|_p)$ *is a Banach space.*

Proof We show that if $f_n \in \mathcal{L}^p$ is a Cauchy sequence, i.e. $\|f_m - f_n\|_p \to 0$ as $m, n \to \infty$, then $\exists\ f \in \mathcal{L}^p$ such that

$$\|f - f_n\|_p \longrightarrow\ 0 \text{ as } n \to \infty.$$

To see this, $\exists\ n_k \uparrow$ such that

$$\|f_m - f_n\|_p < \frac{1}{4^k}\ \forall\ m, n > n_k.$$

It follows that

$$m([|f_{n_k} - f_{n_{k+1}}| > \frac{1}{2^k}]) \leq 2^{pk}\|f_{n_k} - f_{n_{k+1}}\|_p^p < \frac{1}{2^{pk}}$$

whence, a.e.,

$$\exists\ \lim_{K\to\infty} f_{n_1} + \sum_{k=1}^{K}(f_{n_{k+1}} - f_{n_k}) := f,$$

and $f \in \mathcal{L}^p$ as

$$\|\sum_{k=1}^{\infty}(f_{n_k} - f_{n_{k+1}})\|_p \leq \sum_{k=1}^{\infty}\|f_{n_k} - f_{n_{k+1}}\|_p \leq 1.$$

It also follows from this that

$$\|f - f_{n_\ell}\|_p \leq \sum_{k=\ell+1}^{\infty}\|f_{n_k} - f_{n_{k+1}}\|_p \to 0$$

as $\ell \to \infty$, whence, choosing $\ell(N)$ such that $n_{\ell(N)} \leq N < n_{\ell(N)+1}$, we have

$$\|f - f_N\|_p \leq \|f - f_{n_{\ell(N)}}\|_p + \|f_N - f_{n_{\ell(N)}}\|_p \to 0$$

as $N \to \infty$. $\square$

Dual spaces.

For $(B, \|\cdot\|)$ a normed linear space over $\mathbb{R}$, the *dual* space is the space of *bounded linear functionals*

$$B^* = \{L : B \to \mathbb{R} : \text{linear}, \ \exists\ M \ni\ |L(f)| \le M\|f\| \ \forall\ f \in\ B\}.$$

It can be shown that $\|L\|_* := \sup_{f\in B,\ \|f\|=1} |L(f)|$ defines a norm on B^*. Here, we identify B^* for some examples.

10.4 Theorem (Riesz)

Suppose that $(X, \mathcal{B}, m)$ is a σ-finite measure space, and suppose that $1 \le p < \infty$, then

$$(\ddagger) \qquad (L^p(m))^* \cong L^q(m)$$

where $q := \frac{p}{p-1}$ and $\cong$ means isomorphism by Banach space isometry.

Proof For $g \in L^q$, by Hölder's inequality, $fg \in L^1\ \forall\ f \in L^p$, and if $A_g(f) = \int_X fgdm\ (f \in L^p)$ then $A_g \in (L^p)^*$, and $\|A_g\|_{(L^p)^*} := \sup\{|A_g(f) : \|f\|_p = 1\} = \|g\|_q$.

It remains to prove $(\ddagger)$: $\forall\ A \in (L^p)^*,\ \exists\ g \in\ L^q\ \ni\ A \equiv A_g$.

We prove $(\ddagger)$ first under the assumption $m(X) < \infty$.

Let $A \in (L^p)^*$, and define $\mu : \mathcal{B} \to \mathbb{R}$ by $\mu(E) = A(1_E)$ for $E \in \mathcal{B}$. By linearity of A. μ is additive, and if $E_n \in \mathcal{B},\ E_n \uparrow E$, then $1_{E_n} \xrightarrow{L^p} 1_E$, whence $\mu(E_n) \to \mu(E)$ as $n \to \infty$, and μ is a signed measure. Clearly $\mu \ll m$, and so by the R-N theorem, $\exists\ g \in L^1\ \ni\ d\mu = gdm$. We claim that (i) $g \in L^q$, and (ii) $A \equiv A_g$.

We treat only the (more difficult) case of $p > 1$. By linearity of A and A_g we have that $A(f) = A_g(f)$ for every simple function f. If $f \in L^\infty$, then $\exists\ f_n$, simple functions, such that $f_n \xrightarrow{L^\infty} f$, whence

$$A(f) \leftarrow A(f_n) = A_g(f_n) \to A_g(f),$$

and $A(f) = A_g(f)\ \forall\ f \in L^\infty$. We use this next to show that $g \in L^q$. Let h =sgn(g), and for $\kappa > 0$ set $f_\kappa = 1_{[|g|\le\kappa]} h|g|^{q-1} \in L^\infty$, then

$$\int_{[|g|\le\kappa]} |g|^q dm = A_g(f_\kappa) = A(f_\kappa) \le$$

$$\|A\|_{(L^p)^*}\|f_\kappa\|_p = \|A\|_{(L^p)^*}\left(\int_{[|g|\le\kappa]} |g|^q dm\right)^{\frac{1}{p}},$$

whence $\|g\|_q \leftarrow \|1_{[|g|\le\kappa]} g\|_q \le \|A\|_{(L^p)^*}$.

To prove (‡) in general, let $h : X \to \mathbb{R}_+$, be such that $\int_X hdm = 1$, and let $d\mu = hdm$. Then $f \in L^p(\mu)$ iff $fh^{\frac{1}{p}} \in L^p(m)$. Let $A \in (L^p(m))^*$, and set $B(f) = A(fh^{\frac{1}{p}})$ for $f \in L^p(\mu)$. By the above, $\exists\ g \in L^q(\mu) \ni B(f) = \int_X fgd\mu = \int_X fghdm$, whence, for $f \in L^p(m)$, $A(f) = B(fh^{-\frac{1}{p}}) = \int_X fgh^{\frac{1}{q}}dm$. □

Dual space of L^∞. Let $(X, \mathcal{B}, m)$ be a σ-finite, measure space. A finitely additive set function $F : \mathcal{B} \to \mathbb{R}$ is called *m-absolutely continuous* if

$$A \in \mathcal{B},\ m(A) = 0 \implies F(A) = 0.$$

Denote by $\mathcal{C}(X, \mathcal{B}, m)$ the collection of m-absolutely continuous, finitely additive set functions $F : \mathcal{B} \to \mathbb{R}$ with finite total variation:

$$\|F\| := \sup\{\sum_{n=1}^{N} |F(A_n)| :\ A_1, A_2, \dots, A_n \in \mathcal{B} \text{ disjoint}\} < \infty.$$

It follows from the Jordan decomposition theorem that

$$\mathcal{C}(X, \mathcal{B}, m) = \{\mu - \nu :\ \mu,\ \nu \in \mathcal{C}(X, \mathcal{B}, m)_+\}$$

where $\mathcal{C}(X, \mathcal{B}, m)_+ := \{F \in \mathcal{C}(X, \mathcal{B}, m) :\ F(A) \ge 0\ \forall\ A \in \mathcal{B}\}$.

This implies that for $\mu \in \mathcal{C}(X, \mathcal{B}, m)$, the integral functional $f \mapsto \int_X fd\mu$:

- can be defined as in proposition 5.4,
- is bounded on L^∞ as in proposition 5.5; and
- $f = g$ m-a.e. $\implies \int_X fd\mu = \int_X gd\mu$.

10.5 Theorem

$$L^{\infty *}(X, \mathcal{B}, m) = \mathcal{C}(X, \mathcal{B}, m).$$

Proof As above, if $\mu \in \mathcal{C}(X, \mathcal{B}, m)$, then $f \mapsto L_\mu(f) = \int_X fd\mu$ defines am element of $L^{\infty *}$. Conversely, let $L \in L^{\infty *}$ and define $\mu : \mathcal{B} \to \mathbb{R}$ by $\mu(A) = L(1_A)$. We need to show that $\|\mu\| < \infty$ and that $L \equiv L_\mu$.

To see that $\|\mu\| \le \|L\|_{L^{\infty *}}$, let $A_1, A_2, \dots, A_n \in \mathcal{B}$ be disjoint and set $f := \sum_{k=1}^n \mathtt{sign}\,(\mu(A_k))1_{A_k}$, then $\|f\|_\infty = 1$ and

$$\sum_{k=1}^{n} |\mu(A_k)| = L(f) \le \|L\|_{L^{\infty *}}$$

whence $\|\mu\| \le \|L\|_{L^{\infty *}}$, $\mu \in \mathcal{C}(X, \mathcal{B}, m)$ and $L_\mu \in L^{\infty *}$.

We have $L(f) = L_\mu(f)$ for simple functions $f \in L^\infty$ and hence $\forall\ f \in L^\infty$ as the collection of simple functions is uniformly dense. ☑

Remark.

Let $(X, \mathcal{B}, m)$ be $X := [0,1]$ equipped with Borel sets and Lebesgue measure, then $L^1(m) \subsetneqq L^\infty(m)^*$. To see this, by the `Hahn-Banach theorem` $\exists\ L \in L^\infty(m)^*$ so that $L(f) := f(0)\ \forall\ f \in C([0,1])$.

It is not hard to show (!) that $\nexists\ g \in L^1(m)$ so that $\int_0^1 f(t)g(t)dt = f(0)\ \forall\ f \in C([0,1])$.

§11 Linear functionals of continuous functions on a compact Hausdorff space

`Topological Background.`

We'll need the following results about a compact Hausdorff space $(X, \mathcal{T})$:

- **Normality or $\mathbf{T}_4$:**
  If $F, G \subset X$ are disjoint closed sets, then $\exists\ U, V \in \mathcal{T}$ so that $U \cap V = \varnothing$, $U \supset F$, & $V \supset G$.
- **Urysohn's lemma:**
  The space $(X, \mathcal{T})$ is normal iff whenever $F \subset U \subset X$, F closed and U open, $\exists\ f \in C(X, [0,1])$ so that $F \prec f \prec U$.

Here, $F \prec f$ means $1_F \le f \le 1$ and $f \prec U$ means $\text{supp} f := \overline{[f \ne 0]} \subset U$ (and hence $0 \le f \le 1_U$).

Linear functionals.

Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of continuous $\mathbb{R}$-valued functions with respect to the norm $\|f\|_C := \sup_{x \in X} |f(x)|$.

A linear functional $L : C(X) \to \mathbb{R}$ is called *bounded* if $\exists\ M > 0$ such that $|L(f)| \le M\|f\|_C\ \forall\ f \in C(X)$ and *positive* if $f \ge 0 \implies L(f) \ge 0$.

Note that positive $\implies$ bounded and (!) that a linear functional L is positive $\iff |L(f)| \le L(1)\|f\|_C\ \forall\ f \in C(X)$

Set

$$C(X)^* := \{L : C(X) \to \mathbb{R} :\ L \text{ linear, bounded}\},$$

$$C(X)^*_+ = \{L \in C(X)^* :\ L \text{ positive }\}.$$

The question arises:

$$L(f) \overset{?}{=} \int_X f d\mu_L$$

for some measure defined on some sufficiently large σ-algebra.

Baire sets and Borel sets. Given a topological space X, call a σ-algebra $\mathcal{A} \subset 2^X$ *admissible* if each $f \in C(X)$ is $\mathcal{A}$-measurable (i.e. $f^{-1}(\mathcal{B}(\mathbb{R})) \subset \mathcal{A}$)

$$\mathfrak{b}(X) := \bigcap_{\mathcal{A}\subset 2^X,\text{admissible } \sigma\text{-algebra}} \mathcal{A} = \sigma\Big(\bigcup_{f\in C(X)} f^{-1}(\mathcal{B}(\mathbb{R}))\Big).$$

The standard proof shows that $\mathfrak{b}(X)$ is itself an admissible σ-algebra. Sets in $\mathfrak{b}(X)$ are called *Baire sets.*

11.1 Proposition *If X is a separable metric space, then $\mathfrak{b}(X) = \mathcal{B}(X)$.*

Proof For any open ball $U \subset X$, $\exists\ f_n \in C(X)$ so that $f_n(x) \to 1_U(x)\ \forall\ x \in X$ and 1_U is $\mathfrak{b}(X)$-measurable as the pointwise limit of a sequence of $\mathfrak{b}(X)$-measurable (continuous) functions. By separability each open set is the union of countably many balls and $\mathcal{B}(X) = \sigma(\{\texttt{balls}\})$.☑

11.2 Example: A compact Hausdorff space with $\mathfrak{b} \neq \mathcal{B}$.

Let $\Omega := S^{\Lambda}$ (where S is a finite set and Λ is an arbitrary, uncountable set) equipped with the product discrete topology, then Ω is a compact Hausdorff space.

Let $\mathcal{S}$ be the semiring of cylinder sets in Ω (see exercise 1.3 p.9) and let

$$\mathfrak{A} := \Big\{\sum_{k=1}^{N} a_k 1_{C_k} :\ a_k \in \mathbb{R},\ C_k \in \mathcal{S}\ \forall\ 1 \le k \le N\Big\},$$

then $\mathfrak{A}$ is a subalgebra of $C(\Omega)$ which separated points. By the Stone-Weierstrass theorem, $\mathfrak{A}$ is dense in $C(\Omega)$, whence $\mathfrak{b}(\Omega) = \sigma(\mathcal{S})$.

A standard monotone class argument shows that if $A \in \sigma(\mathcal{S})$ then $\exists\ \Gamma \subset [0,1]$ countable, $A' \subset \{0,1\}^{\Gamma}$ so that $A = \{x : [0,1] \to \{0,1\} : x|_{\Gamma} \in A'\}$.

Thus any singleton $\{x\} \in \mathcal{B}(X) \setminus \mathfrak{b}(X)$.

11.3 Frechet Lemma

Let X be a compact Hausdorff space, let $p \in \mathcal{P}(X, \mathfrak{b}(X))$ and let $f : X \to \mathbb{R}$ be bounded, Baire measurable, then $\forall\ \varepsilon > 0,\ \exists\ h \in C(X)$ with $p([|f - h| > \varepsilon]) < \varepsilon$.

Proof for $f = 1_A$ Let $\mathcal{A}$ denote the algebra of finite unions of subintervals of $\mathbb{R}$. Let $\eta > 0$.

Since $A \in \mathfrak{b}(X)$, $\exists\ f_1, \dots, f_N \in C(X)$ and $I_1, \dots, I_N \in \mathcal{A}$ so that

$$\frac{\varepsilon}{2} > p\Big(A \Delta \bigcap_{k=1}^{N} f_k^{-1}(I_k)\Big) = p([1_A \neq F])$$

where $F := \prod_{k=1}^{N} 1_{I_k} \circ f_k$.

For each k, $\exists\ G_{n,k} \in C_B(\mathbb{R})$, $G_{n,k}(z) \xrightarrow[n\to\infty]{} 1_{I_k}(z) \quad \forall\ z \in \mathbb{R}$ and it follows that $H_n := \prod_{k=1}^N G_{n,k} \circ f_k \in C(X) \quad (n \ge 1)$ and

$$H_n \xrightarrow[n\to\infty]{} F(x) \quad \forall\ x \in X.$$

Thus, for $n \ge 1$ large

$$p([|H_n - 1_A| > \varepsilon]) < p([|H_n - F| > \varepsilon] + P(([F \ne 1_A]) < \varepsilon. \quad ☑$$

Open Baire sets and uniqueness of weak representation.

Let $(X, \mathcal{T})$ be a compact Hausdorff space. A F_σ *set* in X is a set of form $A = \bigcup_{n=1}^\infty K_n$ where $K_1, K_2, \dots$ are closed.

Let $\mathcal{T}_\sigma$ be the collection of open F_σ sets.

11.4 Proposition

(i) $\mathfrak{b}(X) = \sigma(\mathcal{T}_\sigma)$,

(ii) $\mu,\ \nu \in \mathfrak{M}(X, \mathfrak{b}(X)),\quad \int_X f d\mu = \int_X f d\nu\ \forall\ f \in C(X)\ \Rightarrow\ \mu = \nu.$

Proof of (i)

We prove first that $\mathcal{T}_\sigma \subset \mathfrak{b}(X)$.[8] Let $U \in \mathcal{T},\ U = \bigcup_{n=1}^\infty K_n$ where $K_1 \subset K_2 \subset \dots$ are closed.

By Urysohn's lemma, $\exists\ f_n \in C(X, [0,1])\ (n \ge 1)$ so that $K_n \prec f_n \prec U$. In particular,

$$1_{K_n} \le f_n \le 1_U.$$

By assumption, $1_{K_n} \uparrow 1_U$, whence $f_n \xrightarrow[n\to\infty]{} 1_U$ on X and $U \in \mathfrak{b}(X)$. Consequently, $\sigma(\mathcal{T}_\sigma) \subset \mathfrak{b}(X)$.

To see $\sigma(\mathcal{T}_\sigma) \supset \mathfrak{b}(X)$, let

$$\mathcal{U} := \{\bigcap_{k=1}^N f_k^{-1}(I_k):\ f_1, \dots, f_N \in C(X)\ \&\ I_1, \dots, I_N \in \mathcal{T}(\mathbb{R})\},$$

then $\mathcal{U} \subset \mathcal{T}_\sigma$ because $\mathcal{T}(\mathbb{R}) = \mathcal{T}_\sigma(\mathbb{R})$. Thus

$$\mathfrak{b}(X) = \sigma(\mathcal{U}) \subset \sigma(\mathcal{T}_\sigma). \quad ☑\text{(i)}$$

Proof of (ii)

For $U \in \mathcal{T}_\sigma$, $\exists\ h_n \in C(X, [0,1]),\ h_n \to 1_U$ on X, whence $\mu(U) = \nu(U)$. It follows from (i) that $\mu = \nu$. ☑(ii)

11.5 Theorem: Structure of compact Hausdorff spaces

Let X be a compact Hausdorff space, then $\exists$ a set Λ and a closed subset $\Omega_X \subset \{0,1\}^\Lambda$ so that X is a continuous image of Ω_X.

Proof

Let $\mathcal{C} \subset C(X, [0,1])$ be uniformly dense and define $\beta : X \to [0,1]^{\mathcal{C}}$ by

$$\beta(x)(g) := g(x).$$

We claim that β is continuous and injective.

`Proof of continuity` A *cylinder* in $[0,1]^{\mathcal{C}}$ is a set of form

$$[U_1, \dots, U_N]_{g_1, \dots, g_n} := \{\omega \in [0,1]^{\mathcal{C}}:\ \omega(g_k) \in U_k\ \forall\ 1 \le k \le N\}$$

[8] The other inclusion also holds. See Halmos' book.

where $g_1,\dots,g_N\in\mathcal{C}$ & $U_1,\dots,U_N\in\mathcal{T}([0,1])$.

The collection of cylinders forms a base for the compact, Hausdorff, product topology on $[0,1]^{\mathcal{C}}$. Thus, continuity of β is established by

$$\beta^{-1}[U_1,\dots,U_N]_{g_1,\dots,g_n}=\bigcap_{k=1}^N g_k^{-1}U_k\in\mathcal{T}.\quad ☑$$

`Proof of injectivity` Suppose that $x,y\in X,\ x\neq y$. By Urysohn's lemma $\exists\ f\in C(X,[0,1])$ so that $f(x)\neq f(y)$, whence by density of $\mathcal{C}$, $\exists\ g\in\mathcal{C}$ so that $g(x)\neq g(y)$. Consequently, $\beta(x)\neq\beta(y)$. ☑

Next $\beta(X)$ is compact in $[0,1]^{\mathcal{C}}$ and $\beta:X\to\beta(X)$ is a homeomorphism.

Now define $\Phi:\{0,1\}^{\mathbb{N}\times\mathcal{C}}\to[0,1]^{\mathcal{C}}$ by

$$\Phi(\omega)(g):=\sum_{n=1}^\infty\frac{\omega(n,g)}{2^n}.$$

It follows that Φ is a continuous surjection.

Let $\Omega_X:=\Phi^{-1}(\beta(X))$, then Ω_X is closed in $\{0,1\}^{\mathbb{N}\times\mathcal{C}}$ and

$$\pi:=\Phi\circ\beta^{-1}:\Omega_X\to X$$

is a continuous surjection. ☑,

11.6 Riesz representation theorem (RRT) (Riesz, Saks, Markov)

Suppose $(X,\mathcal{T})$ is a compact Hausdorff space, and suppose that $\mathcal{L}\in C(X)^*$, then $\exists$ a unique, signed, Baire measure $\mu:\mathfrak{b}(X)\to\mathbb{R}$ such that

$$\mathcal{L}(f)=\int_X fd\mu\ \ \forall\ f\in C(X).$$

Proof For $\nu\in\mathfrak{M}(X,\mathfrak{b}(X))$, $f\mapsto L_\mu(f):=\int_X d\mu$ defines an element $L_\mu\in C(X)^*$. By proposition 11.4(iii), $\nu\mapsto L_\mu$ is injective. We must show that it is surjective.

By theorem 11.5 (`structure theorem for compact Hausdorff spaces`) $\exists$ a set $\mathcal{C}$ and a closed subset $\Omega\subset\{0,1\}^{\mathcal{C}}$ and $\pi:\Omega\to X$ continuous and onto. Define $\pi_*:C(X)\to C(\Omega)$ by $\pi_*(f):=f\circ\pi$, then $D_X:=\pi_*(C(X))$ is a closed linear subspace of $C(\Omega)$.

Now let $L\in C(X)^*$ and define $L_1\in(D_X)^*$ by $L_1(f\circ\pi):=L(f)$.

By the `Hahn-Banach theorem` $\exists$ $L_2\in C(\Omega)^*$ such that $L_2|_{D_X}=L_1$ whence $L_2(f\circ\pi)=L(f)\ \forall\ f\in C(X)$.

We claim that

☺ $\exists$ a signed measure $\mu:\mathfrak{b}(\Omega)\to\ [0,\infty)$ such that $L_2(g)=\int_\Omega gd\mu$.

Proof of ☺

Let $\mathcal{A}$ be the algebra of subsets of Ω generated by cylinders.

Since cylinders are clopen in Ω, this algebra has the finite subcover property.

Define $\nu : \mathcal{A} \to \mathbb{R}$ by $\nu(A) := L(1_A)$, then ν is additive by the linearity of L.

We claim that $\|\mu\| < \infty$. To see that in fact $\|\mu\| \le \|L\|_{C(\Omega)^*}$, let $A_1, A_2, \dots, A_n \in \mathcal{A}$ be disjoint and set $f := \sum_{k=1}^n \mathtt{sign}\,(\mu(A_k))1_{A_k}$, then $f \in C(\Omega)$ & $\|f\|_{C(\Omega)} = 1$. Moreover

$$\sum_{k=1}^{n} |\mu(A_k)| = L(f) \le \|L\|_{C(\Omega)^*}$$

whence our claim.

By the Jordan decomposition, $\mu = \mu_+ - \mu_-$ where $\mu_\pm : \mathcal{A} \to [0, \infty)$ are additive, whence countable subadditive and extend to measures $\overline{\mu}_\pm$ on $\sigma(\mathcal{A}) = \mathfrak{b}(\Omega)$. Thus $\overline{\mu} := \overline{\mu}_+ - \overline{\mu}_-$ is a signed, Baire measure satisfying

$$L_2(F) = L_\mu(F) := \int_\Omega F d\overline{\mu} \quad \forall \ F = \sum_{k=1}^{N} a_k 1_{A_k}$$
$$\text{with } a_1, \dots, a_N \in \mathbb{R},\ A_1, \dots, A_N \in \mathcal{A}.$$

By the Stone-Weierstrass theorem, such functions are uniformly dense in $C(\Omega)$ whence $L_2 \equiv L_\mu$. ☑☺

It follows from ☺ that $\nu := \mu \circ \pi^{-1} : \mathfrak{b}(X) \to [0, \infty)$ is also a signed measure, and for $f \in C(X)$,

$$\int_X f d\nu = \int_\Omega f \circ \pi d\mu = L_2(f \circ \pi) = L(f).$$

Uniqueness follows because the indicator of a Baire set is the a.s. pointwise limit of a uniformly bounded sequence of continuous functions. ☑

Regular Borel measures. Let $(X, \mathcal{T})$ be a topological space. A Borel measure $\mu \in (X, \mathcal{B}(X))$ is *regular* if

$$\mu(A) = \inf \{\mu(U) :\ A \subset U \in \mathcal{T}\}.$$

11.7 Kakutani's extension theorem

Suppose $(X, \mathcal{T})$ *is a compact Hausdorff space, and suppose that* $\mu \in \mathcal{P}(X, \mathfrak{b}(X))$*, then there is a unique, regular* $m \in \mathcal{P}(X, \mathcal{B}(X))$ *such that* $m|_{\mathfrak{b}(X)} \equiv \mu$.

`Proof` See exercises.

Exercise N$^{\underline{\text{o}}}$7

1. Hölder's, and Minkowski's inequalities.

Let $1 < p < \infty$, and $q = \frac{p}{p-1}$.

a) Prove that for $a, b \geq 0$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ with equality iff $a^p = b^q$.

(*Hint*: One way to do this is to prove that $\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta \;\; \forall \; \alpha, \beta \geq 0, \; 0 < \lambda < 1$, and substitute $\alpha = a^p, \; \beta = b^q$ with appropriate λ...)

Now suppose that $(X, \mathcal{B}, m)$ is a σ-finite measure space and that $f, g : X \to I\!R$ are measurable functions satisfying $\int_X |f|^p dm, \int_X |g|^q dm < \infty$.

b) Prove that fg is integrable, and

$$|\int_X fg dm| \leq \frac{1}{p} \int_X |f|^p dm + \frac{1}{q} \int_X |g|^q dm.$$

c) Using b), or otherwise, prove `Hölder's inequality`: $|\int_X fg dm| \leq \|f\|_p \|g\|_q$ where

$$\|f\|_p := \left(\int_X |f|^p dm \right)^{\frac{1}{p}}.$$

with equality iff $fg \geq 0$ a.e. and $\exists$ a constant $c > 0$ so that $|g|^q = c|f|^p$ a.e..

d) Prove that $\|f\|_p = \max\{\int_X fg dm : \|g\|_q \leq 1\}$.

e) Prove `Minkowski's inequality`: $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ with equality iff f & g are linearly dependent

2. Regular Borel measures.

Suppose $(X, \mathcal{T})$ is a compact Hausdorff space, and suppose that $\mu \in \mathcal{P}(X, \mathfrak{b}(X))$. Here, in a series of exercises, you prove Kakutani's theorem that:

$\exists$ a unique, regular $m \in \mathcal{P}(X, \mathcal{B}(X))$ such that $m|_{\mathfrak{b}(X)} \equiv \mu$.

Write $L(f) := \int_X f d\mu$ $(f \in C(X))$.

(i) Show that for $U \in \mathcal{T}$,

$$m(U) := \sup\{\mathcal{L}(f) : f \in C(X), \; f \prec U\} = \sup\{\mu(A) : \; U \supseteq A \in \mathfrak{b}(X)\}.$$

(ii) Show that $m : \mathcal{T} \to [0, \infty)$ satisfies

(a) $U, V \in \mathcal{T},\ U \subset V \ \Rightarrow\ m(U) \le m(V),$

(b) $U, V \in \mathcal{T},\ U \cap V = \varnothing \ \Rightarrow\ m(U \uplus V) = m(U) + m(V),$

(c) $m\left(\bigcup_{n=1}^{\infty} U_n\right) \le \sum_{k=1}^{\infty} m(U_n)\ \forall\ U_1, U_2, \dots \in \mathcal{T}.$

For $E \subset X$, let

$$\overline{\mu}(E) := \inf\{m(U) : E \subset U \in \mathcal{T}\}.$$

(iii) Show that $\overline{\mu}$ is an outer measure and that
(a) $\overline{\mu}(U) = m(U)\ \forall\ U \in \mathcal{T}$, (b) $\overline{\mu}(A) = \mu(A)\ \forall\ A \in \mathfrak{b}(X)$.

(iv) Show that if $U \in \mathcal{T}$, then

$$\forall\ \ a < m(U),\ \exists\ F \subset U,\ F \text{ closed, such that } \ \overline{\mu}(F) > a.$$

(v) Show that $\overline{\mu}(F_1 \cup F_2) \ge \overline{\mu}(F_1) + \overline{\mu}(F_2)\ \forall\ F_1, F_2$ closed, disjoint.
Hint: Use normality (T_4) to get $U_i \in \mathcal{T}$ $(i = 1, 2)$ disjoint, so that $F_i \subset U_i$ $(i = 1, 2)$.

Let

$$\mathcal{C} := \{A \subset X :\ \forall\ \ a < \overline{\mu}(A),\ \exists\ F \subset A,\ F \text{ closed } \ni\ \overline{\mu}(F) > a\}.$$

(vi) Show that if $A, B \in \mathcal{C}$ are disjoint, then $\overline{\mu}(A \cup B) = \overline{\mu}(A) + \overline{\mu}(B)$.

(vii) Show that $A, B \in \mathcal{C} \ \Rightarrow\ A \setminus B \in \mathcal{C}$.
Hint: Let $\varepsilon > 0$, and $F \subset A \subset U$, $G \subset B \subset V$, F, G closed, U, V open be such that $\overline{\mu}(U) - \overline{\mu}(F),\ \overline{\mu}(V) - \overline{\mu}(G) < \varepsilon$.

(viii) Show that

$$\overline{\mu}(A) \ge \overline{\mu}(A \cap U) + \overline{\mu}(A \cap U^c)\ \ \forall\ U \in \mathcal{T},\ A \subset X.$$

Hint Fix $A \subset W \in \mathcal{T}$, then $W \cap U,\ W \cap U^c \in \mathcal{C}$.

(ix) Using the above exercises, prove Kakutani's theorem.

3. Non-regular Borel measures.

Here, you show that X compact, Hausdorff, $p : \mathcal{B}(X) \to [0,1]$ a probability $\not\Rightarrow$ p regular.

Let X be the collection of ordinals up to and including Ω, the least uncountable ordinal, and let $\prec$ be the usual well ordering of X. Let $\mathcal{T}$ be the topology generated by open intervals, i.e. sets of form $(a,b) = \{x \in X : a \prec x \prec b\}$, $[0,a) = \{x \in X : x \prec a\}$, and $(b,\Omega] = \{x \in X : x \succ b\}$.

Show that:

(a) $(X,\mathcal{T})$ is a compact Hausdorff space;

(b) $[0,\Omega)$ is open and not σ-compact;

(c) $\forall\ f \in C(X)\ \exists\ a \in [0,\Omega)$ so that f is constant on $(a,\Omega]$;

(d) if $A \subset [0,\Omega)$ is countable, then $\exists\ a \prec \Omega$ such that $A \subset [0,a)$;

(e) if K is compact, and $a_n \in K$ $(n \ge 1)$, then $\sup_n a_n \in K$;

(f) if K_n is compact, and uncountable $(n \ge 1)$, then so is $\bigcap_{n=1}^{\infty} K_n$;

(g) if $E \in \mathcal{B}(X)$, then either E, or E^c contains an uncountable compact set (but not both).

Define $p : \mathcal{B}(X) \to [0,1]$ by $p(E) = 1$ if E contains an uncountable compact set, and $p(E) = 0$ otherwise. Show that

f) p is a probability, and is not regular.

g) Exhibit a regular probability $q : \mathcal{B}(X) \to [0,1]$ such that

$$\int_X f dp = \int_X f dq\ \forall\ f \in C(X).$$

Week # 8

§12 Haar measure.

A *topological group* is a group G, which is a topological space such that $(g,h) \mapsto gh^{-1}$ is continuous $(G \times G \to G)$.

A measure $m : \mathcal{B}(G) \to [0,\infty]$ is called a *left Haar measure* on G if

(i) $m(U) > 0\ \forall\ U$ open,

(ii) $m(K) < \infty\ \forall\ K$ compact,

and

(iii) $m(xA) = m(A)\ \forall\ x \in G,\ A \in \mathcal{B}(G)$.

An analogous definition can be given for *right Haar measure* on G. For Abelian groups, the definitions coincide.

Examples

1) For $G = \mathbb{R}^d$ under addition, Lebesgue measure is Haar measure.

2) For $G = \mathbb{C} \setminus \{0\}$ under multiplication, a Haar measure is given by $dm(x+iy) = \frac{dxdy}{\sqrt{x^2+y^2}}$.

3) In exercise 6.5, you identified both left and right Haar measures on $G = \{\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) : (x,y) \in \mathbb{R}^2,\ x > 0\}$ equipped with the topology inherited from $\mathbb{R}^2$, and matrix multiplication.

It follows from exercise 9.3 (!) that left Haar measure is unique up to constant multiplication. Our next result is existence of Haar measure on any locally compact topological group.

Theorem (A. Haar) *If G is a locally compact, Hausdorff, topological group, then $\exists$ a left Haar measure on G. This measure is regular.*
If, in addition, G is σ-compact, then the left Haar measure is σ-finite.

Example
Let $G = \mathbb{R}$ equipped with the discrete topology, then G is a locally compact topological group, and Haar measure is counting measure. This shows that Haar measure on a locally compact topological group need not be σ-finite.

`Proof of Haar's theorem` (A.Weil).
Let $\mathcal{C}$ be the collection of compact subsets of G.

Step 1 $\exists\ \lambda : \mathcal{C} \to [0,\infty)$ *which is non-zero, left invariant, monotone, subadditive, and additive.*

Define $(\varnothing : B) := 0\ \ (\varnothing \neq B \subseteq G)$ and

$$(A : B) = \min\{n \in \mathbb{N} : \exists\, x_1, x_2, \dots, x_n \in G,\ A \subset \bigcup_{k=1}^{n} x_k B\}\ \ (\varnothing \neq A,\ B \subseteq G)$$

where it is understood that $\inf \varnothing = \infty$. Evidently $(A : B) < \infty$ in case $A \in \mathcal{C}$ and $B^o \neq \varnothing$.
Also, for $A,, A', B,, B', C \subset G$,

$$(0)\qquad A \subseteq A',\ B \supseteq B' \ \Rightarrow\ (A : B) \le (A' : B').$$

$$(1)\qquad (A : B) = (xA : yB)\ \forall\ x, y \in G,$$

$$(2)\qquad (A : C) \le (A : B)(B : C),$$

$$(3)\qquad (A \cup B : C) \le (A : C) + (B : C),$$

with equality if $A(C)^{-1} \cap B(C)^{-1} = \varnothing$.
Let $\mathcal{C}_e = \{K \in \mathcal{C} : K^o \ni e\}$. Fix $\Omega \in \mathcal{C}_e$, and define, for $U \in \mathcal{C}_e$, $\lambda_U : \mathcal{C} \to [0,\infty)$ by

$$\lambda_U(K) := \frac{(K : U)}{(\Omega : U)}.$$

Evidently $\lambda_U(\Omega) = 1 \ \forall \ U \in \mathcal{C}_e$. By (1), λ_U is left invariant $\forall \ U \in \mathcal{C}_e$, and by (2), $\lambda_U(K) \le (K : \Omega)$. By (3), λ_U is subadditive, and sometimes additive:

$$\text{(4)} \qquad C(U)^{-1} \cap D(U)^{-1} = \varnothing \ \Rightarrow \ \lambda_U(C \cup D) = \lambda_U(C) + \lambda_U(D).$$

Let

$$\Phi := \{\phi : \mathcal{C} \to [0, \infty) : \phi(\Omega) = 1, \ \phi(C) \le (C : \Omega) \ \forall \ C \in \mathcal{C}\},$$

then by Tychonov's theorem, Φ is a compact subset of $\mathbb{R}^{\mathcal{C}}$ (equipped with the product topology). For $K \in \mathcal{C}_e$, let

$$\Lambda(K) = \{\lambda_L : K \supset L \in \mathcal{C}_e\} \subset \Phi.$$

The family $\{\Lambda(K) : K \in \mathcal{C}_e\}$ has the `finite intersection property`:

$$\bigcap_{k=1}^{n} \Lambda(K_k) \supset \Lambda\left(\bigcap_{k=1}^{n} K_k\right) \neq \varnothing \ \forall \ K_1, \dots, K_n \in \mathcal{C}_e,$$

and by compactness of Φ,

$$\exists \ \lambda \in \bigcap_{K \in \mathcal{C}_e} \overline{\Lambda(K)}.$$

We claim that λ is as required for step 1.

• To prove that λ is monotone and subadditive, let $X = \{\nu \in \Phi : \nu$ is left invariant, monotone and subadditive.$\}$, then $\forall \ U \in \mathcal{C}_e, \ \Lambda(U) \subset X$ which is closed, whence $X \supset \overline{\Lambda(U)} \ni \lambda$.

• To prove additivity of λ, suppose that $B, C \in \mathcal{C}$ and $B \cap C = \varnothing$. We show that $\exists \ K \in \mathcal{C}_e$ such that $BK^{-1} \cap CK^{-1} = \varnothing$, obtaining by (4) that $\mu(B \cap C) = \mu(B) + \mu(C) \ \forall \ \mu \in \Lambda(K)$, whence (!) also for $\mu = \lambda$.

Now for every $b \in B$, $\exists \ U_b \in \mathcal{C}_e$ such that $bU_b \subset C^c$. By continuity of $(g,h) \mapsto gh$, $\exists \ V_g \in \mathcal{C}_e$ such that $V_g^2 \subset U_g$. By compactness of B, $\exists \ b_1, \dots, b_n \in B \ \ni \ B \subset \bigcup_{k=1}^n b_k V_{b_k}$. Set $W = \bigcap_{k=1}^n V_{b_k}$, then $BW \subset \bigcup_{k=1}^n b_k V_{b_k} W \subset \bigcup_{k=1}^n b_k V_{b_k}^2 \subset C^c$. Lastly, by continuity of $(g,h) \mapsto gh^{-1}$, $\exists \ K \in \mathcal{C}_e \ \ni \ K^{-1}K \subset W$, and it follows that $BK^{-1} \cap CK^{-1} = \varnothing$. ☐ Step 1

Define

$$\underline{\lambda}(U) := \sup_{U \supset K \in \mathcal{C}} \lambda(K) \text{ for } U \text{ open, and}$$

$$\overline{\mu}(B) := \inf\{\underline{\lambda}(U) : B \subset U \text{ open}\} \ \forall \ B \subset G.$$

Step 2

$$\text{(1)} \qquad \overline{\mu}(K^o) \le \lambda(K) \le \overline{\mu}(K) < \infty \ \forall \ K \in \mathcal{C}$$

(2) $\overline{\mu}$ *is a left invariant outer measure,*

and

(3) $$\mathcal{M}_{\overline{\mu}} \supset \mathcal{B}(G).$$

Proof

• 1) If $K \in \mathcal{C}$, then for every open set $U \supset K$, $\lambda(K) \le \underline{\lambda}(U) = \overline{\mu}(U)$, whence $\lambda(K) \le \overline{\mu}(K)$; and if $K^o \supset K_1 \in \mathcal{C}$, then $\lambda(K_1) \le \lambda(K)$, whence $\overline{\mu}(K^o) = \underline{\lambda}(K^o) \le \lambda(K)$.

To see that $\overline{\mu}(K) < \infty$, note that (!) $\exists\ L \in \mathcal{C},\ K \subsetneq L^o$ whence $\overline{\mu}(K) \le \underline{\lambda}(L^o) \le \lambda(L) < \infty$ and (1) is established.

Clearly, both $\underline{\lambda}$ and $\overline{\mu}$ are left invariant.

• We show that $\underline{\lambda}$ is additive, and subadditive.

Suppose that U, V are open sets.

Let $U \cup V \supset B \in \mathcal{C}$, then $\exists\ C, D \in \mathcal{C} \ni C \subset U,\ D \subset V$ and $B = C \cup D$; whence $\lambda(B) \le \lambda(C) + \lambda(D) \le \underline{\lambda}(U) + \underline{\lambda}(V)$ &

$$\underline{\lambda}(U \cup V) \le \underline{\lambda}(U) + \underline{\lambda}(V).$$

Now suppose that U, V are disjoint open sets. Let $C, D \in \mathcal{C},\ C \subset U,\ D \subset V$, then $\lambda(C) + \lambda(D) = \lambda(C \cup D) \le \underline{\lambda}(U \cup V)$, whence $\underline{\lambda}(U) + \underline{\lambda}(V) \le \underline{\lambda}(U \cup V)$, and $\underline{\lambda}$ is additive.

• Next, we show that $\overline{\mu}$ is countably subadditive. Suppose that $A = \bigcup_{n=1}^\infty A_n$, and let $\varepsilon > 0$. There are open sets $U_n \supset A_n$ such that $\underline{\lambda}(U_n) \le \overline{\mu}(A_n) + \frac{\varepsilon}{2^n}$. Let $\mathcal{C} \ni K \subset \bigcup_{n=1}^\infty U_n$. By compactness $\exists\ N < \infty$ such that $K \subset \bigcup_{n=1}^N U_n$, and by subadditivity, $\lambda(K) \le \sum_{n=1}^N \underline{\lambda}(U_n)$.

Thus,

$$\overline{\mu}(A) \le \underline{\lambda}\left(\bigcup_{n=1}^\infty U_n\right) \le \sum_{n=1}^\infty \underline{\lambda}(U_n) \le \sum_{n=1}^\infty \underline{\lambda}(A_n) + \varepsilon.$$

Finally $\overline{\mu}(\varnothing) = \underline{\lambda}(\varnothing) = \lambda(\varnothing) = 0$ and $\overline{\mu}$ is an outer measure. ☑(2)

We now show that

• (3) $\mathcal{M}_{\overline{\mu}} \supset \mathcal{B}(G)$, equivalently,

$$\overline{\mu}(F) \ge \overline{\mu}(F \cap V) + \overline{\mu}(F \cap V^c)\ \forall\ F \subset G,\ V \text{ open}.$$

Suppose first, that U, V are open.

We claim that $\underline{\lambda}(U) \ge \underline{\lambda}(V \cap U) + \overline{\mu}(V^c \cap U)$.

To see this, fix $V \cap U \supset D \in \mathcal{C}$, and $D^c \cap U \supset E \in \mathcal{C}$, then

$$\underline{\lambda}(U) \ge \lambda(D \cup E) = \lambda(D) + \lambda(E).$$

Leaving D fixed, we have $\underline{\lambda}(U) \ge \lambda(D) + \underline{\lambda}(D^c \cap U)$, and noting that $V^c \cap U \subset D^c \cap U$, an open set, we have

$$\underline{\lambda}(U) \ge \lambda(D) + \overline{\mu}(V^c \cap U).$$

Now let $F \subset G$, and V be open. For any $F \subset U$ open, we have that $\underline{\lambda}(U) \ge \underline{\lambda}(V \cap U) + \overline{\mu}(V^c \cap U) \ge \overline{\mu}(F \cap V) + \overline{\mu}(F \cap V^c)$,

$$\therefore \overline{\mu}(F) \ge \overline{\mu}(F \cap V) + \overline{\mu}(F \cap V^c). \quad ☑$$

To complete the proof of the theorem:

Step 3 $\mu := \overline{\mu}|_{\mathcal{B}(G)}$ *is a left Haar measure.*

Proof By step 2, $\mu(xA) = \mu(A) \ \forall \ A \in \mathcal{B}(G)$ and $\mu(K) < \infty \ \forall \ K \in \mathcal{C}$. It suffices to show that $\mu(U) > 0 \ \forall \ U \in \mathcal{C}_e$. By step 1, $\exists \ \Omega \in \mathcal{C}_e$ with $\mu(\Omega) \ge \lambda(\Omega) > 0$. Let $U \in \mathcal{C}_e$, then $(\Omega : U) = N \in \mathbb{N}$ and $\exists \ x_1, \dots, x_N \in G$ so that $\Omega \subseteq \bigcup_{n=1}^N x_n U$. Thus

$$0 < \mu(\Omega) \le \sum_{n=1}^N \mu(x_n U) = N\mu(U)$$

and $\mu(U) \ge \frac{\mu(\Omega)}{N} > 0$. □

§13 Applications to probability theory

12.1 Kolmogorov's existence theorem *Let $\mathfrak{A}$ be an index set, and for $a \in \mathfrak{A}$ let X_a be a polish space. For $F \subset \mathfrak{A}$ let $X_F = \prod_{a \in F} X_a$, and for $F \subset G \subset \mathfrak{A}$ let $\pi_{G,F} : X_G \to X_F$ be the canonical projection.*

Suppose that for $F \subset \mathfrak{A}$ finite, there is a probability $p_F : \mathcal{B}(X_F) \to [0,1]$, and that these probabilities are consistent in the sense

$$p_G \circ \pi_{G,F}^{-1} = p_F \text{ whenever } F \subset G,$$

then there is a unique probability

$$p : \mathcal{B}_{\mathfrak{A}} := \sigma\left(\bigcup_{F \subset \mathfrak{A}, \text{ finite}} \pi_{\mathfrak{A},F}^{-1} \mathcal{B}(X_F)\right) \to [0,1] \ \ni \ p \circ \pi_{\mathfrak{A},F}^{-1} = p_F \ \forall \ F \subset \mathfrak{A} \text{ finite}.$$

Proof

Up to measurable isomorphism, X_a is discrete whenever X_a is finite, $X_a = X_{\mathbb{N}}$ (as on p.5) whenever X_a is countable and using Kuratowski's theorem, $X_a = \Omega = \{0,1\}^{\mathbb{N}}$ for X_a uncountable. Accordingly, consider the algebras

$$\mathcal{A}(X_a) := \begin{cases} 2^{X_a} & A \text{ finite}, \\ \mathcal{A}_{\mathbb{N}} & A \text{ countable}, \\ \mathcal{A}(\{\texttt{cylinders}\}) & X_a = \Omega. \end{cases}$$

For $F \subsetneq \mathfrak{A}$ let

$$X_F := \prod_{a \in F} X_a \quad (F \subseteq \mathfrak{A}).$$

A *cylinder* in X_F be a set of form

$$\{x \in X_F : \ x(a) \in B(a) \ \forall \ a \in T\}$$

where $T \subset F$ is finite and $B(a) \in \mathcal{A}(X_a) \ \forall \ a \in T$.

For $F \subseteq \mathfrak{A}$, let $\mathcal{A}_F \subset \mathcal{B}(X_F)$ be the collection of finite, disjoint unions of cylinders in X_F, then $\mathcal{A}_F$ is an algebra.

For $F \subset G \subseteq \mathfrak{A}$, $\pi_{G,F}^{-1}\mathcal{A}_F \subset \mathcal{A}_G$ and

$$\widetilde{\mathcal{A}} := \mathcal{A}_{\mathfrak{A}} = \bigcup_{F \subset \mathfrak{A}, \text{ finite}} \pi_{\mathfrak{A},F}^{-1}\mathcal{A}_F.$$

We claim first that

¶1 $p(\pi_{\mathfrak{A},F}^{-1}B) := p_F(B)$ defines an additive set function $p : \widetilde{\mathcal{A}} \to [0,1]$.

`Proof` To show that this is a definition, we must show that

$$\pi_{\mathfrak{A},F}^{-1}A = \pi_{\mathfrak{A},G}^{-1}B \ \Rightarrow \ p_F(A) = p_G(B).$$

To see this, again let $H = F \cup G$, and note that

$$\pi_{\mathfrak{A},F}^{-1}A = \pi_{\mathfrak{A},G}^{-1}B \ \Rightarrow \pi_{H,F}^{-1}A = \pi_{H,G}^{-1}B \ \Rightarrow$$
$$p_F(A) = p_H(\pi_{H,F}^{-1}A) = p_H(\pi_{H,G}^{-1}B) = p_G(B).$$

To see that $p : \widetilde{\mathcal{A}} \to [0,1]$ is additive, let $A, B \in \widetilde{\mathcal{A}}$ be disjoint, $A = \pi_{\mathfrak{A},F}^{-1}A'$, $B = \pi_{\mathfrak{A},G}^{-1}B'$ where $A' \in \mathcal{A}_F$, $B' \in \mathcal{A}_G$. As before, let $H = F \cup G$, and note that $\pi_{H,F}^{-1}A' \cap \pi_{H,G}^{-1}B' = \varnothing$, whence

$$p(A \cup B) = p_H(\pi_{H,F}^{-1}A' \cup \pi_{H,G}^{-1}B') = p_H(\pi_{H,F}^{-1}A') + p_H(\pi_{H,G}^{-1}B') = p(A) + p(B). \ ☑¶1$$

Next, we claim that

¶2 $\widetilde{\mathcal{A}}$ has the finite subcover property.

`Proof` By Tychonov's theorem $X_{\mathfrak{A}}$ is a compact Haussdorf space and sets of form $A = \pi_{\mathfrak{A},F}^{-1}B$ are both open and compact. ☑¶2

Thus $p : \widetilde{\mathcal{A}} \to [0,1]$ is countable subadditive and by Caratheodory's theorem, $\exists$ a probability $\overline{p} : \sigma(\widetilde{\mathcal{A}}) \to [0,1]$ such that $\overline{p}|_{\widetilde{\mathcal{A}}} = p$.

To conclude the proof, it suffices to note that $\sigma(\widetilde{\mathcal{A}}) = \mathcal{B}(X_{\mathfrak{A}})$ and that since each p_F ($F \subset \mathfrak{A}$ finite) is a measure, we have $\overline{p} \circ \pi_{\mathfrak{A},F}^{-1} = p_F$.

□

Weak convergence of probability measures on Polish spaces

Let X be a topological space and let $\mu_n,\ \mu \in \mathcal{P}(X,\mathcal{B}(X))$. We say that μ_n *tends to* μ *weakly* (written $\mu_n \underset{n\to\infty}{\Longrightarrow} \mu$), if

$$\int_X f d\mu_n \underset{n\to\infty}{\longrightarrow} \int_X f d\mu$$

$$\forall\ f \in C_B(X) := \{\texttt{bounded continuous functions } : X \to \mathbb{R}\}.$$

If X is compact then

- $\mu_n \underset{n\to\infty}{\Longrightarrow} \mu$ iff $\mu_n \to \mu$ weak $*$ in $C(X)^*$.

12.3 Examples.

¶1 Let $X := [0,1]$ and define $\mu_n \in \mathcal{P}(X)$ by $\mu_n(A) = \frac{1}{n}\sum_{k=1}^n \delta_{\frac{k}{n}}(A) = \frac{1}{n}\sum_{k=1}^n 1_A(\frac{k}{n})$, then

$$\mu_n \underset{n\to\infty}{\Longrightarrow} \texttt{Leb}.$$

`Proof` Any $f \in C([0,1])$ is Riemann integrable, so by Darboux's theorem

$$\int_{[0,1]} f d\mu_n = \frac{1}{n}\sum_{k=1}^n f(\frac{k}{n}) \underset{n\to\infty}{\longrightarrow} \int_0^1 f(x)dx.$$

¶2 Let $X = \mathbb{R}$, $(\Omega,\mathcal{A},P)$ be a probability space and let $Z_1, Z_2, \dots : \Omega \to \mathbb{R}$ be independent, identically distributed, $\mathbb{R}$-valued, random variables random variables with $E(Z_i) = 0$ and $E(Z_i^2) = 1$. Let $S_n := \sum_{k=1}^n Z_k$ and define $\mu_n \in \mathcal{P}(\mathbb{R})$ by $\mu_n(A) := \texttt{Prob}([\frac{S_n}{\sqrt{n}} \in A])$, then

$$\mu_n \underset{n\to\infty}{\Longrightarrow} \mathcal{N},\ \ \mathcal{N}(A) = \frac{1}{\sqrt{2\pi}}\int_A e^{-\frac{t^2}{2}}dt.$$

This result is aka the `central limit theorem` (CLT) and will be proved in the sequel.

¶3 Let $(\Omega,\mathcal{A},P)$ & $Z_1, Z_2, \dots,\ S_n$ be as above, and let $X = C([0,1])$. Define $B_n : \Omega \to C([0,1])$ by

$$B_n(s\frac{k}{n} + (1-s)\frac{k+1}{n}) := s\frac{S_k}{\sqrt{n}} + (1-s)\frac{S_{k+1}}{\sqrt{n}}. \quad (1 \le k \le n,\ 0 \le s \le 1),$$

then (!) B_n is measurable. Define $\mu_n \in \mathcal{P}(C([0,1]))$ by $\mu_n(A) := \texttt{Prob}([B_n \in A]) \quad (A \in \mathcal{B}(C([0,1])))$, then

$$\exists\ \mathcal{W} \in \mathcal{P}(C([0,1])) \text{ such that } \mu_n \underset{n\to\infty}{\Longrightarrow} \mathcal{W}.$$

The measure is called `Wiener measure` and is the distribution of `Brownian motion`. The result is aka the `functional central limit theorem` (FCLT) and is proved in advanced courses on probability theory.

12.4 Helly's theorem for compact spaces

If X is a compact metric space, then $\forall\ \{\nu_n : n \ge 1\} \subset \mathcal{P}(X)$, $\exists\ n_k \to \infty,\ \mu \in \mathcal{P}(X)$ such that $\mu_{n_k} \underset{k\to\infty}{\Longrightarrow} \mu$.

In view of the `Riesz representation theorem`, this follows from the `Banach-Alaoglu` theorem which says that the bounded sets in the dual of a separable Banach space are weak $*$ sequentially compact.

Remark.

If (X,d) is a metric space which is not pre-compact, Helly's theorem fails.

`Sketch proof of remark`

Since (X,d) is not pre-compact, $\exists\ \varepsilon > 0$ and an infinite set $\Gamma \subset X$ which is *ε-separated* in the sense that $d(x,x') \ge \varepsilon\ \forall\ x,x' \in \Gamma,\ x \ne x'$. If $x_n \in \Gamma,\ x_n \ne x_{n'}\ \ (n \ne n')$ then

$$\forall\ (\omega_1, \omega_2, \dots) \in \{0,1\}^{\mathbb{N}},\quad \exists\ f \in C_B(X) \text{ such that } f(x_n) = \omega_n.$$

It follows that the sequence of point masses $\mu_n = \delta_{x_n}$ has no weakly convergent subsequence. ☐

12.5 Weak convergence proposition *Let (X,d) be a Polish space and let $\mu,\ \mu_1, \mu_2, \dots \in \mathcal{P}(X)$. TFAE:*

(i) $\mu_n \underset{n\to\infty}{\Longrightarrow} \mu$;

(ii) $\overline{\lim}_{n\to\infty} \mu_n(F) \le \mu(F)\ \forall\ F \subset X$ *closed;*

(iii) $\underline{\lim}_{n\to\infty} \mu_n(G) \ge \mu(G)\ \forall\ G \subset X$ *open;*

(iv) $\mu_n(A) \underset{n\to\infty}{\longrightarrow} \mu(A)\ \forall\ A \subset X$ *such that* $\mu(\partial A) = 0$.

Proof

Evidently (ii) $\iff$ (iii).

To see (i) $\Rightarrow$ (ii), fix $F \subset X$ closed and define $f_k : X \to [0,1]$ by $f_k(x) := (1 - kd(x,F)) \vee 0$, then $f_k \in C_B(X)$, $1_F \le f_k \downarrow 1_F$, whence

$$\mu_n(F) \le \mu_n(f_k) \xrightarrow[n\to\infty]{(i)} \mu(f_k) \underset{k\to\infty}{\longrightarrow} \mu(F).$$

To see (iii) $\Rightarrow$ (iv), fix $A \subset X$ such that $\mu(\partial A) = 0$, then

$$\varliminf_{n\to\infty} \mu_n(A) \ge \varliminf_{n\to\infty} \mu_n(A^o) \overset{(iii)}{\ge} \mu(A^o) = \mu(A)$$

and

$$\varlimsup_{n\to\infty} \mu_n(A) \le \varlimsup_{n\to\infty} \mu_n(\overline{A}) \overset{(ii)}{\le} \mu(\overline{A}) = \mu(A).$$

To see (iv) $\Rightarrow$ (i), fix $f : X \to [0,1]$ continuous, then

$$\nu(f) = \int_0^1 \nu([f > t])dt \ \forall \ \nu \in \mathcal{P}(X).$$

By continuity, for $t \in (0,1)$ $\partial[f > t] \subseteq [f = t]$, thus $\varGamma := \{t \in [0,1] : \mu(\partial[f > t]) > 0\}$ is at most countable. Moreover, by (iv)

$$\mu_n([f > t]) \xrightarrow[n\to\infty]{} \mu([f > t]) \ \forall \ t \in [0,1] \setminus \varGamma.$$

Since the Lebesgue measure of $\varGamma$ is zero, by bounded convergence,

$$\mu_n(f) = \int_0^1 \mu_n([f > t])dt \xrightarrow[n\to\infty]{} \int_0^1 \mu([f > t])dt = \mu(f). \quad ☑$$

EXERCISE N$^{\underline{\text{O}}}$8

0. Riesz representation on a locally compact space.

Let $(X,\mathcal{T})$ be a locally compact, Hausdorff space and let

$$C_C(X) := \{f \in C_B(X) : \overline{[f \neq 0]} \ \text{ compact}\}.$$

Suppose that $L : C_C(X) \to \mathbb{R}$ is linear and `positive` in the sense that

$$f \in C_C(X),\ f \geq 0 \implies L(f) \geq 0,$$

then there is a Borel measure $\mu : \mathcal{B}(X) \to [0,\infty]$ so that

(i) $$\mu(C) < \infty \ \forall \ C \subset X \ \text{ compact},$$

(ii) $$L(f) = \int_X f d\mu \ \ \forall \ f \in C_C(X).$$

1. Cartan's proof of uniqueness of Haar measure. Suppose that μ, ν are both left Haar measures on the locally compact, topological group G, and let $m = \mu \times \nu$.

¶1 Show that $m \circ S \equiv m \equiv m \circ T$ where $S, T : G \times G \to G \times G$ are defined by $S(x,y) = (x,xy)$, $\ T(x,y) = (yx,y)$.

¶2 Let $g : G \to \mathbb{R}$ be non-negative, and measurable. Show that

$$\mu(E)\int_G g d\nu = \int_G g(x^{-1})\nu(Ex^{-1})d\mu(x) \ \forall \ E \in \mathcal{B}(G).$$

`Hint:` Show first that $m \circ R \equiv m$ where $R(x,y) := S^{-1} \circ T(x,y) = (yx, x^{-1})$.

¶3 Let $E \subset G$ be compact with $E^o \neq \varnothing$.

Show that for $f : G \to \mathbb{R}$ non-negative, and measurable:

$$\mu(E)\int_G \frac{f(y^{-1})}{\nu(Ey)} d\nu(y) = \int_G f d\mu.$$

`Hint:` Set $g(y) = \frac{f(y^{-1})}{\nu(Ey)}$ and use ¶2.

¶4 Show (using ¶3 or otherwise) that $\int_G f d\nu = c \int_G f d\mu$ where $c = \frac{\nu(E)}{\mu(E)}$.

2. The modular function and right Haar measure.

Let m be a left Haar measure on the locally compact, topological group G.

a) Show that $m_R(A) := m(A^{-1})$ defines a right Haar measure on $(G, \mathcal{B}(G))$, where $A^{-1} := \{x^{-1} : \ x \in A\}$.

b) Show that $\exists$ a continuous, multiplicative homomorphism $\Delta : G \to \mathbb{R}_+$ so that $m(Ax) = \Delta(x)m(A) \ \forall \ x \in G, \ A \in \mathcal{B}(X)$.

`Hint:` Uniqueness and regularity of Haar measure.

c) Show that $\Delta(x) = 1$ for $x \in Z(G) := \{x \in G : \ xz = zx \ \forall \ z \in G\}$ or $x \in [G,G]$, the group generated by $\{ghg^{-1}h^{-1} : \ g,h \in G\}$.

d) Show that

$$\int_G m(x^{-1}A\cap B)dm(x) = m(A)\int_B \frac{dm}{\Delta} = m(A)m(B^{-1}) \ \forall\ A,B\in\mathcal{B}(G).$$

e) Let m_R be as in a). Show that $m_R \ll m$, that $\frac{dm_R}{dm} = \frac{1}{\Delta}$ and that $m_R(xA) = \frac{m_R(A)}{\Delta(x)}\ \forall\ x\in G,\ A\in\mathcal{B}(X)$.

f) Show that G is compact iff $m(G) < \infty$ and that in this case m is also a right Haar measure.
`Hint`: If G is not compact, then $\exists\ K\in\mathcal{C}_e$ and $x_n\in G$ so that $\{x_nK : n\in\mathbb{N}\}$ are pairwise disjoint.

3. Ulam's converse to Haar's theorem. Let m be a σ-finite, left-invariant measure on the Polish topological group G. Using the hints below (or otherwise) prove the `Ulam-Weil theorem`: that G is locally compact.
`Hints`:
(i) For $K\subset G$ compact with $0 < m(K) < \infty$, let $C_1 := KK^{-1},\ C+n+1 := C_nC_n$, then C_n is compact and $H := \bigcup_{n=1}^\infty C_n$ is a subgroup of G.
(ii) If $\Gamma\subset G$ satisfies $G = \biguplus_{g\in\Gamma} gH$, then Γ is at most countable;
(iii) $\exists\ n\ge 1$ such that $C_n^o \ne \varnothing$.

4. Finite dimensional distributions but no stochastic process.
Let $(X,\mathcal{B},m)$ be the unit interval equipped with Borel sets and Lebesgue measure and let $\overline{\mu}$ denote Lebesgue outer measure on $[0,1]$.
(a) Show that $\exists\ E_n\subset[0,1]\ \ (n\ge 1)$ such that
- $E_n\supset E_{n+1},\ \ \overline{\mu}(E_n) = 1\ \forall\ n\ge 1$; and
- $\bigcap_{n=1}^\infty E_n = \varnothing$.

(b) For $n\ge 1$, show that $\exists\ p_n\in\mathcal{P}(E_n,\mathcal{B}_n)$ with $p_n(A) = \overline{\mu}(A)$ where $\mathcal{B}_n := \mathcal{B}\cap E_n := \{A\cap E_n :\ A\in\mathcal{B}\}$.

For $n\ge 1$ define $\overline{p}_n : \prod_{k=1}^n \mathcal{B}_k \to [0,1]$ by $\overline{p}_n(\prod_{k=1}^n A_k) := p_n(\bigcap_{k=1}^n A_k)$. Show that
(d) $\exists\ P_n\in\mathcal{P}(\prod_{k=1}^n E_k, \bigotimes_{k=1}^n \mathcal{B}_k := \sigma(\prod_{k=1}^n \mathcal{B}_k))$ such that $P_n(\prod_{k=1}^n A_k) = \overline{p}_n(\prod_{k=1}^n A_k)\ \forall\ A_1\in\mathcal{B}_1,\dots,A_n\in\mathcal{B}_n$;
(e) $P_{n+1}\circ\pi_n^{-1} = P_n$ where $\pi_n : \prod_{k=1}^{n+1} E_k \to \prod_{k=1}^n E_k$ is defined by $\pi_n(x_1,\dots,x_{n+1}) := (x_1,\dots,x_n)$;
(f) $P_n(D_n) = 1$ where

$$D_n := \{x\in\prod_{k=1}^n E_k :\ x_1\in E_n\ \&\ x_1 = x_2 = \cdots = x_n\in E_n\}.$$

(g) $\nexists\ P\in\mathcal{P}(\prod_{k=1}^\infty E_k, \bigotimes_{k=1}^\infty \mathcal{B}_k)$ with $P\circ\phi_n^{-1} = P_n$ where

$\phi_n : \prod_{k=1}^\infty E_k \to \prod_{k=1}^n E_k$ is defined by $\phi_n(x_1,\dots) := (x_1,\dots,x_n)$.
`Hint for (g):` Consider $\phi_n^{-1}D_n$.

5. Riemann integrability & weak convergence. Let (X,d) be a Polish space and let $\mu,\ \mu_1,\mu_2,\dots \in \mathcal{P}(X),\ \mu_n \underset{n\to\infty}{\Longrightarrow} \mu$.

Let $f : X \to \mathbb{R}$ be bounded, measurable and μ-*Riemann integrable* in the sense that $\mu(C_f) = 1$ where $C_f := \{x \in X : \text{f continuous at } x\}$.

Show that

$$\mu_n(f) \xrightarrow[n\to\infty]{} \mu(f).$$

`Hints` Show that (i) WLOG, $f : X \to [0,1]$; (ii) For $t \in [0,1]$, $\mu(\partial[f > t]) = \mu([f = t])$; .....

6. Riemann integrability on a metric space.

Let X be a metric space.

`A. Semicontinuity` revision from topology.

The function $f : X \to \mathbb{R}$ is said to be

- *lower semicontinuous* (`lsc`) at $x \in X$ if $\varliminf_{y\to x} f(y) \ge f(x)$ and *upper semicontinuous* (`usc`) at $x \in X$ if $-f$ is `lsc` at x.
- Call f `lsc` *on* $A \subseteq X$ if it is `lsc` at every $x \in A$.

Show that

(a) The function $f : X \to \mathbb{R}$ is continuous at x iff it is both `lsc` and `usc` at x.

(b) The function $f : X \to \mathbb{R}$ is `lsc` at $x \in X$ if $x \in \{z \in X : \ f(z) > a\}^o\ \forall\ a < f(x)$, whence `lsc` on X if $\{z \in X : \ f(z) > a\}$ is open $\forall\ \ a \in \mathbb{R}$;

(c) If A is open, then 1_A is `lsc` on X.

(d) The supremum of functions which are `lsc` at $x \in X$ is also `lsc` at x.

Suppose that $f : X \to \mathbb{R}$ is `lsc` on X and $f \ge 0$. Define $U_q\ \ (q \in \mathbb{Q})$ by $U_q := \{x \in X : \ 0 < q < f(x)\}$. Show that

(e) U_q is open and $\exists$ closed sets $F_{q,n}$ such that $U_q = \bigcup_{n\ge1} F_{q,n}$;

(f) $\exists\ f_{q,n} : X \to [0,q]$ continuous such that $f_{q,n}|_{F_{q,n}\uplus U_q^c} \equiv q1_{F_{q,n}}$.

(g) $\sup_{q,n} f_{q,n}(x) = f(x)$.

(h) $\exists\ f_n : X \to \mathbb{R}$ continuous such that $f_n(x) \uparrow f(x)\ \ \forall\ x \in X$.

(i) For $f : X \to \mathbb{R}$ bounded below, show that $\underline{f} : X \to \mathbb{R}$ is `lsc` on X where $\underline{f}(x) := \varliminf_{y\to x} f(y)$.

The function $\underline{f}$ is known as the `lsc` *envelope* of f. The `usc` *envelope* of the bounded function f is the function $x \mapsto \overline{f}(x) := -\underline{(-f)}(x)$.

(j) Show that if $f : X \to \mathbb{R}$ is bounded, then $\exists$ continuous functions $f_{n,\pm}$ $(n \ge 1)$ such that

$$f_{n,-}(x) \uparrow \underline{f}(x) \ \& \ \ f_{n,+}(x) \downarrow \overline{f}(x) \ \forall \ x \in X$$

`B. Riemann integrability.` Let $p \in \mathcal{P}(X)$. Call a bounded function $f : X \to \mathbb{R}$ *p-Riemann integrable* if $p(X \setminus C_f) = 0$ where $C_f := \{x \in X : x_n \to x \Rightarrow f(x_n) \to f(x)\}$ (the continuity points of f).

Show that $f : X \to \mathbb{R}$ is p- Riemann integrable iff $\forall \ \varepsilon > 0, \ \exists \ f_+, \ f_- \in C(X)$ with $f_- \le f \le f_+$ and $\int_X f_+ dp - \int_X f_- dp < \varepsilon$.

`Hint`: $p(X \setminus C_f) = 0$ iff $\underline{f} = \overline{f}$ p-a.e..

Week # 9
TIGHTNESS

Let X be a Polish space. A collection $\mathcal{K} \subset \mathcal{P}(X)$ is called *tight* if $\forall \ \varepsilon > 0 \ \exists \ C \subset X$ compact such that

$$\mu(C) > 1 - \varepsilon \ \forall \ \mu \in \mathcal{K}.$$

• By exercise 5.1, a singleton (whence any finite collection) in $\mathcal{P}(X)$ is tight.

A collection $\Pi \subset \mathcal{P}(X)$ is called *weakly precompact* (`WPC`) if $\forall \ \mu_n \in \Pi \ \exists \ n_k \to \infty, \ Q \in \mathcal{P}(X)$ so that $\mu_{n_k} \implies Q$.

12.7 Prohorov's tightness theorem

Let X be a Polish space and let $\Pi \subset \mathcal{P}(X)$, then Π is weakly precompact if and only if Π is tight.

Illustration:

`Prohorov's tightness theorem on locally compact metric spaces`

The metric space (X, d) is called *locally compact* if $\forall \ x \in X \ \exists \ \varepsilon_x > 0$ so that $B(x, \varepsilon_x)$ is compact.

`e.g.`: any compact metric space, any set equipped with the discrete metric, $\mathbb{R}^d$, $\cdots$

• If the metric space (X, d) is separable and locally compact, then (by the Lindelöf property) there is a countable cover of compact balls and

X is *σ-compact* in the sense that $\exists\ K_n$ compact so that $K_n \subset K_{n+1}^o \uparrow X$. Set $N(x) := \min\{n \ge 1 : \ x \in K_n\}$. Note that X is compact iff N is bounded.

• In case X is not compact, the *one point compactification* of (X, d) is the compact metric space $(\widehat{X} = X \cup \{\infty\}, \rho)$ with

$$\rho(x, \infty) := \frac{1}{N(x)} \ \ \& \ \ \rho(x, y) := \min\{d(x, y), \frac{1}{N(x)} + \frac{1}{N(y)}\}.$$

Let X be a locally compact, separable metric space which is not compact (e.g. $X = \mathbb{R}^d$).

If $\Pi \subset \mathcal{P}(X)$, then Π is weakly precompact in $\mathcal{P}(\widehat{X})$ by Helly's theorem.

It is not hard to show that Π is weakly precompact in $\mathcal{P}(X)$ iff $\mu(\{\infty\}) = 0$ for every limit point μ of Π in $\mathcal{P}(\widehat{X})$, and that this latter property holds iff Π is tight in $\mathcal{P}(X)$.

Proof of Prohorov's theorem

Proof of `WPC` $\Longrightarrow$ tightness

For each $k \ge 1$, $\exists$ countable set of balls $\Gamma_k = \{B_{n,k} : \ n \ge 1\}$ so that

$$\bigcup_{n\ge1} B_{n,k} = X, \ \mathtt{diam}\, B_{n,k} < \frac{1}{k} \ \forall\ n, \ k \ge 1.$$

Let $G_{N,k} := \bigcup_{n=1}^N B_{n,k}$. Evidently $Q(G_{N,k}) \underset{N\to\infty}{\longrightarrow} 1 \ \forall\ Q \in \mathcal{P}(X)$. We claim first that

¶ For $k \ge 1$ fixed, $\mu(G_{N,k}) \underset{N\to\infty}{\longrightarrow} 1$ uniformly in $\mu \in \Pi$.

`Proof` Fix k and set $G_N := G_{N,k}$. If ¶1 fails, then $\exists\ \varepsilon > 0$ and $\mu_n \in \Pi$ such that $\mu_n(G_n) \le 1 - \varepsilon \ \forall\ n \ge 1$. By weak precompactness, $\exists\ n_k \to \infty$ and $\mu \in \mathcal{P}(X)$ such that $\mu_{n_k} \Rightarrow \mu$. It follows that $\forall\ N \ge 1$:

$$\varliminf_{k\to\infty} \mu_{n_k}(G_N) \le \varliminf_{k\to\infty} \mu_{n_k}(G_{n_k}) \le 1 - \varepsilon$$

whence

$$1 - \varepsilon \ge \varliminf_{k\to\infty} \mu_{n_k}(G_N) \ge \mu(G_N) \underset{N\to\infty}{\longrightarrow} 1. \ \boxtimes¶$$

By ¶,

• $\forall\ \varepsilon > 0, \ k \ge 1, \ \exists\ N_{k,\varepsilon}$ such that $\mu(G_{N_{k,\varepsilon},k}) > 1 - \frac{\varepsilon}{2^k} \ \forall\ \mu \in \Pi$.

It follows that (!)

$$K_\varepsilon := \bigcap_{k=1}^\infty \overline{G_{N_{k,\varepsilon},k}}$$

is compact and $\mu(K_\varepsilon) > 1 - \varepsilon \ \forall\ \mu \in \Pi$. ☑

Proof of tightness $\implies$ WPC

Let $\Pi \subset \mathcal{P}(X)$ be tight.

Choose $K_n \subset X$ compact, such that $K_n \subset K_{n+1}$ and $\mu(K_n) > 1 - \frac{1}{n}$ $\forall$ $\mu \in \Pi,\ n \ge 1$.

Let $\mathcal{A}$ be a countable base for the topology on X and let

$$\mathcal{H} := \{K_N \cap \bigcup_{A\in F} \overline{A} : \ N \in \mathbb{N} \ \&\ F \subset \mathcal{A} \texttt{ finite}\} \subset \{\texttt{compact sets}\}.$$

Now let $\{\mu_n\}_{n\ge1} \subset \Pi$. By diagonalization, $\exists\ n_k \to \infty,\ \alpha : \mathcal{H} \to [0,1]$ such that

$$\mu_{n_k}(H) \underset{k\to\infty}{\longrightarrow} \alpha(H) \ \ \forall\ H \in \mathcal{H}.$$

¶1 It suffices to show $\exists\ P \in \mathcal{P}(X)$ such that

(♀) $$P(G) = \sup\{\alpha(H) : \ G \supseteq H \in \mathcal{H}\} \ \ \forall\ G \subset X \ \ \texttt{open}.$$

`Proof` In this case, for G open, $G \supseteq H \in \mathcal{H}$,

$$\mu_{n_k}(G) \ge \mu_{n_k}(H) \underset{k\to\infty}{\longrightarrow} \alpha(H)$$

whence $\underline{\lim}_{k\to\infty} \mu_{n_k}(G) \ge P(G)$ and $\mu_{n_k} \implies P$. ☑¶1

The rest of the proof is to show $\exists\ P \in \mathcal{P}(X)$ satisfying (♀). Evidently

(a) $\alpha(H) \le \alpha(H')$ for $H,\ H' \in \mathcal{H},\ H \subset H'$;

(b) $\alpha(H \cup H') \le \alpha(H) + \alpha(H')$ for $H,\ H' \in \mathcal{H}$ with equality when $H \cap H' = \varnothing$.

Define $\beta : \{\texttt{open sets}\} \to [0,1]$ by

$$\beta(G) := \sup\{\alpha(H) : \ G \supseteq H \in \mathcal{H}\}.$$

¶2 If F is closed and $F \subset G$ open, $F \subset H \in \mathcal{H}$, then $\exists\ H_0 \in \mathcal{H}$ such that $F \subset H_0 \subset G$.

`Proof`

Since $F \subset H \in \mathcal{H}$, F is compact and $\exists\ u \ge 1,\ H \subset K_u$.

$\forall\ x \in F\ \exists\ A_x \in \mathcal{A}$ such that $x \in A_x \subset \overline{A}_x \subset G$.

By compactness $\exists\ \{x_k\}_{k=1}^N \subset F$ such that $F \subset \bigcup_{k=1}^N A_{x_k}$.

Set $H_0 := \bigcap_{k=1}^N \overline{A}_{x_k} \cap K_u$. ☑¶2

¶3 $\beta(\bigcup_{k=1}^n G_k) \le \sum_{k=1}^n \beta(G_k)\ \forall\ G_1, \dots, G_n$ open.

`Proof for` $n = 2$: Let $G_1,\ G_2$ be open and suppose $G_1 \cup G_2 \supset H \in \mathcal{H}$. Set

$$F_1 := \{x \in H : \ d(x, G_1^c) \ge d(x, G_2^c)\}, \ \ F_2 := \{x \in H : \ d(x, G_2^c) \ge d(x, G_1^c)\}.$$

We claim that $F_1 \subset G_1$:
Else $\exists\ x \in F_1 \setminus G_1 \ \Rightarrow \ d(x, G_1^c) = 0 \ \& \ \ x \in G_2$.
But then $d(x, G_2^c) > 0$ contradicting $x \in F_1$.
Similarly $F_2 \subset G_2$.
By ¶2 $\exists\ H_i \in \mathcal{H} \ \ (i = 1, 2)$ such that $F_i \subset H_i \subset G_i \ \ (i = 1, 2)$. Thus

$$\alpha(H) \le \alpha(H_1 \cup H_2) \le \alpha(H_1) + \alpha(H_2) \le \beta(G_1) + \beta(G_2). \ \text{☑¶3}$$

¶4 $\beta(\bigcup_{k=1}^{\infty} G_k) \le \sum_{k=1}^{\infty} \beta(G_k) \ \forall\ G_1, \dots$ open.

`Proof`
Suppose $\bigcup_{k=1}^{\infty} G_k \supset H \in \mathcal{H}$, then by compactness of H, $\exists\ n$ such that $H \subset \bigcup_{k=1}^{n} G_k$, whence using ¶3,

$$\alpha(H) \le \beta(\bigcup_{k=1}^{n} G_k) \le \sum_{k=1}^{n} \beta(G_k) \le \sum_{k=1}^{\infty} \beta(G_k). \ \text{☑¶4}$$

Now define $\overline{\mu} : 2^X \to [0, 1]$ by

$$\overline{\mu}(E) := \inf\{\beta(G) : \ E \subset G \ \texttt{open}\}.$$

Evidently $\overline{\mu}(G) = \beta(G)$ for G open.

¶5 $\overline{\mu}$ is an outer measure.

`Proof of sub-`σ`-additivity`:
Let $E_n \subset X$ and let $\varepsilon > 0$. Fix $E_n \subset G_n$ open such that $\overline{\mu}(E_n) > \beta(G_n) - \frac{\varepsilon}{2^n}$. It follows that $\bigcup_{n\ge1} E_n \subset \bigcup_{n\ge1} G_n$ and

$$\overline{\mu}(\bigcup_{n\ge1} E_n \le \beta(\bigcup_{n\ge1} G_n) \overset{\text{¶4}}{\le} \sum_{n\ge1} \beta(G_n) \le \sum_{n\ge1} \overline{\mu}(E_n) + \varepsilon. \ \text{☑¶5}$$

We complete the proof of (♀) by showing that $\mathcal{M}_{\overline{\mu}} \supset \mathcal{B}(X)$.

¶6 For G open and F closed,

$$\beta(G) \ge \overline{\mu}(G \cap F) + \overline{\mu}(G \cap F^c).$$

`Proof` Fix $\varepsilon > 0$. $\exists\ H_1 \in \mathcal{H}$ such that $H_1 \subset G \cap F^c \ \& \ \alpha(H_1) > \beta(G \cap F^c) - \varepsilon$. $\exists\ H_2 \in \mathcal{H}$ such that $H_2 \subset G \cap H_1^c \ \& \ \alpha(H_2) > \beta(G \cap H_1^c) - \varepsilon$.

Evidently $H_1 \cap H_2 = \varnothing$ so

$$\begin{aligned}\beta(G) &\geq \alpha(H_1 \cup H_2)\\ &= \alpha(H_1) + \alpha(H_2)\\ &> \beta(G \cap F^c) + \beta(G \cap H_1^c) - 2\varepsilon\\ &\geq \overline{\mu}(G \cap F) + \overline{\mu}(G \cap F^c) - 2\varepsilon. \ \boxtimes ¶6\end{aligned}$$

¶7 $\{$`closed sets`$\} \subset \mathcal{M}_{\overline{\mu}}$.

`Proof` Fix F closed and $L \in 2^X$. For $L \subset G$ open,

$$\beta(G) \geq \overline{\mu}(L \cap F) + \overline{\mu}(L \cap F^c),$$

whence

$$\overline{\mu}(L) \geq \overline{\mu}(L \cap F) + \overline{\mu}(L \cap F^c). \ \boxtimes$$

Thus, $\overline{\mu}|_{\mathcal{B}(X)}$ is a measure with $\overline{\mu}(G) = \beta(G)\ \forall\ G$ open. This is (♀). □

Corollary: `(Central limit theorem)`

Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space and that $X_1, X_2, \cdots : \Omega \to \mathbb{R}$ are independent, identically distributed, $\mathbb{R}$-valued, random variables random variables with $E(X_i) = 0$ and $E(X_i^2) = 1$. Let $S_n := \sum_{k=1}^n X_k$ and define $\mu_n \in \mathcal{P}(\mathbb{R})$ by $\mu_n(A) := P([\frac{S_n}{\sqrt{n}} \in A])$, then

$$\text{(CLT)} \qquad \mu_n \underset{n\to\infty}{\Longrightarrow} \mathcal{N}, \quad \mathcal{N}(A) = \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2}} dt.$$

Proof sketch We use the exercises 9.2 & 9.3 (below).

By ex. 9.2(v), $\varphi_{\frac{S_n}{\sqrt{n}}} \underset{n\to\infty}{\longrightarrow} \varphi_{\mathfrak{g}}(t) = \widehat{\mathcal{N}}(t)\ \ \forall\ t \in \mathbb{R}$ where $\mathfrak{g}$ is the `standard Gaussian` random variable on $\mathbb{R}$ defined by `Prob`$([\mathfrak{g} \in A]) = \mathcal{N}(A)$.

If $n_k \to \infty$ and $\mu_{n_k} \underset{k\to\infty}{\Longrightarrow} \nu \in \mathcal{P}(\mathbb{R})$ then $\widehat{\nu} \equiv \widehat{\mathcal{N}}$ whence by ex. 9.3(v), $\nu = \mathcal{N}$.

To complete the proof of (CLT) we show that $\{\mu_n : n \geq 1\}$ is a tight family in $\mathcal{P}(\mathbb{R})$. To see this, for $\varepsilon > 0$ let $K = K_\varepsilon := [-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}]$, then for

$n \ge 1$

$$\begin{aligned}\mu_n(K^c) &= P([|\textstyle\sum_{k=1}^n X_k| \ge \sqrt{\frac{n}{\varepsilon}}|])\\ &\le \frac{\varepsilon}{n}\mathbb{E}((\sum_{k=1}^n X_k)^2)\\ &= \frac{\varepsilon}{n}\sum_{1\le k\le \ell\le n}\mathbb{E}(X_kX_\ell)\\ &= \frac{\varepsilon}{n}\sum_{1\le k\le n}\mathbb{E}(X_k^2) \quad \because\ \mathbb{E}(X_kX_\ell) = 0\ \forall\ k \ne \ell;\\ &= \varepsilon.\end{aligned}$$

By Prohorov's theorem, $\{\mu_n : n \ge 1\}$ is a tight family in $\mathcal{P}(\mathbb{R})$ and by the above $\mu_n \underset{n\to\infty}{\Longrightarrow} \mathcal{N}$. ☑

§14 Geometric measure theory

Covering and differentiation theorems.

Recall that

$$L^1_{\texttt{loc}}(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} \ \ \texttt{measurable} \ , f|_A \in L^1(\mathbb{R}^d) \ \forall \ \texttt{bounded} A \in \mathcal{B}(\mathbb{R}^d)\}.$$

Differentiation theorem *Fix a norm $\|\cdot\|$ on $\mathbb{R}^d$. Suppose that $f \in L^1_{\texttt{loc}}(\mathbb{R}^d)$, then for a.e. $x \in \mathbb{R}$,*

$$\limsup_{\texttt{diam}(B) \to 0, \ x \in B, \ B \ \texttt{a ball}} \frac{1}{m(B)} \int_R |f - f(x)| dm = 0$$

where m is Lebesgue measure on $\mathbb{R}^d$.

To prove this, we need the *Hardy Littlewood maximal inequality*:

Fix a bounded, open set $U \subset \mathbb{R}^d$. For $f : U \to \mathbb{R}$ integrable, define the *Hardy-Littlewood maximal function* on U by

$$Mf(x) = M_U f(x) := \sup_{x \in B \ \texttt{a ball}, \ B \subset U} \frac{1}{m(B)} \int_B |f| dm.$$

The Hardy-Littlewood maximal function is measurable since the set $U \cap [Mf > \lambda]$ is open $\forall \ \lambda > 0$ (!).

Note also (!) that the definition gives the same function if we restrict to the family of rational balls (rational centers and rational radii).

Hardy-Littlewood maximal inequality

$$m(U \cap [M_U f > \lambda]) \le \frac{3^d}{\lambda} \|f\|_{L^1(U)}.$$

Proof of the differentiation theorem given the maximal inequality The theorem is evident for $f : \mathbb{R}^d \to \mathbb{R}$ continuous. Suppose that $f \in L^1_{\texttt{loc}}(\mathbb{R}^d)$ and fix a bounded, open set $U \subset \mathbb{R}^d$. We'll prove the theorem for m-a.e. $x \in U$.

Given $\varepsilon > 0$ choose $g : U \to \mathbb{R}$ continuous such that $\int_U |f - g| dm < \varepsilon^2$.

We have that for $x \in B \subset U$, B a ball:

$$\frac{1}{m(B)} \int_B |f - f(x)| dm \le \frac{1}{m(B)} \int_B |f - g| dm + \frac{1}{m(B)} \int_B |g - g(x)| dm + |f(x) - g(x)|$$

and $\frac{1}{m(B)}\int_B |f-g|dm \le M_U(|f-g|)(x)$, whence

$$\begin{aligned}L(x) &:= \limsup_{\mathtt{diam}(B)\to 0,\ x\in B,\ B \text{ a ball}} \frac{1}{m(B)}\int_B |f-f(x)|dm \\ &= \limsup_{\mathtt{diam}(B)\to 0,\ x\in B\subset U,\ B \text{ a ball}} \frac{1}{m(B)}\int_B |f-f(x)|dm \\ &\le M_U(|f-g|)(x) + |f(x)-g(x)|.\end{aligned}$$

Thus

$$\begin{aligned}m(U\cap[L>2\varepsilon]) &\le m(U\cap[M(|f-g|)>\varepsilon]) + m(U\cap[|f-g|>\varepsilon]) \\ &\le \frac{3^d+1}{\varepsilon}\int_U |f-g|dm < (3^d+1)\varepsilon\end{aligned}$$

and $L = 0$ a.e. on U. ☐

To prove the maximal inequality, we need

Vitali's covering lemma *Let X be a metric space, and let $\mathcal{C}$ be a finite collection of balls (with positive radius) in X. There are disjoint balls $B_1,\dots,B_n \in \mathcal{C}$ such that*

$$\bigcup_{B\in\mathcal{C}} B \subset \bigcup_{k=1}^n \tilde{B}_k$$

where

$$\tilde{B}(x,r) := B(x,3r).$$

`Proof`

Choose $B_1 \in \mathcal{C}$ with maximum diameter. Let $d_2 = \max\{\text{diam}(B) : B \in \mathcal{C},\ B\cap B_1 = \varnothing\} \ge 0$ where $\max\varnothing := 0$. Note that if $d_2 = 0$ then $\forall\ B \in \mathcal{C}$ we have $\text{diam}(B) \le \mathtt{diam}(B_1)$ and either $B\cap B_1 \ne \varnothing$, whence $B \subset \tilde{B}_1$ or $B \cap B_1 = \varnothing$ whence $\text{diam}(B) \le \mathtt{diam}(B_2)$, $B\cap B_2 \ne \varnothing$ and $B \subset \tilde{B}_2$. Continuing, we obtain $n \ge 1$, disjoint balls $B_1,\dots,B_n \in \mathcal{C}$, such that

$$d_k = \max\{\text{diam}(B) : B\in\mathcal{C},\ B\cap B_i = \varnothing,\ i = 1,\dots,k-1\} > 0$$

for $2 \le k \le n$ with $\mathtt{diam}(B_k) = d_k$ and

$$d_{n+1} = \max\{\text{diam}(B) : B\in\mathcal{C},\ B\cap B_i = \varnothing,\ i = 1,\dots,n\} = 0.$$

To show that $\bigcup_{B\in\mathcal{C}} B \subset \bigcup_{k=1}^n \tilde{B}_k$, suppose that $B \in \mathcal{C}$. Since $d_{n+1} = 0$, $\exists\ 1 \le j \le n$ so that $B \cap B_j \ne \varnothing$. Let j be minimal, i.e. $B\cap B_i = \varnothing\ \forall\ i = 1,\dots,j-1$ and $B\cap B_j \ne \varnothing$. Thus $\mathtt{diam}(B) \le d_j$ and $B \subset \tilde{B}_j$. ☑

Proof of the Hardy-Littlewood maximal inequality

Let $K \subset U \cap [M_U f > \lambda]$ be compact. For each $x \in K$, $\exists$ an open ball $B_x \subset U$, $x \in B_x$ with $\int_{B_x} |f| dm > \lambda m(B_x)$. By compactness, $\exists\ F \subset K$ finite so that $K \subset \bigcup_{x \in F} B_x$. By Vitali's covering lemma with $\mathcal{C} = \{B_x : x \in F\}$, there are disjoint open balls $B_1, \dots, B_N \in \mathcal{C}$ such that

$$K \subset \bigcup_{n=1}^{N} \tilde{B}_n, \ \text{ and } \int_{B_n} |f| dm > \lambda m(B_n) \ \forall\ 1 \le n \le N.$$

Here $\tilde{B}_n := B(x_n, 3r_n)$ where $B_n := B(x_n, r_n)$.

By exercise 5.5(iii) (or otherwise), $m(\tilde{B}_n) = 3^d m(B_n)$, whence

$$m(K) \le \sum_{n=1}^{N} m(\tilde{B}_n)| = 3^d \sum_{n=1}^{N} m(B_n) \le \frac{3^d}{\lambda} \sum_{n \ge 1} \int_{B_n} |f| dm \le \frac{3^d}{\lambda} \int_U |f| dm.$$

□

Vitali's Covering theorem

Fix a norm $\|\cdot\|$ on $\mathbb{R}^d$. Let $U \subset \mathbb{R}^d$ be open and bounded. Let $A \subset U$, and let $\mathcal{B}$ be a collection of balls, each contained in U and with positive radius, such that

$$(1) \qquad \forall\ x \in A,\ \varepsilon > 0,\ \exists\ x \in B(u, r) \in \mathcal{B},\ r < \varepsilon,$$

then there are disjoint $B_n \in \mathcal{B}$ such that

$$\overline{\mu}(A \setminus \bigcup_{n=1}^{\infty} B_n) = 0.$$

Proof

For $B = B(u, r)$ we write $u(B) := u$ and $r(B) := r$.

Let $r_1 = \sup\{r(B) : B \in \mathcal{B}\}$, choose $B_1 \in \mathcal{B}$ with $r(B_1) \ge \frac{r_1}{2}$, and define $r_2 = \sup\{r(B) : B \in \mathcal{B},\ B \cap B_1 = \varnothing\}$.

In case $r_2 > 0$, choose $B_2 \in \mathcal{B}$, $B_2 \cap B_1 = \varnothing$ with $r(B_2) \ge \frac{r_2}{2}$, and continue to get:

- $\omega \in \mathbb{N} \cup \{\infty\}$;
- a sequence $r_n > 0$, $(1 \le n < \omega)$ and disjoint balls $B_n \in \mathcal{B}$, $(1 \le n < \omega)$ such that for $1 \le n < \omega$,

$$r_n = \sup\{r(B) :\ B \in \mathcal{B},\ B \cap B_k = \varnothing, 1 \le k \le n-1\}, \text{ and } r(B_n) \ge \frac{r_n}{2}.$$

- for $\omega < \infty$, $r_\omega = 0$.

In case $\omega < \infty$, we claim that $A \subset \bigcup_{k=1}^{\omega-1} \overline{B_k}$.

To see this, suppose otherwise, and let

$$x \in A \setminus \bigcup_{k=1}^{\omega-1} \overline{B_k} \subset U \setminus \bigcup_{k=1}^{\omega-1} \overline{B_k}$$

which latter is an open set, whence $\exists\ \delta > 0$ such that $B(x,\delta) \subset U \setminus \bigcup_{k=1}^{\omega-1} \overline{B_k}$. By assumption $x \in B \in \mathcal{B}$ with $r(B) < \frac{\delta}{3}$, whence $r_\omega \geq r(B) > 0$.⊠

Now suppose that $\omega = \infty$. Since U is bounded, $\sum_{n\geq 1} r_n^d < \infty$ and $r_n \to 0$.

We claim that

$$A \setminus \bigcup_{k=1}^{N} B_k \subset \bigcup_{k=N+1}^{\infty} \widehat{B}_k \ \forall\ N \geq 1$$

where $u(\widehat{B}) = u(B)$ and $r(\widehat{B}) = 5r(B)$. To see this, let $x \in A \setminus \bigcup_{k=1}^{N} B_k$, then $\exists\ x \in B \in \mathcal{B}$ so that $B \subset U \setminus \bigcup_{k=1}^{N} B_k$, whence $0 < r(B) \leq r_{N+1}$. Since $r_n \to 0$, $\exists\ K \geq N+1$ so that $r_K \geq r(B) > r_{K+1}$, whence $\exists\ j \in [N+1, K]$ so that $B \cap B_j \neq \varnothing$. Since $r(B_j) \geq \frac{r_{j-1}}{2} \geq \frac{r(B)}{2}$, we have $B \subset \widehat{B}_j$. Thus

$$m(A \setminus \bigcup_{k=1}^{N} B_k) \leq \sum_{k=N+1}^{\infty} m(\widehat{B}_k) \xrightarrow[N\to\infty]{} 0.$$

□

Exercise No9

1. Well distributed sequences.

Let X be a metric space and let $p \in \mathcal{P}(X)$. A sequence $(w_1, w_2, \dots) \in X^{\mathbb{N}}$ is called *p-well-distributed* if $\frac{1}{n}\sum_{k=1}^{n} \delta_{w_k} \underset{n\to\infty}{\Longrightarrow} p$.

Let $X := \mathbb{T}^N$ (some $N \in \mathbb{N}$) where $\mathbb{T} := \mathbb{R}/\mathbb{Z} \cong [0,1)$ equipped with the metric $d(x,y) := \min\{|x-y|, 1-|x-y|\}$ and let m be Lebesgue measure on $\mathbb{T}^N$.

(i) Prove `Weyl's theorem` that

- $(w_1, w_2, \dots) \in X^{\mathbb{N}}$ is m-well-distributed iff

$$\frac{1}{n}\sum_{k=1}^{N} e^{2\pi i\langle \nu, w_k\rangle} \xrightarrow[n\to\infty]{} 0 \ \forall\ \nu \in \mathbb{Z}^N \setminus \{0\}$$

and, in this case $\overline{\{w_k :\ k \geq 1\}} = X$.

(ii) Define $w_k = (w_k^{(1)}, \dots, w_k^{(N)}) \in X := \mathbb{T}^N$ by $w_k^{(j)} := kx_j \mod 1$.

Show that $(w_1, w_2, \dots) \in X^{\mathbb{N}}$ is m-well-distributed iff $\{1, x_1, \dots, x_N\}$ are linearly independent over $\mathbb{Q}$.

2. Characteristic function of a random variable on $\mathbb{R}$.

The *characteristic function* of a random variable X on $\mathbb{R}$ is $\varphi = \varphi_X : \mathbb{R} \to \mathbb{C}$ is defined by

$$\varphi_X(t) := E(e^{itX}) = \widehat{\mu}(t)$$

where $\mu = \mu_X \in \mathcal{P}(\mathbb{R})$ is defined by $\mu(A) = \texttt{Prob}([X \in A])$ and

$$\widehat{\nu}(t) := \int_{\mathbb{R}} e^{itx} d\nu(x) \quad (t \in \mathbb{R},\ \nu \texttt{ a signed measure on } \mathbb{R}).$$

(i) Show that $\varphi_X \in C_B(\mathbb{R})$ and that $\mu_{X_n} \implies \mu_X$ implies that $\varphi_{X_n} \longrightarrow \nu_X$ uniformly on compact subsets of $\mathbb{R}$.

(ii) Show that if X is a random variable on $\mathbb{R}$ with $E(X^2) < \infty$, then

$$\varphi_X(t) = 1 + itE(X) - \frac{1}{2}E(X^2)t^2 + o(t^2) \quad \texttt{as } t \to 0.$$

(iii) Show that if $\mathfrak{g}$ is a $\texttt{standard Gaussian}$ random variable on $\mathbb{R}$ (ie $\texttt{Prob}([\mathfrak{g} \in A]) = \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2}} dt$), then $\varphi_{\mathfrak{g}}(t) = e^{-\frac{t^2}{2}}$.

(iv) Show that if $X_1, X_2, \dots, X_n$ are independent random variables on $\mathbb{R}$, that is:

$$\texttt{Prob}\,(\bigcap_{k=1}^n [X_k \in A_k]) = \prod_{k=1}^n \texttt{Prob}\,([X_k \in A_k])\ \forall\ A_1, A_2 \dots, A_n B \in \mathcal{B}(\mathbb{R});$$

then $\varphi_{S_n}(t) = \prod_{k=1}^n \varphi_{X_k}(t)$ where $S_n := \sum_{k=1}^n X_k$.

(v) Now suppose that $X_1, X_2, \dots$ are $\texttt{independent, identically distributed}$ random variables on $\mathbb{R}$ with $E(X_i) = 0,\ E(X_i^2) = 1$. Let $S_n := \sum_{k=1}^n X_k$. Show that

$$\varphi_{\frac{S_n}{\sqrt{n}}} \xrightarrow[n\to\infty]{} \varphi_{\mathfrak{g}}(t) \quad \forall\ t \in \mathbb{R}.$$

3. The inversion and uniqueness for characteristic functions.

In this exercise, you show that if X is a random variable on $\mathbb{R}$, and $(a,b) \subset \mathbb{R}$ satisfies $\mu_X(\{a,b\}) = 0$, then

$$(🚘) \qquad \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi_X(t) dt \xrightarrow[T\to\infty]{} \mu_X((a,b)).$$

(i) Show that $S(T) := \int_0^T \frac{\sin x}{x} dx \xrightarrow[T\to\infty]{} \frac{\pi}{2}$.

Write $I(T) := \int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it} \varphi_X(t) dt$. Show that:

$$\text{(ii)}\qquad I(T) = E\Big(\int_{-T}^{T} \frac{e^{it(X-a)} - e^{-it(X-b)}}{it}dt\Big)$$
$$= 2E\Big(\mathtt{sgn}\,(X-a)S(T|X-a|) - \mathtt{sgn}\,(X-b)S(T|X-b|)\Big)$$

(iii) Prove (🚗).

(iv) Show that if $\varphi_X \in L^1(m)$ ($m =$ `Lebesgue measure`), then $\mu_X \ll m$ and
$$\frac{d\mu_X}{dm}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi_X(t)e^{-itx}dt.$$

(v) Show that if $X,\ Y$ are random variables on $\mathbb{R}$ with $\varphi_X = \varphi_Y$, then $\mu_X = \mu_Y$.

4. Differentiation revision.

(A) Suppose that $F : I := [0,1] \to \mathbb{R}$ is non decreasing. Here you show that $\exists\ F' \in L^1([0,1]$ so that

☺ for m-a.e. $x \in [0,1]$, $\frac{1}{h}(F(x+h) - F(x)) \xrightarrow[h\to 0]{} F'(x)$

(where $m :=$ `Lebesgue measure`).

Let
$$\overline{D}F(x) := \varlimsup_{h\to 0} \tfrac{1}{h}(F(x+h) - F(x)),\quad \underline{D}F(x) := \varliminf_{h\to 0} \tfrac{1}{h}(F(x+h) - F(x)).$$

Show that

(0) `WLOG` $F : [0,1] \to [0,1]$ is a homeomorphism;

`Hint`: $F(x) \rightsquigarrow \frac{x+F(x)}{2}\dots$

(i) $\overline{D}F,\ \underline{D}F : [0,1] \to \mathbb{R}$ are measurable;

`Hint`: $\overline{D}F(x) \overset{?}{=} \varlimsup_{h\to 0,\ h\in\mathbb{Q}} \frac{1}{h}(F(x+h) - F(x))$.

$$\text{(ii)}\qquad \underline{D}F < r \quad \text{on} \quad A \in \mathcal{B}(I) \Rightarrow\ m(FA) \le rm(A).$$

$$\text{(iii)}\qquad \overline{D}F > s \quad \text{on} \quad A \in \mathcal{B}(I) \Rightarrow\ m(FA) \ge s\overline{\mu}(A).$$

`Hint`: Vitali's covering theorem.

(iv) Show that $\overline{D}F = \underline{D}F$ m-a.e. on $[0,1]$.

`Hint`: Let $A(r,s) := \{x \in I : \underline{D}F(x) < r < s < \overline{D}F(x)\}$ $\cdots$

(v) Show that $F(1) - F(0) \ge \int_0^1 DFdm$ `where` $DF := \overline{D}F = \underline{D}F$. When is there equality?

(B) Show that if $F : [0,1] \to \mathbb{R}$ has `bounded variation` ie $\exists\ M$ such that $\sum_{k=0}^{n-1} |F(t_{k+1}) - F(t_k)| \le M$ whenever $0 = t_0 < t_1 < \cdots < t_n = 1$, then F is differentiable at a.e. $x \in [0,1]$.

`Hint`: $F = G - H$ where $G,\ H : I := [0,1] \to \mathbb{R}$ are non decreasing.

(C) The function $f : [0,1] \to \mathbb{R}$ is said to be *absolutely continuous* if given $\varepsilon > 0,\ \exists\ \delta > 0$ such that for any collection $\{[a_j, b_j]\}_{j\ge1}$ of disjoint subintervals of $[0,1]$,

$$\sum_{j\ge1} |b_j - a_j| < \delta \implies \sum_{j\ge1} |f(b_j) - f(a_j)| < \varepsilon.$$

Show that if $f : [0,1] \to \mathbb{R}$ is absolutely continuous, then $\exists$ a signed measure $\mu_f : \mathcal{B}([0,1]) \to \mathbb{R},\ \mu_f \ll m$ such that $\mu_f([a,b]) = f(b) - f(a)$ for any interval $[a,b] \subset [0,1]$.

5. Fundamental theorem of calculus?

Let $(X, \mathcal{B}, m) = ([0,1], \mathtt{Borel}, \mathtt{Leb})$.

A measurable function $f : X \to \mathbb{R}$ is called *locally integrable at* $x \in (0,1)$ if $\exists\ \varepsilon > 0$ such that $\int_{(x-\varepsilon,x+\varepsilon)} |f| dm < \infty$. Show that

(i) for $f : X \to \mathbb{R}$ measurable,

$$\mathtt{loc\text{-}int}\,(f) := \{x \in [0,1] : f \ \mathtt{locally\ integrable\ at}\ \ x\}$$

is open in $[0,1]$;

(ii) if $f : [0,1] \to \mathbb{R}$ is differentiable on $[0,1]$, then $\mathtt{loc\text{-}int}\,(f')$ is dense in $[0,1]$;

`Hint` $f' \in B_1$.

(iii) By suitably modifying Volterra's construction (ex. 3.6 on p. 36), or otherwise, show that $\forall\ 0 < \lambda < 1,\ \exists\ f : [0,1] \to \mathbb{R}$ differentiable on $[0,1]$ so that $m([0,1] \smallsetminus \mathtt{loc\text{-}int}\,(f')) > \lambda$.

6. Fractal differentiation theorem.

Suppose that (X, ρ) is a metric space and $\mu \in \mathcal{P}(X)$ satisfies the condition

$$(\star) \qquad \overline{\lim_{r\to0}} \tfrac{\mu(B_\rho(x,3r))}{\mu(B_\rho(x,r))} \le M \quad \text{for } \mu\text{-a.e.} \quad x \in X.$$

a) Prove the differentiation theorem: If $f \in L^1(\mu)$, then for μ-a.e. $x \in X$,

$$\limsup_{\rho\text{-diam}\, B\to0,\ x\in B,\ B \text{ a } \rho\text{-ball}} \frac{1}{\mu(B)} \int_B |f - f(x)| d\mu = 0.$$

`Hint`: Generalize the Hardy-Littlewood maximal inequality.

b) Let $\Omega = \{0,1\}^{\mathbb{N}}$ and let $p \in \mathcal{P}(\Omega)$ be defined by $p([a_1, \dots, a_n]) = \frac{1}{2^n}$ and define *Cantor-Lebesgue measure* μ on $\mathbb{R}$ by

$$\mu(A) := p(\{\omega = (\omega_1, \omega_2, \dots) \in \Omega : \sum_{n=1}^\infty \tfrac{2\omega_n}{3^n} \in A\}).$$

Show that μ satisfies $(\star)$ with respect to the regular metric on $\mathbb{R}$.

Week # 10

Lipschitz maps. Let $\|\cdot\| = \|\cdot\|_2$ and let m be Lebesgue measure on $\mathbb{R}^d$.

• A function $f : U \to \mathbb{R}$ ($U \subset \mathbb{R}^d$) is called *Lipschitz continuous on U* if $\exists\ M > 0$ st

$$\frac{|f(y) - f(x)|}{\|x - y\|} \le M\ \ \forall\ x, y \in U.$$

• The *Lipschitz constant of f on U* is $M_{f,U} := \sup_{x,y\in U,\ x\neq y} \frac{|f(y)-f(x)|}{\|x-y\|}$.

• The function $f : U \to \mathbb{R}$ ($U \subset \mathbb{R}^d$ open) is called *differentiable at $z \in U$* if $\exists\ L \in \mathbb{R}^d$ so that

$$f(z+h) - f(z) - \langle L, h\rangle = o(\|h\|)\ \texttt{as}\ \ h \to 0.$$

• In this case $\exists\ \lim_{t\to 0} \frac{f(z+te_k)-f(x)}{t} =: f_{x_k}(z)\ \ \ (1 \le k \le d)$, and L (above) is given by $L = (f_{x_1}(z), \dots, f_{x_d}(z))$. It is called the *derivative* *(aka gradient) of f at z* and denoted $\nabla f(z) := (f_{x_1}(x), \dots, f_{x_d}(x))$.

13.1 Rademacher's theorem

Suppose that $f : U \to \mathbb{R}$ ($U \subset \mathbb{R}^d$ open) is Lipschitz continuous on U, then f is differentiable at m-a.e. point in U and $\|\nabla f\| \le M_{f,U}$ a.e..

We first prove theorem 1 and then use this to prove theorem $d\ \forall\ d \ge 2$.

¶1 `Rademacher's theorem when` $d = 1$

`Proof` The function $f : U \to \mathbb{R}$ is absolutely continuous and ¶1 follows from exercise 11.7(C).

Now fix $d \ge 2$.

¶2 $\forall\ v \in \mathbb{S}^{d-1}$, for m_d-a.e. $x \in \mathbb{R}^d$,

$$(\star_v)\qquad \exists\ \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t} =: D_v f(x) \in [-M_{f,U}, M_{f,U}].$$

Here m_k is Lebesgue measure on $\mathbb{R}^k$.

`Proof`

Set $L_v = \{tv :\ t \in \mathbb{R}\} \cong \mathbb{R}$ and $L_v^\perp := \{x \in \mathbb{R}^d :\ \langle x, v\rangle = 0\} \cong \mathbb{R}^{d-1}$, then

$$\mathbb{R}^d = L_v \oplus L_v^\perp \cong L_v \times L_v^\perp.$$

Define the measures μ_v on L_v, ν_v on $L_v^\perp$ by $\mu_v(A) := m_1(\{t \in \mathbb{R} :\ tv \in A\})$ and $\nu_v := m_{d-1} \circ T^{-1}$ where m_{d-1} is Lebesgue measure on $\mathbb{R}^{d-1}$ and $T : \mathbb{R}^{d-1} \to L_v^\perp$ is a linear isometry. Note (!) that ν_v does not depend on the isometry T.

By ¶1, $\forall\ y \in L_v^\perp$, $(\star_v)$ holds at $y + x$ for μ_v-a.e. $x \in L_v \cap U$.
By Fubini's theorem, $(\star_v)$ holds q-a.e. on U where

$$q(A) := \int_{L_v^\perp} \mu_v((A - y) \cap L_v) d\nu_v(y).$$

To see that $q = m_d$, note that by definition $q = m_d \circ S$ where $S \in \hom(\mathbb{R}^d, \mathbb{R}^d)$ is defined by $S(x_1, \dots, x_d) := x_1 v + T(x_2, \dots, x_d)$. Since S is a linear isometry, $|\det S| = 1$ and by exercise 6.1 (p. 66), $q = m \circ S = |\det S| m = m$. Thus $(\star_v)$ holds m-a.e. on U. ☐

¶3 $D_v f = \langle v, \nabla f \rangle$ m-a.e. $\forall\ v \in \mathbb{S}^{d-1}$.

`Proof`

• Let $g : U \to \mathbb{R}$ be C^1 and suppose $\overline{[g \neq 0]} \subset U$ is compact. Since f is Lipschitz and g is C^1 with compact support,
$|\frac{f(x+tv)-f(x)}{t}| \le M_{f,U}$, $|\frac{g(x+tv)-g(x)}{t}| \le \sup_{z \in \mathbb{R}^d} \|\nabla g(z)\|_2$ $\forall\ x \in U$, $t \in \mathbb{R}$ with $x + tv \in U$.

By the bounded convergence theorem,

$$\begin{aligned}\int_U D_v f g dm = \int_{\mathbb{R}^d} D_v f g dm \xleftarrow[t \to 0+]{} \int_{\mathbb{R}^d} \frac{f(x+tv) - f(x)}{t} g(x) dx \\ = \int_{\mathbb{R}^d} f(x) \frac{g(x-tv) - g(x)}{t} dx \xrightarrow[t \to 0+]{} -\int_{\mathbb{R}^d} f D_v g dm \\ = -\int_U f D_v g dm.\end{aligned}$$

Since g is C^1, $D_v g \equiv \langle v, \nabla g \rangle$ and so using the above for $e_1, \dots, e_d$,

$$\int_U D_v f g dm = -\sum_{k=1}^d v_k \int_U f D_{e_k} g dm = \sum_{k=1}^d v_k \int_U D_{e_k} f g dm = \int_U \langle v, \nabla f \rangle g dm.$$

Approximating, we get $\int_U D_v f g dm = \int_U \langle v, \nabla f \rangle g dm$ $\forall\ g \in L^1(U)$ whence ¶3. ☐

Fix $\Gamma \subset \mathbb{S}^{d-1}$ countable and dense, $\Gamma \supset \{e_1, \dots, e_d\}$ and let

$$U_\Gamma := \{x \in U : (\star) \texttt{ holds at } x \ \&\ D_v f(x) = \langle v, \nabla f(x) \rangle\ \forall\ v \in \Gamma\}.$$

By ¶2 and ¶3, U_Γ is the intersection of countable many sets of full measure in U and so $m(U \setminus U_\Gamma) = 0$.

¶4 f is differentiable at each $x \in U_\Gamma$.

`Proof` To prove differentiability at $x \in U_\Gamma$, (!) it suffices to show that

$$(\text{☿}) \qquad \sup_{v\in\mathbb{S}^{d-1}} \left|\frac{f(x+tv)-f(x)}{t} - \langle v, \nabla f(x)\rangle\right| \xrightarrow[t\to 0]{} 0.$$

To see this, fix $x \in U_\Gamma$, $\varepsilon > 0$ and choose $\Gamma_\varepsilon \subset \Gamma$ finite so that $\mathbb{S}^{d-1} = \bigcup_{v\in\Gamma_\varepsilon} B(v,\varepsilon)$;

- find $\delta > 0$ so that $|\frac{f(x+tv)-f(x)}{t} - \langle v, \nabla f(x)\rangle| < \varepsilon \ \forall\ 0 < |t| < \delta,\ v \in \Gamma_\varepsilon$.

We claim that

$$\left|\frac{f(x+tv)-f(x)}{t} - \langle v, \nabla f(x)\rangle\right| < (1+M(1+\sqrt{d}))\varepsilon \ \forall\ 0<|t|<\delta,\ v\in\mathbb{S}^{d-1}.$$

Indeed, fix $v \in \mathbb{S}^{d-1}$, then $\exists\ w \in \Gamma_\varepsilon,\ \|v-w\| < \varepsilon$. For $0 < |t| < \delta|$

$$\begin{aligned}&\left|\frac{f(x+tv)-f(x)}{t} - \langle v, \nabla f(x)\rangle\right| \le \\ &\frac{|f(x+tv)-f(x+tw)|}{|t|} + |\langle v-w, \nabla f(x)\rangle| + \left|\frac{f(x+tw)-f(x)}{t} - \langle w, \nabla f(x)\rangle\right| \\ &\le (M + \|\nabla f(x)\|)\|v-w\| + \varepsilon \\ &\le (1+2M)\varepsilon. \quad \Box\end{aligned}$$

In the sequel, we'll need the:

13.2 Uniform differentiation lemma:

Suppose that $f : U \to \mathbb{R}$ ($U \subset \mathbb{R}^d$ open) is Lipschitz continuous on U and let $A \in \mathcal{B}(U)$, $0 < m(A) < \infty$, then $\forall\ \eta > 0$, $\exists\ F \subset A$ closed st $m(A \setminus F) < \eta$ and

$$(\text{♁}) \qquad \sup_{x\in F,\ v\in\mathbb{S}^{d-1}} \left|\frac{f(x+tv)-f(x)}{t} - \langle v, \nabla f(x)\rangle\right| \xrightarrow[t\to 0+]{} 0.$$

`Proof` Let $\delta_n := \frac{1}{n}$. We claim first that

☺ $\forall\ \eta > 0$, $\exists\ F \subset A$ closed st $m(A \setminus F) < \eta$ and

$$(\text{♀}) \qquad \sup_{x\in F,\ v\in\mathbb{S}^{d-1}} \left|\frac{f(x+\delta_n v)-f(x)}{\delta_n} - f_v(x)\right|| \xrightarrow[n\to\infty]{} 0$$

where $f_v(x) := \langle v, \nabla f(x)\rangle$.

`Proof of` ☺:

Fix $\Gamma \subset \mathbb{S}^{d-1}$ countable and dense.

- By (☿) (see ¶4 in the proof of Rademacher's theorem),

$$\Phi_t(x) := \sup_{v\in\mathbb{S}^{d-1}} \left|\frac{f(x+tv)-f(x)}{t} - \langle v, \nabla f(x)\rangle\right| \xrightarrow[t\to 0+]{} 0$$

for a.e. $x \in U$.

- By Egorov's theorem, $\exists\ F \subset A$ closed so that $m(A \setminus F) < \eta$ and

$$\sup_{x\in F} \Phi_{\delta_n}(x) \xrightarrow[n\to\infty]{} 0.$$

This is (♀). ☑ ☺

To complete the proof of (♁) let, for $0 < t < 1$, $n_t := \lfloor \frac{1}{t} \rfloor$, then

$$\delta_{n_t} \ge t > \delta_{n_t+1}, \quad \delta_{n_t} \sim t \ \&\ 0 \le \delta_{n_t} - t \le \frac{1}{n_t^2} \sim t^2 \ \ \texttt{as} \quad t \to 0+.$$

Thus

$$\begin{aligned}
&|f(x+tv) - f(x) - tf_v(x)| \le \\
&|f(x+tv) - f(x+\delta_{n_t}v)| + |f(x+\delta_{n_t}v) - f(x) - \delta_{n_t}f_v(x)| + (\delta_{n_t} - t)|f_v(x)| \\
&\le 2M_{f,U}(\delta_{n_t} - t) + \delta_{n_t}S_{n_t} \\
&\ll t^2 + t\sup_{x\in F} \Phi_{\delta_{n_t}}(x) \\
&= o(t) \quad \text{as} \quad t \to 0. \quad \Box
\end{aligned}$$

Change of variables.

13.3 Lemma (non-singularity of Lipschitz equivalences)

Suppose that $U \subset \mathbb{R}^d$ is a domain, and that $T : U \to TU \subset \mathbb{R}^d$ is a Lipschitz equivalence (i.e. both T and T^{-1} are Lipschitz), then $T : U \to TU$ is non-singular and

$$M_{T^{-1},TU}^{-d} \le T' := \frac{dm \circ T}{dm} \le M_{T,U}^d.$$

`Proof` We'll prove that $T' \le M_{T,U}^d$, the other inequality following from $T^{-1\prime} \le M_{T^{-1},TU}^d$.

It suffices to show that $m(TA) \le M_{T,U}^d m(A) \quad (A \in \mathcal{B}(U))$. Fix $A \in \mathcal{B}(U),\ 0 < m(A) < \infty$.

If $A \subset \bigcup_{k=1}^\infty B(x_k, r_k)$, then $TA \subset \bigcup_{k=1}^\infty B(Tx_k, Mr_k)$, whence

$$m(TA) \le \sum_{k=1}^\infty m(B(Tx_k, Mr_k)) = M^d \sum_{k=1}^\infty m(B(x_k, r_k)).$$

By Vitali's covering theorem,

$$m(A) = \inf\{\sum_{k=1}^\infty m(B(x_k, r_k)) : \ A \subset \bigcup_{k=1}^\infty B(x_k, r_k)\}.$$

Thus $m(TA) \le M^d m(A)$. $\Box$

By the lemma, $\exists\ T' : U \to [M^{-d}, M^d]$ (the *Radon-Nikodym derivative* of T) so that $m(TA) = \int_A T'dm \ \ (A \in \mathcal{B}(U))$. It follows from the chain rule for RN derivatives (ex. 7.2.(ii)) that

• If $U,\ V \subset \mathbb{R}^d$ are open, and $A : U \to V,\ B : V \to \mathbb{R}^d$ are Lipschitz equivalences, then so is $B \circ A$ and $(B \circ A)' = B' \circ A \cdot A'$.

Write $T = (T_1, \dots, T_d)$. By Rademacher's theorem, each T_k is differentiable a.e. on $U_T \in \mathcal{B}(U),\ m(U \setminus U_T) = 0$.

Define $DT : U_T \to \hom(\mathbb{R}^d, \mathbb{R}^d)$ by $DT(x)_{i,j} := (T_i)_{x_j}(x)$, then T is differentiable at each $x \in U_T$ in the sense that

$$\|T(x+h) - T(x) - DT(x)h\| = o(\|h\|) \texttt{ as } h \to\ 0.$$

13.4 Lemma (chain rule for Lipschitz equivalences)

Suppose that $U,\ V \subset \mathbb{R}^d$ are open, and $A : U \to V,\ B : V \to \mathbb{R}^d$ are Lipschitz equivalences, then

(a) *so is $B \circ A$;*

(b) $$U_0 := \{x \in U :\ A \texttt{ is diffble at } x \ \&\ B \texttt{ is diffble at } A(x)\}$$

has full measure in U;

(c) $D(B \circ A)(x) = DB(A(x))DA(x)\ \forall\ x \in U_0$.

`Proof` By non-singularity the set

$$U_0 := \{x \in U :\ A \texttt{ is diffble at } x\} \cap A^{-1}\{y \in V :\ B \texttt{ is diffble at } y\}$$

has full measure in U.

Fix $x \in U_0$ and $\varepsilon > 0$. Since B is differentiable at $A(x),\ \exists\ \delta_1 > 0$ such that

$$\|B(A(x) + h) - B(A(x)) - DB(A(x))h\| \le \varepsilon\|h\|\ \forall\ \|h\| < \delta_1.$$

Since A is differentiable (whence continuous) at x, $\exists\ \delta_2 > 0$ such that $\forall\ \|h\| < \delta_2$,

$$\|A(x+h) - A(x) - DA(x)h\| \le \varepsilon\|h\|, \texttt{ and } \|A(x+h) - A(x)\| < \delta_1.$$

It follows that for $\|h\| < \delta_2$, setting $k := A(x+h) - A(x)$, we have

$$\begin{aligned}
&\|B(A(x+h)) - B(A(x)) - DB(A(x))DA(x)h\| \\
&\le \|B(A(x+h)) - B(A(x)) - DB(A(x))k\| + \|DB(A(x))(k - DA(x)h)\| \\
&\le \|B(A(x)+k) - B(A(x)) - DB(A(x))k\| + \\
&\qquad\qquad + \|DB(A(x))\|\,\|A(x+h) - A(x) - DA(x)h\| \\
&\le \varepsilon(\|k\| + \|DB(A(x))\|\,\|h\|) \\
&\le \varepsilon(\|A(x+h) - A(x) - DA(x)h\| + \|DA(x)h\| + \|DB(A(x))\|\,\|h\|) \\
&\le \varepsilon\|h\|(\varepsilon + \||DA(x)\| + \|DB(A(x))\|). \quad \square
\end{aligned}$$

13.5 Change of variables formula for Lipschitz equivalences

Let $U \subset \mathbb{R}^d$ be open and let $T : U \to TU$ be a Lipschitz equivalence, then

$$(⛢) \qquad T' = |\det DT|.$$

`Proof` It suffices to prove

$$(🚘) \qquad \frac{m(TB(x,r))}{m(B(x,r))} \underset{r\to 0+}{\longrightarrow} |\det DT(x)| \texttt{ for } m\texttt{-a.e. } x \in U$$

because, assuming (🚘), we also have by the Lebesgue differentiation theorem that for m-a.e. $x \in U$,

$$T'(x) \underset{r\to 0+}{\longleftarrow} \frac{1}{m(B(x,r))}\int_{B(x,r)} T'dm = \frac{m(TB(x,r))}{m(B(x,r))} \underset{r\to 0+}{\longrightarrow} |\det DT(x)|.$$

Proof of (🚘)

Let U_0 denote the set of points $x \in U$ so that

$$T \texttt{ is diffble at } x, \;\; T^{-1} \texttt{ is diffble at } Tx.$$
$$(T^{-1})'(Tx) = T'(x)^{-1} \;\&\; DT^{-1}(Tx) = DT(x)^{-1}.$$

By non singularity and the chain rules for T' (exercise 7.6(ii)) and DT (13.4 above), $m(U \setminus U_0) = 0$.

By exercise 5.4(iii)

$$T' = |\det T| \;\forall\; T \in GL(d, \mathbb{R}).$$

where $GL(d,\mathbb{R}) := \{\texttt{invertible, linear maps} : \mathbb{R}^d \to \mathbb{R}^d\}$.

We claim

¶3 $\forall x \in U_0,\ \varepsilon > 0, \exists\ \delta > 0$ such that $\forall\ 0 < r < \delta$,

$$T(x) + DT(x)B(0, r(1-\varepsilon)) \overset{\text{(i)}}{\subseteq} TB(x,r) \overset{\text{(ii)}}{\subseteq} T(x) + DT(x)B(0, r(1+\varepsilon)).$$

`Proof of` ¶3:

Let $M := \|DT(x)\|, \vee \|DT(x)^{-1}\|$.

- $\exists\ \delta > 0$ such that

$$\|T(x+h) - T(x) - DT(x)h\| < \frac{\varepsilon}{M}\|h\|\ \forall\ \ \|h\| < \delta.$$

To see inclusion (ii), let $a \in TB(x,r)$, then $a = T(x+h)$ where $\|h\| \le r < \delta$. We show that $\|DT(x)^{-1}(a - Tx)\| \le r(1+\varepsilon)$. Indeed,

$$\begin{aligned}\|DT(x)^{-1}(a - Tx)\| &= \|DT(x)^{-1}(T(x+h) - Tx)\| \\ &\le \|DT(x)^{-1}(T(x+h) - Tx - DT(x)h)\| + \|h\| \\ &\le M\|T(x+h) - Tx - DT(x)h\| + \|h\| \\ &< (1+\varepsilon)\|h\| \le r(1+\varepsilon). \quad \square\texttt{(ii)}\end{aligned}$$

- $\exists\ 0 < \delta_1 < \delta$ st
  $\|T^{-1}(Tx + k) - x - DT(x)^{-1}k\| \le \varepsilon\frac{\|k\|}{M}\ \forall\ \|k\| < M\delta_1$, whence

$$\|T^{-1}(Tx + DT(x)h) - x - h\| \le \varepsilon\|h\|\ \forall\ \|h\| < \delta_1.$$

To see inclusion (i), choose $r < \delta_1$, and $b \in T(x) + DT(x)B(0, r(1-\varepsilon))$, then $b = Tx + DT(x)h$ for some $\|h\| < (1-\varepsilon)r$ and

$$\begin{aligned}\|T^{-1}b - x\| &= \|T^{-1}(Tx + DT(x)h) - x\| \\ &\le \|T^{-1}(Tx + DT(x)h) - x - h\| + \|h\| \le (1+\varepsilon)\|h\| \\ &\le (1-\varepsilon^2)r. \quad \square\texttt{(i)}\end{aligned}$$

This proves ¶3, whence follows (🙈) using ¶2. ☑

Steiner symmetrization.

The `isodiametric inequality` says that the measurable set with maximum volume among those with given Euclidean diameter is a ball. A `Steiner symmetrization` is a transformation of measurable sets preserving volume and decreasing diameter which is used to prove the isodiametric inequality.

14.1 Isodiametric inequality[9]

[9] Bieberbach, L. Über eine Extremaleigenschaft des Kreises, Jber. DMV, 24 (1915), pp. 247-250

For $A \in \mathcal{B}(\mathbb{R}^d)$,

$$m(A) \le m(B(1))(\frac{\mathtt{diam}\, A}{2})^d$$

where $B(1)$ is the unit ball and diam *is diameter, both with respect to* $\|\cdot\|_2$.

Steiner symmetrization.

For $u \in \mathbb{S}^{d-1}$, $v \in \mathbb{R}^d$ let $L_{u,v} : \mathbb{R} \to \mathbb{R}^d$ be defined by $L_{u,v}(t) := v + tu$ and let $L(u,v) := L_{u,v}(\mathbb{R})$. Define the corresponding linear measure $\lambda_{u,v} : \mathbb{R} \to [0,\infty]$ by $\lambda_{u,v}(A) := m(L_{u,v}^{-1}A)$.

The *Steiner symmetrization of $A \in \mathcal{B}(\mathbb{R}^d)$ in the direction $u \in \mathbb{S}^{d-1}$* is

$$\mathfrak{s}_u(A) := \{v + tu : v \perp u,\ \lambda_{u,v}(A) > 0,\ |t| < \frac{1}{2}\lambda_{u,v}(A)\}.$$

For example if $u = e_1$ then $u^\perp = \{0\} \times \mathbb{R}^{d-1}$ and

$$\mathfrak{s}_{e_1}(A) := \{(t,v) : v \in \mathbb{R}^{d-1},\ \lambda_{e_1,v}(A) = m_1(A^v) > |t|\}.$$

Given $u \in \mathbb{S}^{d-1}$, define *reflection through* $u^\perp$ by $R_u : \mathbb{R}^d \to \mathbb{R}^d$ by $R_u(x) := x - 2\langle x, u\rangle u$.

For example, $R_{e_j}(x) = x - 2x_j e_j$ i.e. $R_{e_i}(x)_j = x_j$ for $j \ne i$ and $R_{e_i}(x)_i = -x_i$.

Note also that $R_u\mathfrak{s}_u(A) = \mathfrak{s}_u(A)$.

14.2 Steiner symmetrization proposition

Suppose that $A \in \mathcal{B}(\mathbb{R}^d)_+$, then
(o) $U\mathfrak{s}_u(A) = \mathfrak{s}_{Uu}(UA)$ *for* $U \in \hom(\mathbb{R}^d, \mathbb{R}^d)$ *an isometry;*
(i) $\mathfrak{s}_u(A)$ *is Lebesgue measurable and* $m(\mathfrak{s}_u(A)) = m(A)$;
(ii) $\operatorname{diam}\mathfrak{s}_u(A) \le \operatorname{diam} A$*;*
(iii) *if $u \perp u'$ then $\mathfrak{s}_u(R_{u'}A) = R_{u'}\mathfrak{s}_u(A)$.*

`Proof of`(o):

$U \circ L_{u,v}(t) = Uv + tUu = L_{Uu,Uv}(t)$ whence

$$\lambda_{u,v} \circ U^{-1} = m \circ (U \circ L_{u,v})^{-1} = m \circ L_{Uu,Uv}^{-1} = \lambda_{Uu,Uv}$$

and, setting $a = Uu$,

$$\begin{aligned}
U\mathfrak{s}_u(A) &= \{Uv + tUu : v \perp u,\ \lambda_{u,v}(A) > 0,\ |t| < \tfrac{1}{2}\lambda_{u,v}(A)\} \\
&= \{b + ta : b \perp a,\ \lambda_{U^{-1}a,U^{-1}b}(A) > 0,\ |t| < \tfrac{1}{2}\lambda_{U^{-1}a,U^{-1}b}(A)\} \\
&= \{b + ta : b \perp a,\ \lambda_{a,b}(UA) > 0,\ |t| < \tfrac{1}{2}\lambda_{a,b}(UA)\} \\
&= \mathfrak{s}_a(UA) = \mathfrak{s}_{Uu}(UA). \quad \boxtimes
\end{aligned}$$

`Proof of` (i): By (o), WLOG $u = e_1 := (1,0,\dots,0)$. $e_1^\perp = \{0\} \times \mathbb{R}^{d-1}$ and for $v \in e_1^\perp$,

$$L_{e_1,v}(t) = (t,v),\ \lambda_{e_1,v}(A) = m_1(A_v)$$

where $A_v := \{t \in \mathbb{R} :\ (t,v) \in A\}$.

By Fubini's theorem, $F : \mathbb{R}^{d-1} = e_1^\perp \to [0,\infty),\ F(x) := m(A_x)$ is measurable, whence

$$\mathfrak{s}_{e_1}(A) = \{(t,x) :\ x \in e_1^\perp,\ |t| < F(x)\}$$

is measurable.

By Fubini's theorem,

$$m(\mathfrak{s}_{e_1}(A)) = \int_{e_1^\perp} m(\mathfrak{s}_{e_1}(A)_v)dv = \int_{e_1^\perp} m(A_v)dv = m(A). \quad \boxtimes\text{(i)}$$

Exercise N$^{\underline{\text{O}}}$10

1. Convexity & Jensen's Inequality. A function $f:(a,b)\to\mathbb{R}$ is called *convex* if

$$f(\alpha x+(1-\alpha)y)\le\alpha f(x)+(1-\alpha)f(y)\quad\forall\ x,y\in(a,b),\ 0\le\alpha\le 1.$$

(a) Show that if $f:(a,b)\to\mathbb{R}$ is convex, then $\forall\ a<x<y<z<b$,

$$\frac{f(y)-f(x)}{y-x}\le\frac{f(z)-f(x)}{z-x}\le\frac{f(z)-f(y)}{z-y},$$

whence f has non-decreasing, one-sided derivatives

$$f'_\pm(x):=\lim_{h\to 0+}\frac{f(x\pm h)-f(x)}{h}\quad\forall\ x\in(a,b).$$

Show by example that maybe $f'_+(x)\ne f'_-(x)$.

(b) Show that the function $f:(a,b)\to\mathbb{R}$ is convex iff

$$\forall\ x\in(a,b),\ \exists\ g(y)=\alpha y+\beta \text{ such that}$$
$$g(x)=f(x),\ \&\ g(y)\le f(y)\ \forall\ y\in(a,b).$$

(c) Prove `Jensen's Inequality`:

Let $f:(a,b)\to\mathbb{R}$ be convex, $(X,\mathcal{B},m)$ be a probability space, and $F:X\to(a,b)$ be integrable on X, then

$$f\left(\int_X F\,dm\right)\le\int_X f\circ F\,dm.$$

2. Which inequality?

Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ be probability spaces, and let $m:\mathcal{B}\otimes\mathcal{C}\to[0,1]$ be the product measure.

Suppose that $f:X\times Y\to[0,\infty)$ is $\mathcal{B}\otimes\mathcal{C}$-measurable, $1<p<\infty$ and for $x\in X$ define $f_x:Y\to[0,\infty)$ by $f_x:=f(x,y)$.

Which of the following (if any) is true:

$$\|f\|_{L^p(m)}\le\int_X\|f_x\|_{L^p(\nu)}d\mu(x),\quad\text{or }\|f\|_{L^p(m)}\ge\int_X\|f_x\|_{L^p(\nu)}d\mu(x)\ \text{?}$$

When is there equality?

3. σ-finiteness and separability.

Let $(X,\mathcal{B},m)$ be a measure space.

(i) Show that m is σ-finite iff $\exists\ f\in L^1(m),\ f>0$ a.e..

(ii) Suppose that m is finite. Show that $L^1(m)$ is separable (as a Banach space) iff the measure algebra of $(X,\mathcal{B},m)$ is separable (as in exercise 3.3).

4. Clarkson's parrallelogram inequalities.

Let $(X,\mathcal{B},m)$ be a σ-finite measure space, and let $f,g \in L^p(m)$. Prove that

$$\|f+g\|_p^p + \|f-g\|_p^p \begin{cases} \le 2(\|f\|_p^p + \|g\|_p^p) \text{ for } p \in [1,2], \\ \ge 2(\|f\|_p^p + \|g\|_p^p) \text{ for } p \in [2,\infty), \end{cases}$$

and that for $p \ne 2$, $f,g \in L^p$, there is equality in the relevant inequality iff $fg = 0$ a.e..

`Hints` (i) By convexity, for $a > 0$, $t > 1$,

$$2t^a \quad \begin{cases} < (t-1)^a + (t+1)^a & a > 1, \\ > (t-1)^a + (t+1)^a & a < 1. \end{cases}$$

Set $f_p(t) := (t-1)^p + (t+1)^p - 2t^p - 2$, then $f_p(1) = 2^p - 4 > 0 (p > 2)$ and < 0 $(p < 2)$. Also $f_p'(t) > 0\ \forall\ t > 1$ $(p > 2)$ and $f_p'(t) < 0\ \forall\ t > 1$ $(p < 2)$. Thus $f_p(t) > 0\ \forall\ t > 1$ $(p > 2)$ and $f_p(t) < 0\ \forall\ t > 1$ $(p < 2)$.

Week # 11.

`Proof of` (ii): Assume WLOG that A is closed and bounded and (by (o)) that $u + e_1$.

Fix $\varepsilon > 0$, $x,y \in \mathfrak{s}_{e_1}(A)$ with $\mathtt{diam}\ \mathfrak{s}_{e_1}(A) < \|x-y\|_2 + \varepsilon$.

Set $x' := (x_2,\dots,x_d)$, $y' := (y_2,\dots,y_d)$, then $(0,x')$, $(0,y') \perp e_1$. Set

$$\underline{x} := \inf\{t:\ (t,x') \in A\},\ \overline{x} := \sup\{t:\ (t,x') \in A\},$$

$$\underline{y} := \inf\{t:\ (t,y') \in A\},\ \overline{y} := \sup\{t:\ (t,y') \in A\}.$$

Suppose (WLOG) that $\overline{y} - \underline{x} \ge \overline{x} - \underline{y}$, then

$$\begin{aligned} \overline{y} - \underline{x} &\ge \frac{1}{2}(\overline{y} - \underline{x}) + \frac{1}{2}(\overline{x} - \underline{y}) \\ &= \frac{1}{2}(\overline{y} - \underline{y}) + \frac{1}{2}(\overline{x} - \underline{x}) \\ &\ge \frac{1}{2}m_1(A_{y'}) + \frac{1}{2}m_1(A_{x'}) \\ &\ge |y_1| + |x_1| \ge |x_1 - y_1| \end{aligned}$$

and

$$\begin{aligned} \mathtt{diam}\ \mathfrak{s}_{e_1}(A) - \varepsilon &< \|x-y\|_2 \\ &= \sqrt{\|x'-y'\|_2^2 + |x_1 - y_1|^2} \\ &\le \sqrt{\|x'-y'\|_2^2 + (\overline{y} - \underline{x})^2} \\ &= \|(\underline{x}, x_2, \dots, x_d) - (\overline{y}, y_2, \dots, y_d)\|_2 \\ &\le \mathtt{diam}\ A.\ \boxtimes\text{(ii)} \end{aligned}$$

`Proof of` (iii): Note that $R_{u'}$ is an isometry, $R_{u'}^{-1} = R_{u'}$ and if $u' \perp u$, then $R_{u'}|_{u'^\perp} \equiv \text{Id}$. Thus by (o), for $u \perp u'$,

$$R_{u'}\mathfrak{s}_u(A) = \mathfrak{s}_{R_{u'}u}(R_{u'}A) = \mathfrak{s}_u(R_{u'}A). \quad ☑\text{(iii)}$$

`Proof of the isodiametric inequality`

Set $A_0 := A$, $A_i := \mathfrak{s}_{e_i}(A_{i-1})$ $(1 \le i \le d)$, then $\texttt{diam}\, A_d \le \texttt{diam}\, A$ and $m(A_d) = m(A)$.

Next $R_{e_i} \circ R_{e_j} = R_{e_j} \circ R_{e_i}$ $\forall\ i, j$ whence

$$R_{e_k} A_{k+1} = R_{e_k}\mathfrak{s}_{ek+1}(A_k) = \mathfrak{s}_{ek+1}(R_{e_k}A_k) = A_{k+1} \ \&$$

by induction, using

$$R_{e_k} A_{k+r+1} = R_{e_k}\mathfrak{s}_{ek+r+1}(A_{k+1}) = \mathfrak{s}_{ek+r+1}(R_{e_k}A_{k+r}) = A_{k+r},$$

we see that $R_{e_k} A_d = A_d$ $\forall\ 1 \le k \le d)$ whence

$$-A_d = A_d; \implies A_d \subseteq B(0, \frac{\texttt{diam}\, A_d}{2}) \ \&$$

$$m(A) = m(A_d) \le m(B(1))(\frac{\texttt{diam}\, A_d}{2})^d \le m(B(1))(\frac{\texttt{diam}\, A}{2})^d. \quad ☑$$

Hausdorff measures.

Let (X, d) be a metric space. For $A \subset X$, let
$|A| = \text{diam} A := \sup_{x,y \in A} d(x, y)$.
A *gauge function* is a function $a : [0, \infty) \to [0, \infty)$ continuous and strictly increasing, and satisfying $a(0) = 0$.

Given a gauge function a, define, for $\varepsilon > 0$, and $A \subset X$,

$$H_a^{(\varepsilon)}(A) := \inf\{\sum_{k=1}^\infty a(|A_k|) : A \subset \bigcup_{k=1}^\infty A_k,\ |A_k| < \varepsilon\ \forall\ k \ge 1\};$$

$$H_{\mathbb{S},a}^{(\varepsilon)}(A) := \inf\{\sum_{k=1}^\infty a(|B(x_k, r_k)|) : A \subset \bigcup_{k=1}^\infty B(x_k, r_k), \quad r_k < \varepsilon\ \forall\ k \ge 1\}$$

It follows that

$$H_a^{(\varepsilon)}(A) \uparrow H_a(A),\ H_{\mathbb{S},a}^{(\varepsilon)}(A) \uparrow H_{\mathbb{S},a}(A) \quad \text{as} \ \ \varepsilon \downarrow 0.$$

The set functions H_a, $H_{\mathbb{S},a}$ are known as *Hausdorff measure* and *spherical measure* (respectively) on (X, d) with *gauge function a*.

- $H_a \le H_{\mathbb{S},a} \le \sup_{t>0} \frac{a(2t)}{a(t)} H_a$.

`Proof` $H_a \le H_{\mathbb{S},a}$ $\because$ the inf is on a supset.
$H_{\mathbb{S},a} \le \sup_{t>0} \frac{a(2t)}{a(t)} H_a \because x \in A \subset X \implies A \subset B(x, |A|)$. ☐

Note that in •, one can replace $\sup_{t>0} \frac{a(2t)}{a(t)}$ with $\lim_{\varepsilon \to 0+} \sup_{0<t<\varepsilon} \frac{a(2t)}{a(t)}$.

Hausdorff measures as measures.

As shown in week 2, for any gauge function a, the Hausdorff measures $H_a,\ H_{a,\mathbb{S}} : 2^X \to [0,\infty]$ are metric outer measures

i.e. $A, B \subset X,\ d(A,B) := \inf_{x\in A, y\in B} d(x,y) > 0 \Rightarrow \overline{\mu}(A\cup B) = \overline{\mu}(A) + \overline{\mu}(B)$,

whence by Caratheodory's theorem

$$\mathcal{B}(X) \subset \mathcal{M}(H_a) \cap \mathcal{M}(H_{a,\mathbb{S}}).$$

Hausdorff dimension.

Let (X,d) be a metric space. For $\alpha > 0$, let $a_\alpha(t) := t^\alpha$, and let $H_\alpha := H_{a_\alpha}$ on (X,d). Note that if $0 < \beta < \alpha$, then

$$H_{a_\alpha}^{(\varepsilon)}(A) \le \varepsilon^{\alpha-\beta} H_{a_\beta}^{(\varepsilon)}(A)\ \forall\ \varepsilon > 0,\ A \subset X,$$

therefore, for $A \subset X,\ 0 < \beta < \alpha$,

$$H_\alpha(A) > 0 \Rightarrow H_\beta(A) = \infty,\ \&\ H_\beta(A) < \infty \Rightarrow H_\alpha(A) = 0.$$

It follows that

$$\exists\ \text{H-dim}(A) \in [0,\infty]\ \ni\ H_\alpha(A) = \begin{cases} \infty & \forall\ \alpha < \text{H-dim}(A), \\ 0 & \forall\ \alpha > \text{H-dim}(A). \end{cases}$$

The number H-dim(A) is called the *Hausdorff dimension* of the set A.

Quasi-isometry of metric spaces and Hausdorff measures.

Let $M \ge 1$. A M*-quasi-isometry* of the metric spaces (X,d) and (Y,ρ) is a bijection $T : X \to Y$ satisfying

$$\frac{d(x,x')}{M} \le \rho(Tx,Tx') \le Md(x,x')\ \ \forall\ x,x' \in X.$$

A surjective map is a Lipschitz equivalence iff it is a M-quasi-isometry for some $M \ge 1$.

Quasi-isometry lemma

Suppose that (X,d) and (Y,ρ) are metric spaces, that $M \ge 1$ and that $a : [0,\infty) \to [0,\infty)$ is a gauge function with $\frac{a(Mt)}{a(t)} \le K\ \forall\ t > 0$.

If $A \subset X$ and $T : A \to TA \subset Y$ is an M-quasi-isometry, then

$$H_{a,\rho}(TA) = K^{\pm 1} H_{a,d}(A),\quad H_{\mathbb{S},a,\rho}(TA) = K^{\pm 1} H_{\mathbb{S},a,d}(A).$$

The proof is straightfoward and standard.

The quasi-isometry lemma will be useful when $\sup_{t>0} \frac{a(Mt)}{a(t)} \to 1$ as $M \to 1$.

Hausdorff measures on $\mathbb{R}^d$.

Let $a_s(t) := t^s \quad (s, t > 0)$.

Theorem (Norm-spherical measure on $\mathbb{R}^d$)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$, then $H_{\mathbb{S},a_d,\|\cdot\|} = \frac{2^d}{m_{\mathbb{R}^d}(B(1))} m_{\mathbb{R}^d}$ where $B(r) = B(0,r)$ where $B(z,r) := \{x \in \mathbb{R}^d : \|x - z\| \le r\}$.

`Proof` Both measures are translation invariant and σ-finite so $H_{\mathbb{S},a,d} = c m_{\mathbb{R}^d}$ for some $c > 0$.

Suppose that $A \subset \bigcup_{k=1}^\infty B(z_k, r_k)$, then

$$\sum_{k=1}^\infty (2r_k)^d = \frac{2^d}{m(B(1))} \sum_{k=1}^\infty m(B(z_k, r_k)) \ge \frac{2^d m(A)}{m(B(1))}$$

whence $c \ge \frac{2^d}{m(B(1))}$.

Let $\varepsilon, \ \delta > 0$. By the Vitali covering theorem, $\exists$ disjoint balls $B(x_k, r_k)$ $(k \ge 1)$ such that $r_k < \delta$ and $A \subset \biguplus_{k=1}^\infty B(x_k, r_k) \mod m$ and $m(\biguplus_{k=1}^\infty B(x_k, r_k) \setminus A) < \varepsilon$. It follows that

$$\begin{aligned} m(A) + \varepsilon &> \sum_{k=1}^\infty m(B(x_k, r_k)) \\ &= m(B(1)) \sum_{k=1}^\infty r_k^d \\ &\ge \frac{m(B(1))}{2^d} H^{(\delta)}_{\mathbb{S},a_d,\|\cdot\|}(A). \quad \square \end{aligned}$$

Round Hausdorff measure Theorem

Let $a_d(t) := t^d$, then on $\mathbb{R}^d$, $H_{\mathbb{S},a_d,\|\cdot\|_2} = H_{a_d,\|\cdot\|_2}$.

- In general, if $\|\cdot\|$ is a norm on $\mathbb{R}^d$ we have that on $\mathbb{R}^d$, $H_{\mathbb{S},a_d,\|\cdot\|} = c H_{a_d,\|\cdot\|}$ for some $1 \le c \le 2^d$.
- Besicovitch showed[10] that $\exists\ A \in \mathcal{B}(\mathbb{R}^2)$ so that $H_{\mathbb{S},a_1,\|\cdot\|_2}(A) > H_{a_1,\|\cdot\|_2}(A)$.

`Proof of the round Hausdorff measure theorem`

Since $H_{a_d} \le H_{\mathbb{S},a_d} = \frac{2^d}{m(B(1))} m$, it suffices to show that

$$\frac{m(B(1))}{2^d} H_{a_d} \ge m.$$

[10] *On the fundamental geometrical properties of linearly measurable plane sets of points*; Math. Ann. **98** 422-464. (1927)

Suppose that $A \subset \bigcup_{n=1}^\infty A_n$, then by the `isodiametric inequality`,

$$m(B(1))\sum_{n=1}^\infty (\frac{\texttt{diam}\, A_n}{2})^d \ge \sum_{n=1}^\infty m(A_n) \ge m(A).$$

Thus,

$$\frac{m(B(1))}{2^d} H_{a_d}(A) \ge m(A). \quad \square$$

Hausdorff measures on submanifolds and the Area Formula.

The "surface area formula" says that if $U \subset \mathbb{R}^2$ and $T : U \to \mathbb{R}^3$ is 1-1, C^1 and *regular* in the sense that $r(DT(x)) = 2\ \forall\ x \in U$, then the surface area of a subset of $TU \subset \mathbb{R}^3$ is `defined` in some calculus courses by

$$(\text{Ш}) \qquad \texttt{Area}\,(TA) := \int_A \|\frac{\partial T}{\partial x} \times \frac{\partial T}{\partial y}\|_2 dm \quad (A \in \mathcal{B}(U)).$$

- Let $k \le d$. We define k-dimensional "area" in $\mathbb{R}^d$ to be

$$\sigma_k = \sigma_{d,k} = \frac{m_{\mathbb{R}^k}(B(1))}{2^k} H_{a_k, \|\cdot\|_2}$$

where $a_k(t) = t^k$ and $B(1) := B_{\|\cdot\|_2}(0,1)$. This has the property that

$$\sigma_k(A \times \{c\}) = m_{\mathbb{R}^k}(A) \ \ (A \in \mathcal{B}(\mathbb{R}^k),\ c \in \mathbb{R}^{d-k}.$$

In this section we prove a generalization of the "surface area formula".

`Lipschitz regularity.`

Suppose that $k < d$, $U \subset \mathbb{R}^k$ is a domain and $T : U \to TU \subset \mathbb{R}^d$ is a Lipschitz equivalence (i.e. M-quasi-isometry), then

- by the quasi-isometry lemma, $\sigma_k(TA) = M^{\pm k} m_{\mathbb{R}^k}(A)\ \forall\ U \in \mathcal{B}(U)$;
- by Rademacher's theorem $m(U \setminus U_\nabla) = 0$ where

$$U_\nabla := \{x \in U :\ T\ \texttt{diffble.}\ \ \texttt{at}\ x\};$$

- for $x \in U_\nabla,\ h \in \mathbb{R}^k$,

$$\|DT(x)h\|_2 \underset{t\to 0+}{\longleftarrow} \frac{\|T(x+th) - T(x)\|_2}{t} = M^{\pm 1}\|h\|_2,$$

whence $r(DT(x)) = k$.

`Linear Algebra.`

Suppose that $k \le d$ and that $T \in \hom(\mathbb{R}^k, \mathbb{R}^d)$ is regular (i.e. injective), then $T^* \circ T \in \hom(\mathbb{R}^k, \mathbb{R}^k)$ is regular and we define the *Jacobian* $\Delta(T) := \sqrt{\det T^* \circ T} \in \mathbb{R}_+$. In case $k = d$, we have $\Delta(T) := |\det T|$.

The area formula

Suppose that $k < d$, $U \subset \mathbb{R}^k$ is a domain and $T : U \to TU \subset \mathbb{R}^d$ is a Lipschitz equivalence, then

$$\sigma_k(TA) = \int_A \Delta(DT) dm_{\mathbb{R}^k} \ \forall \ A \in \mathcal{B}(U).$$

• `The area formula generalizes` (♊): if $T \in \hom(\mathbb{R}^2, \mathbb{R}^3)$ is of form $T(x, y) := xa + yb$ where $a, \ b \in \mathbb{R}^3$, then (!)

$$\Delta(T) := \sqrt{|\det(T^*T)|} = \|a \times b\|_2.$$

We first prove the area formula in case $T \in \hom(\mathbb{R}^k, \mathbb{R}^d)$. We'll need the

Polar decomposition lemma

Suppose that $k \le d$ and that $T \in \hom(\mathbb{R}^k, \mathbb{R}^d)$ has rank k, then $\exists \ U \in Hom(\mathbb{R}^k, \mathbb{R}^d)$ orthogonal and $S \in Hom(\mathbb{R}^k, \mathbb{R}^k)$ symmetric, non-singular such that $T = U \circ S$ and $\Delta(S) = \Delta(T)$.

`Proof` Let $C := T^* \circ T \in \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^k)$, then C is symmetric and positive definite and $\exists$ an orthonormal basis $\{z_i\}_{i=1}^k \subset \mathbb{R}^k$ with $C(z_i) = \lambda_i^2 z_i$ where $\lambda_i \ne 0 \ \ (1 \le i \le k)$.

Now set $x_j := \frac{1}{\lambda_j} T z_j \in \mathbb{R}^d \ \ (1 \le j \le k)$.

We claim that $\{x_i\}_{i=1}^k \subset \mathbb{R}^d$ is orthonormal. To see this,

$$\begin{aligned} \langle x_i, x_j \rangle &= \frac{1}{\lambda_i \lambda_j} \langle T z_i, T z_j \rangle = \frac{1}{\lambda_i \lambda_j} \langle C z_i, z_j \rangle \\ &= \frac{\lambda_i}{\lambda_j} \langle z_i, z_j \rangle = \delta_{i,j} := \begin{cases} 0 & i \ne j, \\ 1 & i = j. \end{cases} \end{aligned}$$

We claim that the required decomposition is $T = U \circ S$ where $S z_j := \lambda_j z_j, \ U z_j := x_j$.

To see that U is orthogonal, $\langle U z_i, U z_j \rangle = \langle x_i, x_j \rangle = \delta_{i,j}$.

To see that $T = U \circ S$,

$$U S z_i = \lambda_i U z_i = \lambda_i x_i =: T z_i.$$

To see $\det S^2 = \det T^* \circ T$,

$$T^* \circ T = (U \circ S)^* \circ U \circ S = S^* \circ U^* \circ U \circ S = S^2. \quad ☑$$

`Proof of the area formula in case` $T \in \hom(\mathbb{R}^k, \mathbb{R}^d)$

By the polar decomposition, $T = U \circ S$ where $U \in \hom(\mathbb{R}^k, \mathbb{R}^d)$ is orthogonal; and $S \in \hom(\mathbb{R}^k, \mathbb{R}^k)$ is symmetric with $\varDelta(S) = \varDelta(T)$. It follows that

$$\begin{aligned}\sigma_k(TA) &= \frac{m_{\mathbb{R}^k}(B(1))}{2^k} H_{a_k,\|\cdot\|_2}(TA)\\ &= \frac{m_{\mathbb{R}^k}(B(1))}{2^k} H_{a_k,\|\cdot\|_2}(USA)\\ &= \frac{m_{\mathbb{R}^k}(B(1))}{2^k} H_{a_k,\|\cdot\|_2}(SA) \ \because\ U \text{ is an isometry;}\\ &= m_{\mathbb{R}^k}(SA) \text{ by the norm-spherical and round Hausdorff measure theorems}\\ &= \varDelta(S) m_{\mathbb{R}^k}(A) \text{ by the change of variables formula}\\ &= \varDelta(T) m_{\mathbb{R}^k}(A). \quad \square\end{aligned}$$

`Proof of the area formula in general:`

As before, let

$$U_\nabla := \{x \in U :\ T \texttt{ diffble. at } x\},$$

then $m(U \setminus U_\nabla) = 0$.

Suppose that $T : U \to TU$ is a M-quasi-isometry, then

- $\|DT(x)h\|_2 = M^{\pm 1}\|h\|_2\ \forall\ x \in U_\nabla,\ h \in \mathbb{R}^k$.

Let $A \in \mathcal{B}(U),\ 0 < m(A) < \infty,\ A \subset U_\nabla$ and fix $\varepsilon \in (0,1)$.

- By Luzin's theorem and the uniform differentiation lemma, $\exists\ F \subset A$ closed and $\eta > 0$ such that

(i) $\sigma_k(T(A \setminus F)),\ \int_{A \setminus F} \varDelta(DT)dm < \varepsilon$;

(ii) for $y, z \in F,\ \|y - z\|_2 \le \eta$,

$$\|DT(y)v - DT(z)v\|_2 \le \frac{\varepsilon}{2}\|v\|_2\ \forall\ v \in \mathbb{R}^d,\ \&$$
$$|\log \varDelta(DT(y)) - \log \varDelta(DT(z))| < \varepsilon;$$

and such that for $x \in F,\ \|h\| < \eta$:

(iii) $\|T(x+h) - T(x) - DT(x)h\|_2 < \frac{\varepsilon}{2M}\|h\|_2 \le \frac{\varepsilon}{2}\|DT(x)h\|_2$,
whence (!)

(iv) $\|T(x+h) - T(x)\|_2 = (1 \pm \frac{\varepsilon}{2})\|DT(x)h\|_2$.

- For $x \in F,\ y, z \in B(x,\eta) \cap F$,

$$\begin{aligned}\|T(y) - T(z)\|_2 &\overset{(iv)}{=} (1 \pm \frac{\varepsilon}{2})\|DT(z)(y-z)\|_2\\ &\overset{(ii)}{=} (1 \pm \frac{\varepsilon}{2})^2\|DT(x)(y-z)\|_2\\ &= e^{\pm 2\varepsilon}\|DT(x)(y-z)\|_2.\end{aligned}$$

- Now fix $x \in F,\ 0 < \delta < \eta$. We claim that

$$R := T \circ DT(x)^{-1} : DT(x)(B(x,\delta) \cap F) \to T(B(x,\delta) \cap F)$$

is a $e^{2\varepsilon}$-quasi-isometry.

`Proof`

$$\begin{aligned}\|R(DT(x)y) - R(DT(x)z)\|_2 &= \|T(y) - T(z)\|_2\\ &= e^{\pm 2\varepsilon}\|DT(x)y - DT(x)z\|_2.\ ☑\end{aligned}$$

By the quasi-isometry lemma, (ii) and the area formula for linear transformations, $\forall\ C \in \mathcal{B}(B(x,\delta) \cap F)$,

$$\begin{aligned}\sigma_k(TC) = \sigma_k(RDT(x)C) &= e^{\pm 2k\varepsilon}\sigma_k(DT(x)C)\\ &= e^{\pm 2k\varepsilon}\varDelta(DT(x))m(C)\\ &= e^{\pm 4k\varepsilon}\int_C \varDelta(DT)dm.\end{aligned}$$

- Now fix G open, $A \subset G \subset U$, $m(G \setminus F) < \frac{\varepsilon}{M}$.

$$\mathcal{C} := \{B(x,\delta) :\ x \in F,\ 0 < \delta < \eta,\ B(x,\delta) \subset G\}.$$

- By Vitali's covering theorem, $\exists\ B_k = B(x_k, r_k) \in \mathcal{C}\ (k \ge 1)$ disjoint st $F \subset H := \biguplus_{k=1}^\infty B_k \mod m_k$ and

$$\sigma_k(T(G \setminus F)),\quad \int_{G \setminus F} \varDelta(DT)dm < \varepsilon.$$

$$\begin{aligned}\sigma_k(TF) &= \sum_{n=1}^\infty \sigma_k(T(B_n \cap F))\\ &= e^{\pm 4k\varepsilon}\sum_{n=1}^\infty \int_{B_n \cap F} \varDelta(DT)dm\\ &= e^{\pm 4k\varepsilon}\int_F \varDelta(DT)dm.\end{aligned}$$

We obtain $\sigma_k(TA) \pm \varepsilon = e^{\pm 4k\varepsilon}(\int_A \varDelta(DT)dm \pm \varepsilon)$. The area formula follows as $\varepsilon \to 0$. ☑

Exercise №11

1. Hausdorff dimension of the Cantor set.

Let $\Omega = \{0,1\}^{\mathbb{N}}$ and let $\pi : \Omega \to [0,1]$ be defined by $\pi(\omega) := \sum_{n=1}^{\infty} \frac{2\omega_n}{3^n}$. The set $C := \pi(\Omega)$ is called the *Cantor set*.

In this exercise, you show that

$$\text{H-dim}(C) = \frac{\log 2}{\log 3}.$$

For $s > 0$, let $H_s = H_{a_s}$ on $\mathbb{R}$ equipped with the metric $d(x,y) = |x-y|$ where $a_s(t) = t^s$ and let $h = \frac{\log 2}{\log 3}$.

(i) Show that $H_h(C) \le 1$.

`Hint:` $\forall\ n \ge 1$, $C \subset \bigcup_{\underline{\varepsilon} \in \{0,2\}^n} I(\underline{\varepsilon})$ where for $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1,2\}^n$, $I(\underline{\varepsilon}) = [x(\underline{\varepsilon}), x(\underline{\varepsilon}) + \frac{1}{3^n}]$, $x(\underline{\varepsilon}) = \sum_{k=1}^{n} \frac{\varepsilon_k}{3^k}$.

Let $P = \prod(\frac{1}{2}, \frac{1}{2}) \in \mathcal{P}(\Omega)$ be symmetric product measure and define *Cantor-Lebesgue* measure $\mu := P \circ \pi^{-1} \in \mathcal{P}(C)$.

(ii) Show that $\mu(I) \le 4|I|^h$ for any subinterval $I \subset [0,1]$.

(iii) Show that $H_h(C) \ge \frac{1}{4}$.

(iv) Show that H-dim$(C) = h$.

2. Cantor-Lebesgue theorem.

In this exercise, you continue the previous exercise and show that $H_h|_C \equiv \mu$.

The first step is to show that

¶1 $H_h(C) = 1$

The *order* of an interval $I(\underline{\varepsilon})$ is n where $\underline{\varepsilon} \in \{0,1,2\}^n$, and the interval $I(\underline{\varepsilon})$ is *basic* if $\underline{\varepsilon} \in \{0,2\}^n$ and *complementary* otherwise.

Call an interval $[a,b]$ *compound-basic* if $\exists\ n, n' \ge 1$, $\underline{\varepsilon} \in \{0,2\}^n$, $\underline{\varepsilon}' \in \{0,2\}^{n'}$ so that $a = x(\underline{\varepsilon})$ and $b = x(\underline{\varepsilon}') + \frac{1}{3^{n'}}$.

a) Show that if $I \subset [0,1]$ is a compound-basic interval, then $I \cap C = \biguplus_{n=1}^{N} I(\varepsilon^{(n)}) \cap C$ where $I(\varepsilon^{(1)}), \dots, I(\varepsilon^{(N)})$ are basic intervals (not necessarily of the same order).

We'll need such a decomposition together with $|I|^h \ge \sum_{n=1}^{N} |I_n|^h$.

b) For $I \subset [0,1]$ a compound-basic interval, let $\#(I)$ be the minimal $N \ge 1$ so that $I \cap C = \biguplus_{n=1}^{N} I_n$ where $I_1, \dots, I_N$ are basic intervals. Show that a compound-basic interval I is basic iff $\#(I) = 1$.

c) Show that if $I = [a,d] \subset [0,1]$ is a compound-basic interval with $\#(I) > 1$, $I \cap C = (J \cup K) \cap C$ where $J = [a,b]$ and $K = [c,d]$ are compound-basic intervals with $\#(J)$, $\#(K) < \#(I)$ and $[b,c]$ is a complementary interval with $c - b \ge (b-a)$, $(d-c)$. Show that in this case $|I|^h \ge |J|^h + |K|^h$.

`Hint`: Choose a complementary interval of maximal length contained in I....

d) Show that if $I \subset [0,1]$ is a compound-basic interval, then $\exists\ I_1, \dots, I_N$ basic intervals so that $I \cap C = \dot\bigcup_{n=1}^N I_n \cap C$ and $|I|^h \ge \sum_{n=1}^N |I_n|^h$.

`Hint`: Use c) successively to reduce $\#(\cdot)$.

e) Show that if $I \subset [0,1]$ is a compound-basic interval, then $\forall\ k \ge 1$ large enough, $I \cap C = \dot\bigcup_{n=1}^{N_k} I_n(k) \cap C$ where $I_1(k), \dots, I_{N_k}(k)$ are basic intervals of order k and $|I|^h \ge \sum_{n=1}^{N_k} |I_n(k)|^h$.

`Hint`: If $I = [a,d]$ is basic of order ν then $I \cap C = (J \cup K) \cap C$ where $J = [a,b]$ and $K = [c,d]$ are basic of order $\nu+1$, $b-a = c-b = d-c$ and $|I|^h = |J|^h + |K|^h$.

f) Show that if $I_1, I_2, \dots$ are intervals, $C \subset \bigcup_{k=1}^\infty I_k$, then $\sum_{k=1}^\infty |I_k|^h \ge 1$. Conclude that $H_h(C) = 1$.

`Hint`: Show first that WLOG, there are only finitely many I_k's and that (WLOG) these are compound-basic. Then "split" using e).

¶2 $H_h|_C = \mu$.

a) Show that $H_h(C \cap I(\underline{\varepsilon})) = \frac{1}{2^n}\ \forall\ n \ge 1,\ \underline{\varepsilon} \in \{0,2\}^n$.

`Hint`: $\varphi_{\underline{\varepsilon}}(C) = I(\underline{\varepsilon}) \cap C\ \ \forall\ \underline{\varepsilon} \in \{0,2\}^n$ where $\varphi_{\underline{\varepsilon}}(y) := x(\underline{\varepsilon}) + \frac{y}{3^n}$.

b) Show that $H_h(A) = \mu(A)\ \forall\ A \in \mathcal{B}(C)$.

End of coursenotes