ON HERMAN'S THEOREM FOR ERGODIC, AMENABLE GROUP EXTENSIONS OF ENDOMORPHISMS

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ABSTRACT. For any ergodic endomorphism of a nonatomic probability space and any Borel generating set of a LCP amenable group, we construct a measurable function taking values in the generating set so that the skew product generated is ergodic.

§1 INTRODUCTION

Let $(X, \mathcal{B}, m, T)$ be an invertible ergodic measure preserving transformation of a standard probability space.

Let $\mathbb{G}$ be a locally compact, Polish (LCP) group. Given $\phi : X \to \mathbb{G}$ measurable, define the $\mathbb{G}$-*skew product* $T_\phi : X \times \mathbb{G} \to X \times \mathbb{G}$ by $T_\phi(x, y) := (Tx, \phi(x)y)$. This preserves the product measure $m \times m_{\mathbb{G}}$ (where $m_{\mathbb{G}}$ denotes a left Haar measure on $\mathbb{G}$) and it is natural to ask when T_ϕ is ergodic, and also for which LCP groups $\mathbb{G}$ is there an ergodic $\mathbb{G}$-skew product, which latter question is the basis for the present paper.

Results of R. Zimmer (in [14]) and M. Herman (in [9]) show that there is an $\mathbb{G}$-skew product ergodic with respect to the product measure $m \times m_{\mathbb{G}}$ if and only if $\mathbb{G}$ is *amenable.*

Here are 3 equivalent defining conditions for amenability of a LCP topological group $\mathbb{G}$:

1) Every continuous action on a compact metric space has an invariant probability;

2) $\exists$ compact sets $F_n \subset \mathbb{G}$ such that $\frac{|F_n \Delta g F_n|}{|F_n|} \to 0\ \forall\ g \in \mathbb{G}$ where $|\cdot|$ denotes a left Haar measure on $\mathbb{G}$;

3) $\exists$ a $\mathbb{G}$-invariant *mean* (i.e. a positive linear functional on $L^\infty(\mathbb{G})$).

The sets $\{F_n\}$ appearing in 2) are called *Følner sets* (see [6]).

Zimmer's Theorem [14]

1991 *Mathematics Subject Classification.* 28D 37A.

Key words and phrases. measure preserving transformation, ergodic, cocycle, amenable group.

Let $\mathbb{G}$ be an LCP topological group. If $\exists\ \phi : X \to \mathbb{G}$-measurable, such that the skew product T_ϕ is ergodic with respect to the product measure $m \times m_{\mathbb{G}}$, then $\mathbb{G}$ is amenable.

The converse is a well-known result of M. Herman in orbital ergodic theory:

Herman's Theorem [9], [7], [8]

Let $\mathbb{G}$ be an LCP, amenable topological group. $\exists\ \phi : X \to \mathbb{G}$-measurable, such that the skew product T_ϕ is ergodic with respect to the product measure $m \times m_{\mathbb{G}}$.

At the end of this introduction, we'll sketch a proof of Zimmer's result using random transformations (based on a conversation with J-P. Conze); and a proof of Herman's theorem for unimodular, amenable groups (an "orbit" proof along the lines of [8]).

The purpose of this note is to extend Herman's theorem in two directions. The first is replacing the base transformation by a possibly non-invertible map. Since ergodicity is preserved by orbit equivalence, once Herman's theorem is established for a single invertible, ergodic, probability preserving transformation, Dye's theorem implies that it holds for all. This ceases to be true for endomorphisms and so a construction has to be provided for each one. Using the natural extension of an endomorphism we get an invertible transformation and so the problem becomes one of constructing a cocycle generated by a ϕ which is measurable with respect to an increasing $\sigma-$algebra.

Our second refinement concerns the nature of ϕ itself. If one uses Dye's theorem to go from one to another then no special property of ϕ will be preserved. Our construction will provide a ϕ which takes values in a *generator set* (i.e. a Borel set $S \subset \mathbb{G}$ such that $\overline{\text{Group}(S)} = \mathbb{G}$) . Even for $\mathbb{G} = \mathbb{Z}$ and invertible transformations this result appears to be new. We combine the two features in our main result:

Endomorphism Theorem

Let $(X, \mathcal{B}, m, T)$ be an invertible, ergodic measure preserving transformation of a standard probability space, let $\mathcal{A} \subset \mathcal{B}$, $T^{-1}\mathcal{A} \subset \mathcal{A}$ be a non-atomic sub-σ-algebra and let $\mathbb{G}$ be a LCP amenable group with Borel generator set $S \subset \mathbb{G}$, then $\exists\ \phi : X \to S$ $\mathcal{A}$-measurable, such that the skew product T_ϕ is ergodic with respect to the product measure $m \times m_{\mathbb{G}}$.

Proof of Zimmer's theorem Let $\mathbb{G}$ be an LCP group and suppose that $\phi : X \to S$ is measurable, such that the skew product T_ϕ is ergodic with respect to the product measure $m \times m_{\mathbb{G}}$.

To show that $\mathbb{G}$ is amenable, it suffices to show whenever $\mathbb{G}$ acts by homeomorphism on a compact metric space Y, there is a $\mathbb{G}$-invariant probability on Y.

Now suppose that $\mathbb{G}$ acts by homeomorphism on the compact metric space Y and consider the skew product $\tau : X \times Y \to X \times Y$ defined by $\tau(x,y) := (Tx, \phi(x)y)$. By theorem 1.5.10 of [3] (c.f. p. 254 of [13]), $\exists\ P \in \mathcal{P}(X \times Y)$ τ-invariant and ergodic such that $P(A \times Y) = m(A)\ \ (A \in \mathcal{B})$.

We may write $P(A \times B) = \int_A P_x(B) dm(x)\ \ (A \in \mathcal{B},\ B \in \mathcal{B}(Y))$ where $x \mapsto P_x$ is measurable $X \to \mathcal{P}(Y)$. The invariance $P \circ \tau = P$ implies that $P_{Tx} = P_x \circ \phi(x)^{-1}$ m-a.e..

We claim that a.e. P_x is invariant for the action of $\mathbb{G}$. To see this, let $Q \in \mathcal{P}(\mathcal{P}(Y))$ be the m-distributution of $P.$, and let $\mu \in \mathcal{P}(Y)$ be in the weak-$*$ closed support of Q. Suppose (in order to prove our claim by contradiction) that μ is not $\mathbb{G}$-invariant. In particular $\exists\ g \in \mathbb{G},\ f \in C(Y)$ such that

$$0 < |\mu(f) - \mu(f \circ g)| =: \epsilon.$$

Since μ is in the weak-$*$ closed support of Q, $\exists\ A \in \mathcal{B}$ with $m(A) > 0$ such that

$$|P_x(f) - \mu(f)|,\ |P_x(f \circ g) - \mu(f \circ g)| < \tfrac{\epsilon}{4}\ \forall\ x \in A.$$

Let $U \subset \mathbb{G}$ be open such that

$$\|f \circ h - f \circ g\|_\infty < \tfrac{\epsilon}{4}\ \forall\ h \in U.$$

Note that

$$T_\varphi^n(x,y) = (T^n x, \phi_n(x)y)$$

where $\phi_n(x) = \phi(T^{n-1}x) \cdots \phi(x)$. By ergodicity of T_ϕ, $\exists\ n \ge 1$ such that $m(A \cap T^{-n}A \cap [\phi_n \in U]) > 0$. Choosing $x \in A \cap T^{-n}A \cap [\phi_n \in U]$ we see that on the one hand $T^n x \in A$ whence

$$|P_{T^n x}(f) - \mu(f)| < \tfrac{\epsilon}{4};$$

whereas on the other hand, $\phi_n \in U$ and

$$P_{T^n x}(f) = P_x(f \circ \phi_n(x)) = \mu(f \circ g) \pm \tfrac{\epsilon}{4},$$

whence

$$\tfrac{\epsilon}{4} > |P_{T^n x}(f) - \mu(f)| \ge |\mu(f) - \mu(f \circ g)| - |P_{T^n x}(f) - \mu(f \circ g)| > \tfrac{3\epsilon}{4}.$$

This is the advertised contradiction. □

Proof of Herman's theorem in the unimodular case

Let $(X, \mathcal{B}, m, T)$ be an invertible, ergodic measure preserving transformation of a standard probability space and let $\mathbb{G}$ be a LCP, unimodular, amenable group. We'll show $\exists\ \phi : X \to \mathbb{G}$ measurable, such that the skew product $T_\phi : X\times\mathbb{G}\to X\times\mathbb{G}$ defined by $T_\phi(x,y) := (Tx,\phi(x)y)$ is ergodic with respect to the product measure $m\times m_{\mathbb{G}}$.

In case $\mathbb{G}$ is discrete, let $Y = \{0,1\}^{\mathbb{G}}$, $\mu := \prod \frac{1}{2}(\delta_0+\delta_1)$ and let, for $g\in\mathbb{G}$, $S_g : Y\to Y$ be defined by $(S_g y)_h := y_{g^{-1}h}$, then S is free, ergodic and μ-preserving.

In case $\mathbb{G}$ is non-discrete, consider (as in [10]) the action S of $\mathbb{G}$ by left translation on the Poisson space $(Y,\mathcal{F},\mu)$ of $\mathbb{G}$ equipped with left Haar measure. As shown in [10] (for $\mathbb{G}=\mathbb{Z}$), S is mixing (hence ergodic).

Let $Z := Y^{\mathbb{Z}}$, $m := \prod \mu$ and define: for $g\in\mathbb{G}$, $\overline{S}_g : Y\to Y$ by $(\overline{S}_g z)(n)_h := y(n)_{g^{-1}h}$ and $\tau : Z\to Z$ by $(\tau z)(n) = z(n+1)$, then $\overline{S}$ and τ are both m-preserving, and τ is ergodic. Moreover $\overline{S}_\gamma\circ\tau = \tau\circ\overline{S}_\gamma$.

Thus, the group $\mathbb{H} := \mathbb{G}\times\mathbb{Z}$ is amenable, and acts freely, ergodically and
m-preservingly on Z by $R_{(\gamma,\kappa)} := \overline{S}_\gamma\circ\tau^\kappa$.

Now define the R-cocycle $\psi : H\times Z\to\mathbb{G}$ by $\psi((\gamma,n),z) := \gamma$ and consider the $\mathbb{H}$-action R_ψ on $Z\times\mathbb{G}$ defined by

$$(R_\psi)_{(\gamma,\kappa)}(z,g) := (R_{(\gamma,\kappa)}(z,\psi(z,(\gamma,\kappa)g) = (\overline{S}_\gamma\circ\tau^\kappa z,\gamma g).$$

The action R_ψ is ergodic, since if $f : Z\times\mathbb{G}\to\mathbb{R}$ is measurable, R_ψ-invariant:

$$f(z,g) = f\circ(R_\psi)_{(e,1)}(z,g) = f(\tau z,g)\ \forall\ g\in\mathbb{G}$$

whence $f(z,g) = F(g)$ and $F(\gamma g) = f\circ(R_\psi)_{(\gamma,0)}(z,g) = f(z,g) = F(g)$ with the conclusion that f is constant.

In case $\mathbb{G}$ is discrete, by [4], $\exists$ a probability space isomorphism $\pi : X\to Z$ so that for a.e. $x\in X$,

$$\{\pi\circ T^n x :\ n\in\mathbb{Z}\} = \{R_h\circ\pi x :\ h\in\mathbb{H}\}.$$

In particular $\exists$ a T-cocycle $\varphi : X\times\mathbb{Z}\to\mathbb{H}$ so that $\pi\circ T^n(x) = R_{\varphi(n,x)}(\pi x)$ and $\{\varphi(n,x) :\ n\in\mathbb{Z}\} = \mathbb{H}$ a.e.. Let $\phi : X\to\mathbb{G}$ be defined by

$$\phi(x) := \psi(\varphi(1,x),\pi x).$$

We claim that $T_\phi : X\times\mathbb{G}\to X\times\mathbb{G}$ is ergodic. This follows from the ergodicity of R_ψ via

$$\begin{aligned}\{T_\phi^n(x,g) :\ n\in\mathbb{Z}\} &= \{(\pi^{-1}R_{\varphi(n,x)}(\pi x),\psi(\varphi(n,x),\pi x)g) :\ n\in\mathbb{Z}\}\\ &= \{\pi^{-1}R_h(\pi x),\psi(h,\pi x)g) :\ h\in\mathbb{H}\}.\end{aligned}$$

In case $\mathbb{G}$ is non-discrete, by [11], $\exists$ a cross section $\Sigma\in\mathcal{B}(Z)$ for the action of $\mathbb{H}$ so that the induced equivalence relation $\mathcal{R} := \{(x,\gamma x) :\ \gamma\in$

$\mathbb{H},\ x,\gamma x\in\Sigma\}$ is probability preserving and has countable equivalence classes. Evidently the skew product equivalence relation

$$\mathcal{R}_\psi := \{((x,y),(\gamma x,\psi(\gamma,x)y) :\ \gamma\in\mathbb{H},\ x,\gamma x\in\Sigma,\ y\in\mathbb{G}\}$$

is also ergodic, $\Sigma\times\mathbb{G}$ being a cross section for the skew product action of $\mathbb{H}$.

It is shown in [4] that $\mathcal{R}$ is hyperfinite, whence $\exists$ a probability space isomorphism $\pi : X\to\Sigma$ so that for a.e. $x\in X$,

$$\{\pi\circ T^n x :\ n\in\mathbb{Z}\} = \mathcal{R}_x := \{\gamma\pi x :\ \gamma\in\mathbb{H},\ x,\gamma x\in\Sigma\}.$$

In particular $\exists$ a T-cocycle $\varphi : X\times\mathbb{Z}\to\mathbb{H}$ so that $\pi\circ T^n(x) = \varphi(n,x)(\pi x)$. Let $\phi : X\to\mathbb{G}$ be defined by

$$\phi(x) := \psi(\varphi(1,x),\pi x).$$

As before, $T_\phi : X\times\mathbb{G}\to X\times\mathbb{G}$ is ergodic. ☐

The reason that the above proof doesn't work for non-unimodular groups is that the equivalence relation that the G orbits induce on the cross section is not of type II_1 unless the group is unimodular. We owe this remark to N. Avni.

§2 Proof of the endomorphism theorem

Let T be an invertible, ergodic probability preserving transformation of the standard probability space $(X,\mathcal{B},m)$, let $\mathcal{A}\subset\mathcal{B},\ T^{-1}\mathcal{A}\subset\mathcal{A}$ be non-atomic sub-σ-algebra and let $\mathbb{G}$ be a locally compact, second countable, amenable topological group with Borel generator set S. We'll construct $\phi : X\to S$ $\mathcal{A}$-measurable with T_ϕ ergodic.

We adapt here from §3 of [2] the *essential value conditions* or EVC's, which give countably many conditions for the ergodicity of the skew product $T_\varphi : X\times\mathbb{G}\to X\times\mathbb{G}$ defined by $T\varphi(x,y) := (Tx,\varphi(x)y)$.

These are best understood in terms of orbit cocycles, and the *groupoid* of T (see [5]).

A *partial probability preserving transformation* of X is a pair (R,A) where $A\in\mathcal{B}$ and $R : A\to RA$ is invertible and $m|_{RA}\circ R^{-1} = m|_A$. The set A is called the *domain* of (R,A). We'll sometimes abuse this notation by writing $R = (R,A)$ and $A = \mathcal{D}(R)$. Similarly, the *image* of (R,A) is the set $\mathfrak{I}(R) = RA$.

The *equivalence relation* generated by T is

$$\mathcal{R} = \{(x,T^n x) : x\in X,\ n\in\mathbb{Z}\}.$$

For $A\in\mathcal{B}(X)$ and $\phi : A\to\mathbb{Z}$, define $T^\phi : A\to X$ by $T^\phi(x) := T^{\phi(x)}x$.

The *groupoid* of T is

$$[T] = \{T^\phi :\ T^\phi \text{ is a partial probability preserving transformation}\}.$$

It's not hard to see that

$[T]=\{R:R$ a partial probability preserving transformation, & $(x,Rx)\in\mathcal{R}$ a.e.$\}$.

For $R=T^\phi\in[T]$, write $\phi^{(R)}:=\phi$. Let

$$[T]_+=\{R\in[T]:\ \phi^{(R)}\ge 1\text{ a.e.}\}.$$

The skew product action of $[T]$ is given by

$$R_\varphi(x,y):=(Rx,\varphi_R(x))\ \text{ where } \varphi_R(x):=\varphi(\phi(x),x)\ \ (R=T^\phi\in[T]).$$

Definition

Let $A\in\mathcal{B}$, U a subset of $\mathbb{G}$, and $c>0$. We say that the measurable cocycle $\varphi:X\to\mathbb{G}$ satisfies $\mathrm{EVC}_T(U,c,A)$ if
$\exists\ R\in[T]_+$ such that

$$\mathcal{D}(R),\ \mathfrak{I}(R)\subset A,\ \varphi_R\in U\text{ on }\mathcal{D}(R),\ m(\mathcal{D}(R)))>cm(A).$$

The following can be extracted from [12] (see also §8.2 of [1] and §3 of [2]).

Ergodicity Proposition

The skew product T_φ is ergodic with respect to the product measure $m\times m_{\mathbb{G}}$ iff $\exists$
- *a neighbourhood base $\mathcal{U}$ for $\mathbb{G}$;*
- *a dense collection $\mathcal{A}\subset\mathcal{B}$;*

and a number $0<c<1$ such that
φ satisfies $EVC_T(U,c,a)\ \forall\ a\in\mathcal{A},\ U\in\mathcal{U}$.

Sketch proof of "if"

Need to prove (see [12]) that $\forall$ neighbourhoods $B\subset\mathbb{G}$ and $A\in\mathcal{B}$, $m(A)>0$, $\exists\ n\in\mathbb{Z}$ such that $m(A\cap T^{-n}A\cap[\varphi_{T^n}\in B])>0$.

Accordingly, assume that φ satisfies $\mathrm{EVC}_T(U,c,a)\ \forall\ a\in\mathcal{A},\ U\in\mathcal{U}$ and let $B\subset\mathbb{G}$ be nonempty and open, and let $A\in\mathcal{B},\ m(A)>0$.

$\exists\ U\in\mathcal{U}$ such that $U\subset B$ and $\exists\ a\in\mathcal{B}$ such that $m(A\Delta a)<\frac{c}{2}m(a)$.

By assumption, $\exists\ R\in[T]_+$ such that $\mathcal{D}(R),\ \mathfrak{I}(R)\subset a,\ \varphi_R\in U$ on $\mathcal{D}(R)$ and $m(\mathcal{D}(R)))>cm(a)$. We have that

$$m(\mathcal{D}(R)\cap A)=m(\mathcal{D}(R))-m(\mathcal{D}(R)\setminus A)\ge m(\mathcal{D}(R))-m(a\setminus A)\ge\tfrac{c}{2}m(a).$$

It follows that $\exists\ n\in\mathbb{Z}$ such that $m(A\cap[\phi^{(R)}=n])>0$, whence

$$m(A\cap T^{-n}A\cap[\varphi_{T^n}\in U])\ge m(A\cap T^{-n}A\cap[\varphi_{T^n}\in U])\ge m(A\cap[\phi^{(R)}=n])>0.$$

$\square$

Essential value conditions are impervious to small changes.

Stability Lemma (c.f. lemma 3.5 of [2]**)**

If $\psi : X \to \mathbb{G}$ is a cocycle satisfying $EVC_T(U, c, A)$ where $A \in \mathcal{B}$, $c > 0$, $U \subset \mathbb{G}$; then $\exists\ \delta > 0$ such that if $\varphi : X \to \mathbb{G}$ is measurable, and $m([\varphi \neq \psi]) < \delta$ then φ satisfies $EVC_T(U, c, A)$.

Proof By possibly restricting $\mathcal{D}(R)$, we ensure that $\exists\ N \in \mathbb{N}$ such that $\phi^{(R)} \le N$, but still $m(\mathcal{D}(R)) = cm(A) + \eta$ where $\eta > 0$. Let $\delta := \frac{\eta}{2N}$.

If $\varphi : X \to \mathbb{G}$ is measurable, and $m([\varphi \neq \psi]) < \delta$ then $m([\varphi_R \neq \psi_R]) < \delta N = \eta$ and further restricting $\mathcal{D}(R)$ to $\mathcal{D}(R) \smallsetminus [\varphi_R \neq \psi_R]$ shows that φ satisfies $\mathrm{EVC}_T(U, c, A)$ since

$$m(\mathcal{D}(R) \smallsetminus [\varphi_R \neq \psi_R]) \ge m(\mathcal{D}(R)) - m([\varphi_R \neq \psi_R]) > cm(A).$$

$\square$

The construction of ϕ is sequential, and at each stage, we'll have an $\mathcal{A}$-measurable coboundary satisfying finitely many EVC's. The coboundary at the next stage will be constructed using the inductive lemma (below) sufficiently close to the present coboundary so as not to affect any of the EVC's already satisfied (by the stability lemma), and will itself satisfy a new (arbitrary) EVC.

The closeness of approximation will also ensure a.s. convergence of the coboundaries to a limit cocycle which will satisfy all the EVC's. The arbitariness of the EVC's means that a countable list may be chosen to ensure ergodicity (by the ergodicity proposition). Modification is by means of a Rokhlin tower, which can be arranged to be factor measurable thus ensuring preservation of $\mathcal{A}$-measurability of the successive coboundaries.

Inductive Lemma

Let $S \in \mathcal{B}(\mathbb{G})$ be a generator set and suppose that $h \ge 1$, $B \in T^{-h}\mathcal{A}$, $m(\biguplus_{j=0}^{h-1} T^j B) < 1$, and that $\phi : X \to S$ is an $\mathcal{A}$-measurable coboundary taking finitely many values and satisfying $\phi \equiv e$ off $\biguplus_{j=0}^{h-1} T^j B$ and $\phi_h \equiv e$ on B.

If $A \in \mathcal{B}$, $\varnothing \neq U \subset \mathbb{G}$ is open and $\epsilon > 0$, then $\exists\ \hat{h} \ge 1$, $\hat{B} \in T^{-3\hat{h}}\mathcal{A}$, $m(\biguplus_{j=0}^{\hat{h}-1} T^j \hat{B}) < 1$ and $\exists$ an $\mathcal{A}$-measurable, coboundary $\hat{\phi} : X \to S$ taking finitely many values and satisfying $EVC_T(U, \frac{1}{3}, A)$; $m(\hat{\phi} \neq \phi) < \epsilon$; $\hat{\phi} \equiv e$ off $\biguplus_{j=0}^{\hat{h}-1} T^j \hat{B}$;

and $\hat{\phi}_{\hat{h}} \equiv e$ on $\hat{B}$.

The proof of the inductive lemma is given in the next (and last) section. We complete this section with the

Proof of the endomorphism theorem, given the inductive lemma

Let $\mathcal{C} \subset \mathcal{B}$ be a countable, dense sub-collection, let $\mathcal{U}$ be a countable neighbourhood base for $\mathbb{G}$ and let $\{(A_n, U_n)\}_{n\ge1} = \mathcal{C} \times \mathcal{U}$.

We construct a sequence of $\mathcal{A}$-measurable coboundaries $\phi^{(r)}\ \ (r \ge 1)$ defined on $\mathcal{T}_r := \biguplus_{j=0}^{h_r-1} T^j B_r \in \mathcal{A}$, so that each $\phi^{(r)}$ satisfies $\mathrm{EVC}_T(U_q, \frac{1}{3}, A_q)$ for $1 \le q \le r$.

Indeed, given such $\phi^{(r)}$ on $\mathcal{T}_r$ satisfying $\mathrm{EVC}_T(U_q, \frac{1}{3}, A_q)$ for $1 \le q \le r$, we find by the stability lemma, $0 < \delta_r < \frac{1}{2^r}$ such that if $\varphi : X \to \mathbb{G}$ is measurable, and $m([\varphi \ne \phi^{(r)}]) < \delta_r$ then φ satisfies $\mathrm{EVC}_T(U_q, \frac{1}{3}, A_q)$ for $1 \le q \le r$.

By the inductive lemma, $\exists\ \ h_{r+1} \ge 1,\ B_{r+1} \in T^{-3h_{r+1}}\mathcal{A}$ such that

$$m(\biguplus_{j=0}^{\hat{h}_{r+1}-1} T^j B_{r+1}) < 1,$$

and $\exists$ a $\mathcal{A}$-measurable, coboundary $\phi^{(r+1)} : X \to S$ such that $\phi^{(r+1)} \equiv e$ off $\mathcal{T}_{r+1} := \biguplus_{j=0}^{\hat{h}_{r+1}-1} T^j B_{r+1}$, $\phi^{(r+1)}_{h_{r+1}} \equiv e$ on B_{r+1}, $m(\phi^{(r+1)} \ne \phi^{(r)}) < \epsilon$ and satisfying $\mathrm{EVC}_T(U_{r+1}, \frac{1}{3}, A_{r+1})$.

Since $\sum_{r\ge1} \delta_r < \infty$, $\exists\ \phi : X \to S$ such that for a.e. $x \in X$, $\phi^{(r)}(x) = \phi(x)\ \forall$ large r. It follows that ϕ is $\mathcal{A}$-measurable, and satisfies $\mathrm{EVC}_T(U_q, \frac{1}{3}, A_q)\ \forall\ q \ge 1$. ☐

§3 Proof of the Inductive lemma

Let d be a metric on $\mathbb{G}$. Fix $\sigma \in U$ and $\delta > 0$ such that $B(\sigma, \delta) := \{y \in \mathbb{G} :\ d(y, \sigma) < \delta\} \subset U$. Let $V := \{\phi_j(x) :\ x \in B,\ 0 \le j \le h\} \subset S^h$, $W := \{v\sigma v^{-1} :\ v \in V\}$.

For $v \in V$ set

$$A^{(v)} := \begin{cases} \biguplus_{j=0}^{h-1} \{x \in A \cap T^j B :\ \phi_j(T^{-j}x) = v\} & (v \ne e), \\ \biguplus_{j=0}^{h-1} \{x \in A \cap T^j B :\ \phi_j(T^{-j}x) = e\} \cup X \setminus \biguplus_{j=0}^{h-1} T^j B & (v = e). \end{cases}$$

Set

$$\eta := \min\{m(A^{(v)}) :\ v \in V,\ m(A^{(v)}) > 0\}.$$

By amenability, $\exists\ F \subset \mathbb{G}$ compact, such that

$$|wF \cap F| > (1 - \tfrac{\epsilon\eta}{4})|F|\ \forall\ w \in W.$$

We now fix some parameters for the construction:

- $\exists\ \delta_1 > 0$ such that if $y, z \in F,\ v \in V$ satisfy $d(y, v\sigma v^{-1}z) < \delta_1$, then $d(v^{-1}yz^{-1}v, \sigma) < \frac{\delta}{4}$,

  and
- $\exists\ K \ge 1$ such that $\forall\ x \in F^{-1}F \cup FF^{-1}$, $\exists\ u_1, u_2, \ldots, u_K \in S \cup \{e\}$ such that $d(y, u_1u_2, \ldots u_K) < \frac{\delta}{4}$.

Fix $N_1 \ge 1$ such that $m(D) > 1 - (\epsilon\eta)^4$ where

$$D := \bigcap_{v\in V} D_v := \bigcap_{v\in V}\Big[\big|\frac{1}{N_1}\sum_{k=0}^{N_1-1} 1_{A^{(v)}}\circ T^k - m(A^{(v)})\big| < \epsilon\eta\Big].$$

Let $\mathcal{F}$ be a finite partition of F into Borel sets, each of diameter small enough so that:

$$\operatorname{diam}(wf) < \delta_1 \ \forall\ f\in\mathcal{F},\ w\in W\cup\{e\},$$

and also

$$|\bigcup_{f\in\mathcal{F},\ wf\subset F} f| > (1-\tfrac{\epsilon\eta}{2})|F| \ \forall\ w\in W.$$

By considering iidrv's on F (distributed according to $\frac{|F\cap\cdot|}{|F|}$), construct $(y_1,y_2,\dots)\in F^{\mathbb{N}}$ and $N_2 > N_1$ such that

$$\tfrac{1}{n}\sum_{k=1}^{n} 1_{wf}(y_k) = (1\pm\epsilon\eta)\tfrac{|f|}{|F|}\ \forall\ f\in\mathcal{F},\ n\ge N_2,\ w\in W\cup\{e\} \text{ whenever } wf\subset F.$$

Fix $L > \frac{N_2K}{\epsilon\eta},\ M > \frac{2|F|^2}{\epsilon\eta}$, set $\hat{h} := LM$ and choose $\hat{B}\in T^{-3\hat{h}}\mathcal{A},\ 1-\frac{\epsilon}{h} < m(\biguplus_{j=0}^{\hat{h}-1} T^j\hat{B}) < 1$. It follows that

$$(\bigstar)\qquad m(\hat{B}\cap\big[\tfrac{1}{\hat{h}}\sum_{i=0}^{\hat{h}-1} 1_D\circ T^i < 1-(\epsilon\eta)^2\big]) < (\epsilon\eta)^2 m(\hat{B}).$$

Define $\overline{\phi}: X\to\ S$ by $\overline{\phi} = e$ on $E :=$

$$\biguplus_{\ell=0}^{M}\biguplus_{i=-h}^{K}\biguplus_{k=0}^{h-1} T^k(T^{\ell L+i}\hat{B}\cap B)$$

and $\overline{\phi} = \phi$ off E.

Evidently $\overline{\phi}$ is $\mathcal{A}$-measurable, $\overline{\phi}\equiv e$ off $\bigcup_{i=0}^{\hat{h}-1} T^i\hat{B}$, $\overline{\phi}_h\equiv e$ on B, $\overline{\phi}_{\hat{h}}\equiv e$ on $\hat{B}$ and

$$m([\overline{\phi}\ne\phi]) \le \tfrac{K+2h}{L}.$$

Next, we define $\psi : [\overline{\phi} = e]\to\ S$. Set $y_0 = y_M = e$ and define $x_\ell := y_{\ell+1}y_\ell^{-1}$. By construction, $\exists\ \underline{u} = u(x_\ell)\in\ S^K$ such that $x_\ell = u_K\dots u_1$. Set

$$\psi(x) = \begin{cases} u_i(x_\ell) & x\in T^{\ell L+i}\hat{B},\ 0\le\ell<M,\ 0\le i<K;\\ e & \text{else.}\end{cases}$$

Note that ψ is $\mathcal{A}$-measurable and

$$m([\psi\ne e]) \le \tfrac{K+2h}{L} < \epsilon.$$

Lastly, define $\hat{\phi} := \overline{\phi}\psi : X\to\ S$. Evidently $[\hat{\phi}\ne\overline{\phi}] = [\psi\ne e]$, whence

$$m([\hat{\phi}\ne\phi]) \le m([\psi\ne e]) + m([\phi\ne\overline{\phi}]) < 2\epsilon.$$

Also $\hat{\phi}_{\hat{h}} = e$ on $\hat{B}$.

We now show that $\hat{\phi}$ satisfies $\mathrm{EVC}_T(U, \frac{1}{3}, A)$. Namely, for $v \in V$ with $m(A^{(v)}) > 0$, $\exists\ R \in [T]_+$, such that $\mathcal{D}(R)$, $\mathfrak{I}(R) \subset A^{(v)}$, $m(\mathcal{D}(R)) > \frac{m(A^{(v)})}{3}$ and $\hat{\phi}_R \in U$.

To this end, fix v and purify the $(\hat{B}, \hat{h})$-tower with respect to $A^{(v)}$ and D. Let β be the partition of $\hat{B}$ into the bases of the pure columns. By (★),

$$\frac{1}{\hat{h}} \sum_{i=0}^{\hat{h}-1} 1_D \circ T^i \geq 1 - (\epsilon\eta)^2$$

on each $b \in \beta$ except for a subcollection of total mass not more than $(\epsilon\eta)^2 m(\hat{B})$. We'll call such bases $b \in \beta$ *good*.

Next, in a *good column* $(b, Tb, \ldots, T^{\hat{h}-1}b)$ (i.e. where b is good), for each $0 \leq j < M$ except for a subcollection of at most $\epsilon\eta M$,

$$\#\{0 \leq i < L := L : \ T^{jL+i}b \subset D\} > (1 - \epsilon\eta)L,$$

whence

$$\#\{K \leq i < L : \ T^{jL+i}b \subset D\} > (1 - \epsilon\eta)L - K > (1 - 2\epsilon\eta)L.$$

Call such $b_j := (T^{jL+K}b, \ldots, T^{(j+1)L-1}b)$ a *good block*.

The definition of ψ ensures that $\psi_K \equiv u_K(x_j)\ldots u_1(x_j) = x_j$ on $T^{jL}\hat{B}$ where $x_{j-1}x_{j-2}\ldots x_0 = y_j$. If $j' > j, K \leq i, i' < L$,

$$T^{jL+i}\hat{B} \subset A^{(v)},\ T^{j'L+i'}\hat{B} \subset A^{(v)}$$

and $r := (j'L + i') - (jL + i)$, then

$$\hat{\phi}_r = v^{-1}\psi_r v = v^{-1}x_{j'-1}x_{j'-2}\ldots x_j v = v^{-1}y_{j'}y_j^{-1}v \quad \text{on } T^{jL+i}\hat{B}.$$

In order to have $\hat{\phi}_r \in U$ on $T^{jL+i}\hat{B}$, we'll need that $d(v^{-1}y_{j'}y_j^{-1}v, \sigma) < \delta$. As arranged above, this will follow from

$$d(y_{j'}, v\sigma v^{-1}y_j) < \delta_1.$$

The rest of the proof shows that there are enough pairs of good blocks which are matchable in this way, and in them there are enough $A^{(v)}$-levels in order to obtain the required R.

We claim that in a good block b_j, we have

$$\#\{K \leq i < L : \ T^{jL+i}b \subset A^{(v)}\} = (1 \pm 7\epsilon)m(A^{(v)})L.$$

To see this, set $\Omega := \{K \leq i < L : \ T^{jL+i}b \subset D\}$. By the above $\#\Omega > 1 - 2\epsilon\eta)L$. We cover Ω by disjoint $[i_k, i_k + N_1)$ $(k = 1, 2, \ldots)$ so that each $i_k \in \Omega$. It follows that

$$\begin{aligned}\#\{K \le i < L :\ T^{jL+i}b \subset A^{(v)}\} &= \#\{i \in \Omega :\ T^{jL+i}b \subset A^{(v)}\} \pm N_1 \pm (L - \#\Omega)\\ &= \sum_k \#\{i \in [i_k, i_k + N_1) :\ T^i b \subset A^{(v)}\} \pm 3\epsilon\eta L\\ &= \sum_k (1 \pm \epsilon) m(A^{(v)}) \#[i_k, i_k + N_1) \pm 3\epsilon\eta L\\ &= (1 \pm \epsilon) m(A^{(v)})(L \pm 3\epsilon\eta L) \pm 3\epsilon\eta L\\ &= (1 \pm 7\epsilon) m(A^{(v)}) L.\end{aligned}$$

Now set $w := v\sigma v^{-1}$. The construction of F ensured that $|\bigcup_{f \in \mathcal{F}_w} f| > (1 - \epsilon\eta)|F|$ where $\mathcal{F}_w := \{f \in \mathcal{F} :\ wf \subset F\}$.

Using $M > 2N_2$, we have

$$\#\{0 \le j < \tfrac{M}{2} :\ y_j \in f\} = \tfrac{M|f|}{2|F|}(1 \pm \epsilon\eta)\ \forall\ f \in \mathcal{F}$$

and

$$\#\{\tfrac{M}{2} \le j < M :\ y_j \in wf\} = \tfrac{M|f|}{2|F|}(1 \pm \epsilon\eta)\ \forall\ f \in \mathcal{F}_w.$$

It follows from the above that

$$\#\{0 \le j < \tfrac{M}{2} :\ b_j \text{ is good}\},\ \#\{\tfrac{M}{2} \le j < M :\ b_j \text{ is good}\} > \tfrac{M}{2}(1 - 2\epsilon\eta).$$

In order to construct R, we claim that

$$|\{f \in \mathcal{F} :\ \#\{0 \le j < \tfrac{M}{2} :\ b_j \text{ is good, and } y_j \in f\} > \tfrac{3M|f|}{8|F|}\}| \ge (1 - 12\epsilon\eta)|F|$$

and (similarly)

$$|\{f \in \mathcal{F}_w :\ \#\{\tfrac{M}{2} \le j < M :\ b_j \text{ is good, and } y_j \in wf\} > \tfrac{3M|f|}{8|F|}\}| \ge (1 - 12\epsilon\eta)|F|$$

(where, for $\mathcal{J} \subset \mathcal{H}$, $|\mathcal{J}| := \sum_{f \in \mathcal{J}} |f|$).

To see the first, let

$$\mathcal{F}_- := \{f \in \mathcal{F} :\ \#\{0 \le j < \tfrac{M}{2} :\ b_j \text{ is good, and } y_j \in f\} \le \tfrac{3M|f|}{8|F|}\},\ F_- := \bigcup_{f \in \mathcal{F}_-} f,$$

then

$$\begin{aligned}\tfrac{M}{2}(1 - 2\epsilon\eta) &< \#\{0 \le j < \tfrac{M}{2} :\ b_j \text{ is good}\}\\ &= \sum_{f \in \mathcal{F}} \#\{0 \le j < \tfrac{M}{2} :\ b_j \text{ is good, and } y_j \in f\}\\ &= \sum_{f \in \mathcal{F}_-} + \sum_{f \in \mathcal{F} \setminus \mathcal{F}_-}\\ &\le |F_-|\tfrac{3M}{8|F|} + \sum_{f \in F \setminus F_-} \#\{0 \le j < \tfrac{M}{2} :\ y_j \in f\}\\ &\le |F_-|\tfrac{3M}{8|F|} + |F \setminus F_-|\tfrac{M}{2}\tfrac{1}{|F|}(1 + \epsilon\eta)\end{aligned}$$

whence

$$|F_-| < 12\epsilon\eta|F|.$$

It follows that

$$|\{f \in \mathcal{F} : \#\{0 \leq j < \tfrac{M}{2} : b_j \text{ is good, and } y_j \in f\} > \tfrac{3M}{8|F|}\}| \geq (1 - 12\epsilon\eta)|F|$$

and

$$|\{f \in \mathcal{F}_w : \#\{\tfrac{M}{2} \leq j < M : b_j \text{ is good, and } y_j \in wf\} > \tfrac{3M}{8|F|}\}| \geq (1 - 12\epsilon\eta)|F|.$$

Thus, we can match a proportion of $(1 - 12\epsilon\eta) \times \frac{3}{4}$ of the good blocks b_j ($j < \frac{M}{2}$) with good blocks $b_{j'}$ ($j' > \frac{M}{2}$) so that $y_{j'}$ and wy_j are in the same element of $w\mathcal{F}$, whence $d(y_{j'}, wy_j) < \delta_1$. Inside each pair of matched good blocks, we match a proportion of $(1 - 7\epsilon)$ of the $A^{(v)}$ levels. Using these, we define $R \in [R]_+$, $\mathcal{D}(R)$, $\mathfrak{I}(R) \subset A^{(v)}$ with $\hat{\phi}_R \in U$ and

$$m(\mathcal{D}(R)) > (1 - 13\epsilon\eta) \times \tfrac{3}{8}(1 - 7\epsilon)m(A^{(v)}) \overset{?}{>} \tfrac{1}{3}m(A^{(v)}).$$

□

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