TAIL-IN Variant MEASURES FOR SOME SUSPENSION SEMIFLOWS

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Abstract. We consider suspension semiflows over abelian extensions of one-sided mixing subshifts of finite type. Although these are not uniquely ergodic, we identify (in the “ergodic” case) all tail-invariant, locally finite measures which are quasi-invariant for the semiflow.

1. Introduction

1.1. The Tail Relations. We start with some background on equivalence relations, (see [F-M] for more detail). Let \((X, \mathcal{B})\) be a standard Borel space, and let \(R \subseteq X \times X\) be an equivalence relation. Assume that \(R \in \mathcal{B} \otimes \mathcal{B}\), and that each equivalence class \(R(x) := \{ y : (x, y) \in R \}\) is countable. Then for any \(A \in \mathcal{B}\), the saturation \(R(A) = \cup\{R(x) : x \in A\}\) is again a Borel set. A \(\sigma\)-finite measure \(\mu\) on \(X\) is called non-singular for \(R\) if \(\mu(R(A)) = 0\) whenever \(\mu(A) = 0\), and is, in addition, called ergodic if any saturated set \(A = R(A)\) has either zero or full measure.

A Borel isomorphism \(\phi\) defined on some \(A \in \mathcal{B}\) with image \(B \in \mathcal{B}\) is a holonomy if \((x, \phi(x)) \in R\) for any \(x \in A\). A measure \(\mu\) is invariant for \(R\), if it is invariant under all the holonomies of \(R\).

Let \(S\) be a finite set, and let \(\Sigma\) be a subshift of finite type over \(S\):

\[\Sigma := \{ x \in S^\mathbb{N} : \forall k \geq 1, A_{x_k, x_{k+1}} = 1 \}\]

where \(A = (t_{ij})_{S \times S}\) with \(t_{ij} \in \{0, 1\}\). We endow \(\Sigma\) with the topology generated by cylinders \([a_1, \ldots, a_n] := \{ x \in \Sigma : x^n_1 = a^n_1 \}\), where \(x^n_i := (x, \ldots, x_j)\). Note that the collection of cylinders of length \(n\) is exactly \(\alpha_{n-1}^\mathbb{N}\) where \(\alpha := \{ [a] : a \in S\}\). Define the left shift \(T : \Sigma \to \Sigma\) by \((Tx)_i = x_{i+1}\). Let \(\mathcal{P}(\Sigma)\) denote the collection of Borel probability measures on \(\Sigma\).

Henceforth we assume that \((\Sigma, T)\) is topologically mixing. It is well-known that this is equivalent to the existence of \(N_0\) such that all the entries of \(A^{N_0}\) are positive (see [Bo]).

Let \(h : \Sigma \to \mathbb{R}_+, \ f : \Sigma \to \mathbb{Z}^d\) be H"older continuous. Set

\[\Sigma^h := \{ (x, s) : x \in \Sigma, 0 \leq s < h(x) \}\]

and define the semiflows \(g_t : \Sigma^h \to \Sigma^h\) and \(G_t : \Sigma^h \times \mathbb{Z}^d \to \Sigma^h \times \mathbb{Z}^d\) by

\[
\begin{align*}
g_t(x, s) &:= (T^n x, s + t - h_n(x)) \quad \text{and} \\
G_t(x, s, \nu) &:= (T^n x, s + t - h_n(x), \nu + f_n(x))
\end{align*}
\]

where \(s + t \in [h_n(x), h_{n+1}(x)]\).
Define the tail equivalence relations $\mathfrak{T}(g)$ on $\Sigma^h$, and $\mathfrak{T}(G)$ on $\Sigma^h \times \mathbb{Z}^d$ as follows:

\[ \mathfrak{T}(g) := \{ ((x, s), (x', s')) | g_t(x, s) = g_t(x', s') \text{ for some } t > 0 \} \]
\[ \mathfrak{T}(G) := \{ ((x, s, \nu), (x', s', \nu')) | G_t(x, s, \nu) = G_t(x', s', \nu') \text{ for some } t > 0 \}. \]

It is not difficult to verify that

\[ ((x, s), (x', s')) \in \mathfrak{T}(g) \iff \exists n, m > 0 \text{ s.t. } \begin{cases} T^n(x) = T^m(x') \\ s - h_n(x) = s' - h_m(x') \end{cases} \]

and that

\[ ((x, s, \nu), (x', s', \nu')) \in \mathfrak{T}(G) \iff \exists n, m > 0 \text{ s.t. } \begin{cases} T^n(x) = T^m(x') \\ s - h_n(x) = s' - h_m(x') \\ \nu + f_n(x) = \nu' + f_m(x') \end{cases} \]

As shown in [B-M], the relation $\mathfrak{T}(g)$ is a symbolic model for the strong stable foliation of a topologically mixing basic set $\Omega_k$ of an Axiom A flow, in the sense that, given such a flow, there exists $\Sigma, h$ as above, and a one-to-one correspondence between invariant measures for the strong stable foliation of $\Omega_k$ and locally-finite invariant measures for $\mathfrak{T}(g)$. The reader is referred to [B-M] for the definition of the these geometric objects.

In the same sense, $\mathfrak{T}(G)$ is a symbolic model for the strong stable foliation of a $\mathbb{Z}^d$-extension of an Axiom A flow, see [B-L], [Po], [C].

1.2. The Babillot–Ledrappier Measures. The relation $\mathfrak{T}(g)$ is uniquely ergodic [B-M], but $\mathfrak{T}(G)$ is not: [B-L] provides a $d$-parameter family of pairwise disjoint $\mathfrak{T}(G)$-invariant measures, called here Babillot–Ledrappier (B-L) measures. These are given as follows. Fix $\alpha \in \mathbb{R}^d$. By [BG], [AR] there exists a unique $\tau_\alpha \in \mathbb{R}$ and a unique Borel probability measure $\mu_\alpha$ on $\Sigma$ which is $(e^{-\tau_\alpha h_+ + (\alpha, f), T})$-conformal in the sense that $\mu_\alpha \circ T \sim \mu_\alpha$ and

\[ \frac{d\mu_\alpha \circ T}{d\mu_\alpha} = e^{-\tau_\alpha h_+ + (\alpha, f)}. \]

The B-L measure indexed by $\alpha \in \mathbb{R}^d$ is the measure on $X = \Sigma^h \times \mathbb{Z}^d$ given by

\[ m_\alpha(A \times B \times \{ \nu \}) := e^{-\langle \alpha, \nu \rangle} \mu_\alpha(A) \int_B e^{\tau \nu} dr. \]

These are $\mathfrak{T}(G)$-invariant measures. They are infinite, but locally finite: compact subsets of $\Sigma^h \times \mathbb{Z}^d$ have finite measure.

1.3. Main Results. It is known that ([C] and [Po])

**Proposition 1.** $m_\alpha$ is $\mathfrak{T}(G)$–ergodic iff $T_{(-h, f)} : \Sigma \times \mathbb{R} \times \mathbb{Z}^d \to \Sigma \times \mathbb{R} \times \mathbb{Z}^d$ given by $T_{(-h, f)}(x, s, \nu) = (Tx, s - h(x), \nu + f(x))$ is ergodic with respect to $\mu_\alpha \times m_{\mathbb{R} \times \mathbb{Z}^d}$, where $m_{\mathbb{R} \times \mathbb{Z}^d}$ denotes Haar measure.

The purpose of this note is

(1) To characterize this situation of ergodicity in terms of a cocycle condition for $(-h, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ by showing that if one of the B-L measures is ergodic, then $(-h, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ is non-arithmetic (as defined below) and that this implies that all the B-L measures are ergodic (see [C], and theorem [I] and corollary [I] below, which imply proposition [I]).
(2) To identify the locally finite $\mathcal{I}(G)$-invariant measures by showing that in the case when the B-L measures are ergodic, that every locally finite, $\mathcal{I}(G)$-invariant, ergodic measure which is $G$-quasi-invariant must be proportional to a B-L measure (Theorem 2 below). Theorem 2.2 in [A-N-S-S] can be viewed as a (more complete) discrete time version of this result.

As shown in [B-L], horocycle foliations of $\mathbb{Z}^d$-covers of compact manifolds of constant negative curvature are ergodic with respect to the B-L measures. This is implied (via theorem 1 below) by ergodicity with respect to Lebesgue measure which was established earlier in [L-S] (see also [K] and [Po]).

It follows from our results that a locally finite measure which is ergodic and invariant for the strong stable foliation of a basic set $\Omega_k$ of an Axiom A flow, and which is quasi-invariant under the flow must be proportional to a B-L measure. (In the case of a surface of constant negative curvature this can also be shown via a geometric argument, [Ba].)

2. Ergodicity and non-arithmeticity of $G$-extensions

Let $G$ be a locally compact, second countable, Abelian topological group; let $(X, B, m, T)$ be a probability preserving transformation and let $\phi : X \to G$ be measurable. Consider the skew product $T_\phi : X \times G \to X \times G$ defined by $T_\phi(x, y) := (Tx, y + \phi(x))$ with respect to the (invariant) product measure $m \times m_G$ where $m_G$ denotes Haar measure.

Following [G], we say that $\phi$ is non-arithmetic if

$$\gamma(\phi) = g \cdot T$$

has no nontrivial solution in $\gamma \in \hat{G}$ and $g : X \to S^1$ measurable; and that $\phi$ is aperiodic if

$$\gamma(\phi) = zg \cdot T$$

has no nontrivial solution in $\gamma \in \hat{G}$, $z \in S^1$ and $g : X \to S^1$ measurable. It is not hard to show that if $T_\phi$ is ergodic, and $T$ is weakly mixing, then $\phi$ is non-arithmetic, and in this case $T_\phi$ is weakly mixing iff $\phi$ is aperiodic (see e.g. [K-N]).

Since $G$ is a locally compact Abelian polish group topological group, there are norms $\|\cdot\|$ generating the topology of $G$ which are Lipschitz in the sense that each character $\gamma : G \to S^1$ is $\|\cdot\|$-Lipschitz. Indeed, if $Y$ is a metric space, and $f : Y \to G$ is such that $\gamma \circ f : Y \to S^1$ is Lipschitz $\forall$ characters $\gamma$, then $\exists$ a Lipschitz norm $\|\cdot\|$ such that $f : Y \to G$ is $\|\cdot\|$-Lipschitz.

Livsic’s theorem (see [L]) states that if $(\Sigma, B, m, T)$ is a mixing subshift of finite type equipped with a Gibbs measure, $\phi : X \to G$ is Hölder continuous (w.r.t some Lipschitz norm), and $\gamma \in \hat{G}$ and $g : X \to S^1$ measurable with $\gamma(\phi) = g \cdot T$ a.e., then $g : X \to S^1$ is also Hölder continuous (w.r.t the same Lipschitz norm). Thus if a Hölder continuous $\phi : X \to G$ is non-arithmetic with respect to some Gibbs measure, then it is non-arithmetic with respect to all Gibbs measures.

Recall that a non-singular subshift of finite type $(\Sigma, B, m, T)$ has the Rényi property if there is a constant $C > 0$ such that for every cylinder of positive measure $a = [a_1, \ldots, a_n]$,

$$\frac{v_a'(x)}{v_a'(y)} \leq C \quad \text{for } m \times m \text{ a.e. } (x, y) \in a \times a,$$
where \( v_a := (T^n|_a)^{-1} \) and \( v'_a := \frac{dnu_a}{dm} \). The following is a generalization of a theorem in [4].

**Theorem 1.** Suppose that \((\Sigma, B, m, T)\) is a mixing subshift of finite type with the Rényi property and that \( \phi \) is Hölder continuous and non-arithmetic; then \( T_\phi \) is ergodic.

**Lemma 1.** Assume \( u : \Sigma \to S^1 \) is Hölder continuous. At least one of the following statements is true:

1. \( u = \overline{\psi} \circ g \circ T \) for some Hölder continuous \( g : \Sigma \to S^1 \).
2. Let \( \epsilon \in (0, 1) \) and \( N \in \mathbb{N} \) be arbitrary constants. There exists \( n \geq N \) such that for every \( z \in \Sigma \) there are \( x \in \Sigma \) and \( k \leq n \) such that

\[
x_1^N = z_1^N, \quad T^k x = T^n z \quad \text{and} \quad |u_n(z) - u_k(x)| \geq \epsilon.
\]

**Proof.** Let \( \mu \) be the Parry measure (i.e. measure of maximal entropy on \( \Sigma \)), then \( d\mu = \psi dv \) where \( \nu \in \mathcal{P}(\Sigma) \) is \((1, T)\)-conformal and \( \psi > 0 \) is Hölder continuous. Let \( P : L^1(\nu) \to L^1(\nu) \) be the transfer operator, then

\[
P f(x) = \sum_{Ty=x} e^{-h_{top}(T)} f(y)
\]

and \( P^n f \to \psi \int_X f d\nu \) uniformly \( \forall \ f \in C(X) \). Define \( P_u : C(\Sigma) \to C(\Sigma) \) by \( P_u(f) := P(u f) \), then \( P^n u f = P^n(u_n f) \) where \( u_n := \prod_{i=0}^{n-1} u \circ T^i \). By [G-H] either \( \exists \phi : \Sigma \to \mathbb{S}^1 \) Hölder continuous such that \( P_u(\phi) = \phi \) (which implies (1) with \( g := \phi / \psi \)), or \( \frac{1}{n} \sum_{k=0}^{n-1} P^k u f \to 0 \) \( \forall \ f \in C(\Sigma) \). If (2) fails, then \( \exists \epsilon \in (0, 1), \ N \geq 1 \) such that \( \forall \ n \geq N, \ \exists \ z = z^{(n)} \) satisfying

\[
k \leq n, \ x \in T^{-k}\{T^n z\}, \ x_1^N = z_1^N \Rightarrow |u_k(x) - u_n(z)| < \epsilon.
\]

There are only finitely many possibilities for the \( N \)-prefix of \( z^{(n)} \). We may therefore assume without loss of generality that \( \exists a = [a_1, \ldots, a_N] \) such that \( z^{(n)} \in a \) for all \( n \).

\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} P_u^k 1_a \right\|_\infty \geq \left\| \frac{1}{n} \sum_{k=0}^{n-1} P_u^k 1_a(T^n z^{(n)}) \right\|_\infty \\
= \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_{top}(T)} \sum_{y \in T^{-k}\{T^n z^{(n)}\}} u_k(y) 1_a(y) \right| \\
= \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_{top}(T)} \sum_{y \in T^{-k}\{T^n z^{(n)}\}} 1_a(y) \left( u_n(z^{(n)}) - [u_n(z^{(n)}) - u_k(y)] \right) \right| \\
\geq \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_{top}(T)} \sum_{y \in T^{-k}\{T^n z^{(n)}\}} 1_a(y) \left( 1 - |u_n(z^{(n)}) - u_k(y)| \right) \right| \\
\geq (1 - \epsilon) \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_{top}(T)} \sum_{y \in T^{-k}\{T^n z^{(n)}\}} 1_a(y) \\
= (1 - \epsilon) \frac{1}{n} \sum_{k=0}^{n-1} P^k 1_a(T^n z^{(n)}).
\]
Now \( \frac{1}{n} \sum_{k=0}^{n-1} P^k 1_a \to \nu(a) \psi \) uniformly, whence

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k 1_a(T^n z^{(n)}) \geq \nu(a) \inf \psi > 0. \]

\( \square \)

Let \( W_n \) denote the collection of admissible words of length \( n \) in \( \Sigma \), that is \( W_n := \{(\epsilon_1, \ldots, \epsilon_n) \in S^n : A_{\epsilon_j, \epsilon_{j+1}} = 1 \forall 1 \leq j \leq n-1 \} \). We denote the concatenation of \( a \in W_n \) and \( b \in W_m \) with \( A_{a_n, b_1} = 1 \), by \( a \cdot b \), and the concatenation of \( a \in W_n \) and \( x \in \Sigma \) with \( A_{a_n, x_1} = 1 \) by \( (a, x) \).

**Lemma 2.** Suppose that \( \phi \) is H"older continuous, \( \gamma \in \hat{G} \) is non-constant, \( \epsilon \in (0,1) \) and \( N \in \mathbb{N} \). If \( \phi \) is non-arithmetic, then there exists \( \ell \geq 1 \) arbitrarily large and infinitely many \( n \geq N \) with the following property:

\[
a \in W_n, \quad c \in W_\ell, \quad a \cdot c \in W_{n+\ell} \quad \Rightarrow \quad \exists k \in [N, n] \text{ s.t. } \exists b \in W_k \quad \forall x \in c, \gamma \circ \phi_n(a, x) - \gamma \circ \phi_k(b, x) \geq \epsilon
\]

**Proof:** Fix \( \gamma \in \hat{G} \) non-constant, \( \epsilon \in (0,1) \), and \( N \geq 1 \). Choose \( 0 < \delta < \frac{1 - \epsilon}{2} \) and \( \ell \geq 1 \) such that

\[
\eta := \sup \{|\gamma \circ \phi_n(x) - \gamma \circ \phi_n(y)| : n \geq 1, x, y \in \Sigma, x_1^n = y_1^n \} < \delta.
\]

By lemma [1] \( \exists n \geq N \) such that \( \forall z \in \Sigma, \exists k \leq n, x \in T^{-k} \{T^nz\}, x_1^n = z_1^N \) such that

\[
|\gamma \circ \phi_n(z) - \gamma \circ \phi_k(x)| \geq \epsilon + 2\delta.
\]

Now fix \( a \in W_n, \ c \in W_\ell \) with \( a \cdot c \in W_{n+\ell} \), choose some \( u \in \Sigma \) such that \( A_{a \cdot c, u_1} = 1 \), and set \( z = (a, c, u) \). Let \( k \leq n \), \( x(z) \in T^{-k} \{T^nz\} \), \( x(z)_1^N = z_1^N \) be such that

\[
|\gamma \circ \phi_n(z) - \gamma \circ \phi_k(x(z))| \geq \epsilon + \delta \quad \text{and let } b = x(z)_1^k.
\]

Since \( T^k x(z) = T^n z, \ x(z) = (b, c, u) \). For any \( v \in \Sigma \) with \( A_{a \cdot c, v_1} = 1 \) we have that

\[
|\gamma \circ \phi_n(a, c, u) - \gamma \circ \phi_k(b, c, u)| < \delta, \quad |\gamma \circ \phi_k(b, c, u) - \gamma \circ \phi_k(b, c, v)| < \delta
\]

whence \( |\gamma \circ \phi_n(a, c, v) - \gamma \circ \phi_k(b, c, v)| \geq \epsilon \). Since this is true for all \( v \in \Sigma \) with \( A_{a \cdot c, v_1} = 1 \), the lemma is proved. \( \square \)

**Proof of theorem [2]** (c.f. §2 “Proof of theorem 1” in [AD]) For a nonsingular transformation \( (Y, \mathcal{C}, \mu, Q) \), define the Grand Tail Relation of \( Q \):

\[
\mathcal{G}(Q) := \{(x, y) \in Y \times Y : \exists n, k > 0, Q^n x = Q^k y\}.
\]

This is an equivalence relation, and if \( (Y, \mathcal{C}, \mu) \) is standard, then \( \mathcal{G}(Q) \in \mathcal{C} \otimes \mathcal{C} \). If \( Q \) is locally invertible, then \( \mathcal{G}(Q) \) has countable equivalence classes and is nonsingular. It is easy to check that every \( Q \)-invariant subset of \( Y \) is \( \mathcal{G}(Q) \)-saturated. It follows that if \( \mathcal{G}(Q) \) is ergodic, then \( Q \) is ergodic.

It is therefore enough to prove that \( \mathcal{G}(T_\phi) \) is ergodic. Define

\[
\tilde{\phi} : \mathcal{G}(T) \setminus \{(x, y) \in X \times X : x \text{ and } y \text{ are pre-periodic } \} \to \mathbb{G}
\]

by \( \tilde{\phi}(x, y) = \phi_n(x) - \phi_k(y) \) whenever \( T^n x = T^k y \). This is independent of the choice of \( n, k \) whenever \( x, y \) are not pre-periodic.
The grand tail relation of $T_\phi$ is given by

$$\mathfrak{G}(T_\phi) = \left\{ ((x, s), (y, t)) \in (X \times G)^2 : \exists n, k > 0 \text{ such that } T^n x = T^k y, \right. $$

$$\left. \text{and } s - t = \phi_n(y) - \phi_k(x) \right\}$$

$$= \left\{ ((x, s), (y, t)) \in (X \times G)^2 : (x, y) \in \mathfrak{G}(T), \; \bar{\phi}(x, y) = s - t \right\}$$

We prove that $\mathfrak{G}(T_\phi)$ is ergodic by the method of Schmidt (explained in [S]), by considering the group of essential values which we now proceed to define. Set $B_+: = \{ B \in B : m(B) > 0 \}$. For every $B \in B_+$, let $\text{Hol}(B) = \text{Hol}(B, \mathfrak{G}(T))$ be the collection of non-singular $\mathfrak{G}(T)$-holonomies with domain $B$:

$$\text{Hol}(B) := \{ \tau : B \to X : \tau \text{ is a non-singular Borel isomorphism } B \to \tau(B) \text{ such that } \forall x \in B, (x, \tau(x)) \in \mathfrak{G}(T) \}.$$

Now define

$$E(\mathfrak{G}(T_\phi)) := \left\{ t \in G : \exists U \text{ open neighborhood of } t \text{ and } \forall A \in B_+, \right. $$

$$\left. \exists B \in B_+ \text{ and } \exists \tau \in \text{Hol}(B) \text{ such that } B, \tau(B) \subseteq A \right. $$

$$\left. \text{and } m(B \cap \tau^{-1}B \cap \{ x \in X : \bar{\phi}(x, \tau(x)) \in U \}) > 0 \right\}.$$

It is shown in [S] that $E(\mathfrak{G}(T_\phi))$ is a closed subgroup of $G$. To prove ergodicity, we show that $E(\mathfrak{G}(T_\phi)) = G$ (see [S]).

Suppose that $E(\mathfrak{G}(T_\phi)) = H \subseteq G$, then $\exists \gamma \in \hat{G}, \; \gamma \neq 0 \text{ with } \gamma|_H \equiv 1$. Fix a precompact neighborhood of the identity $V \subseteq G$, and let $N \in \mathbb{N}$ be so large that $j \geq 1, n \geq N, \; x_1^{j+n} = y_1^{j+n} \Rightarrow \phi_j(x) - \phi_j(y) \in V$.

Fix $\epsilon \in (0, 1)$ and let $\ell \geq 1$ and $n \geq N$ be as in lemma 2 with $\ell$ so large that

$$\eta_\ell := \sup \left\{ |\gamma \circ \phi_j(x) - \gamma \circ \phi_j(y)| : j \geq 1, \; x, y \in \Sigma, \; x_1^{j+\ell} = y_1^{j+\ell} \right\} < \epsilon 5.$$

It follows that $\forall a \in W_n, \; \forall c \in W_\ell \text{ s.t. } a \cdot c \in W_{n+\ell}, \; \exists k \leq n, \; b \in W_k$ with

$$b_1^N = a_1^N, \; b_k = a_n \text{ such that } \forall j \geq 1, \forall u \in W_j \text{ s.t. } A_{u_j, a_1} = 1,$$

$$|\gamma \circ \phi_{j+k}(u, a, c, x) - \gamma \circ \phi_{j+k}(u, b, c, x)| \geq \frac{4\epsilon}{5} \forall \; x \in Tc_\ell.$$

Let

$$K := \left\{ \phi_{j+k}(u, a, c, x) - \phi_{j+k}(u, b, c, x) : \; j \geq 1, \; u \in W_j, \; a \in W_n, \; A_{u_j, a_1} = 1, \right. $$

$$c \in W_\ell, a \cdot c \in W_{n+\ell}, \; k \leq n, \; b \in W_k, \; b_1^N = a_1^N, \; b_k = a_n, \right.$$  

$$x \in Tc_\ell, \; |\gamma \circ \phi_{n+j}(u, a, c, x) - \gamma \circ \phi_{j+k}(u, b, c, x)| \geq \frac{4\epsilon}{5} \left\}.$$

By the choice of $N$ and $\gamma$, $\bar{K} \subset V \setminus E(\mathfrak{G}(T_\phi))$ and $\bar{K}$ is compact. The methods of [S] show that $\exists A \in B_+$ such that

$$(A \times A) \cap \mathfrak{G}(T) \cap \{ \bar{\phi} \in K \} = \emptyset.$$
By the Rényi property, ∃ M > 1 such that

\[ M^{-1} m(u) m(v) \leq m(u \cap T^{-k} v) \leq M m(u) m(v) \ \forall \ u \in \alpha_{0}^{k}, \ v \in \alpha_{0}^{k-1}, \ [u_{1}] \subset T[u_{k}]. \]

Given \( j \geq 1, \ u = [u_{1}, \ldots, u_{j}] \subset \Sigma \) and \( a \in W_{n}, \ b \in W_{k}, \ c \in W_{l} \) as above, define \( \tau : [u \cdot a \cdot c] \to [u \cdot b \cdot c] \) by

\[ \tau(u, a, c, y) := (u, b, c, y). \]

It follows that \( \tau : [u, a, c] \to [u, b, c] \) is invertible, nonsingular and \( \frac{dm\tau}{dm} = M^{\pm 4} \frac{m(b)}{m(a)}. \)

Let \( \delta > 0 \) be so small that for all \( k \leq n, a \in W_{n}, b \in W_{k}, c \in W_{l} , k \leq n, \)

\[ \delta < \frac{m(b)}{M^{4} m(a)} \left( \frac{m([a, c])}{M} - \delta \right) \]

\[ \exists j \geq 1 \text{ and } u = [u_{1}, \ldots, u_{j}] \subset \Sigma \text{ such that } m(u \setminus A) < \delta m(u). \text{ Let } a \in W_{n} \text{ be such that } [u, a] \neq \emptyset \text{ and let } k \leq n, b \in W_{k}, c \in W_{l} \text{ be as above. Consider} \]

the corresponding \( \tau : [u, a, c] \to [u, b, c]. \) Evidently \( T^{j+k} \circ \tau \equiv T^{j+n} \) so \( (x, \tau(x)) \in \mathcal{O}(T) \ \forall \ x \in [u, a, c], \) and \( \phi_{j+k} \circ \tau(x) - \phi_{j+n}(x) \in K \ \forall \ x \in [u, a, c]. \)

To complete the proof we claim that \( \exists B \in B_{+} \ B \subset A \cap [u, a, c] \) such that \( \tau B \subset A. \) To see this we show that \( m(\tau([u, a, c] \cap A)) \geq m(u \setminus A), \) because this implies \( m(A \cap \tau([u, a, c] \cap A)) > 0 \) since \( \tau([u, a, c] \cap A) \subset u. \) Now

\[ m(\tau([u, a, c] \cap A)) \geq \frac{m(b)}{M^{4} m(a)} \left( m([a, c]) - m(u \setminus A) \right) \]

\[ > \frac{m(b)}{M^{4} m(a)} \left( \frac{m([a, c])}{M} - \delta \right) m(u) \]

\[ > \delta m(u) > m(u \setminus A). \]

and this shows that \( (A \times A) \cap \mathcal{O}(T_{\phi}) \cap [\emptyset \in K] \neq \emptyset \) which is a contradiction. \( \square \)

The following amplifies proposition 1:

**Corollary 1.** Let \( m_{\alpha} \) be a B-L measure on \( \Sigma^{h} \times \mathbb{Z}^{d}. \) The following are equivalent:

(1) \( (\Sigma^{h} \times \mathbb{Z}^{d}, m_{\alpha}, \mathcal{G}(G)) \) is ergodic;
(2) the cocycle \( (-h, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^{d} \) is non-arithmetic;
(3) \( T_{(-h, f)} \) is ergodic on \( \Sigma \times \mathbb{R} \times \mathbb{Z}^{d} \) with respect to \( \mu_{\alpha} \times m_{\mathbb{R} \times \mathbb{Z}^{d}} \) where \( m_{\mathbb{R} \times \mathbb{Z}^{d}} \) denotes Haar measure and \( \mu_{\alpha} \) is as in [1,2].

**Proof.** Set \( X = \Sigma^{h} \times \mathbb{Z}^{d}. \) As shown in [1,2],

\[ \mathcal{O}(T_{(-h, f)}) \cap (X \times X) = \mathcal{G}(G) \]

(1) \implies (2). Suppose (1) and that \( s \in \mathbb{R}, \gamma \in \mathbb{R}^{d} \) and \( g : \Sigma \to \mathbb{S} \) satisfy \( e^{-is_{h} + i\gamma_{f}} = \frac{g(x)}{\sqrt{|x|^{2}}} \), and define \( F : X \to \mathbb{C} \) by \( F(x, y, z) := g(x)e^{-iy + i\gamma_{f}} \), then

\[ F \circ T_{(-h, f)}(x, y, z) = F(Tx, y - h(x), z + f(x)) \]

\[ = g(Tx)e^{-i(y + is_{h}) + i(\gamma_{f}) + i(\gamma_{f})} \]

\[ = g(Tx)e^{-i(\gamma_{f})}F(x, y, z) = F(x, y, z). \]

It follows that \( F \) is constant, since \( F \circ T_{(-h, f)} = F \) and so every set of the form \( [F \leq \ell] \) is \( \mathcal{O}(T_{(-h, f)}) \)-saturated whence also \( \mathcal{G}(G) \)-saturated.
Now consider $F_0 : X \to \mathbb{C}$ the restriction of $F$ to $X$. It follows that for $(x, y, z) \in X$, $t \geq 0$ (choosing $n \geq 0$ such that $h_n(x) \leq t < h_{n+1}(x)$):

$$F_0 \circ G_t(x, y, z) = F_0(T^n x, y + t - h_n(x), z + f_n(x)) = F \circ T^n_{(h_n, f)}(x, y + t, z) = F(x, y + t, z) = e^{-ist} F_0(x, y, z)$$

and $F_0$ is $T(G)$-invariant, whence constant. It follows that $s = 0$, $\gamma = 0$ and $g \equiv 1$, so $(-h, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ is non-arithmetic. (2) $\Rightarrow$ (3) by theorem 1 (3) $\Rightarrow$ (1) follows from 1.

Thus:

**Corollary 2.** If $T(G)$ is ergodic with respect to some B-L measure, then the cocycle $(-h, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ is non-arithmetic and $T(G)$ is ergodic with respect to all B-L measures.

3. **Identification of ergodic, locally finite $T(G)$-invariant measures**

**Theorem 2.** Let $X := \Sigma^h \times \mathbb{Z}^d$ and let $G_t$ ($t \geq 0$) be the suspension semiflow. Assume that $(-h, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ is non-arithmetic and Hölder continuous. Suppose that $m$ is a locally finite, $T(G)$-invariant, ergodic measure on $X$ and that $m \circ G_t^{-1} \sim m \forall \ t > 0$, then $m$ is proportional to a B-L measure.

**Proof.** By assumption, $f : \Sigma \to \mathbb{Z}^d$ is Hölder continuous, and every such function is of the form $f(x) = f(x_1, \ldots, x_m)$ for some $m$. Recoding $\Sigma$ if necessary, we assume without loss of generality that $f(x) = f(x_1, x_2).

For $t > 0$, define the measure $m \circ G_t$ by $m \circ G_t(A) := \sum_{a \in \alpha} m(G_t(A \cap a))$ where $\alpha$ is a countable partition of $X$ such that $G_t|_a$ is 1-1 $\forall \ a \in \alpha$. Evidently $m \circ G_t \sim m$. Let $\mathcal{M}(\Sigma \times \mathbb{Z}^d)$ denote the collection of all (possibly infinite) Borel measures on $\Sigma \times \mathbb{Z}^d$.

**Claim 1:** $\exists \tau \in \mathbb{R}$ such that \(\frac{dm \circ G_t}{dm} = e^{rt}\), and $\exists \mu \in \mathcal{M}(\Sigma \times \mathbb{Z}^d)$ locally finite, such that

\[
m(A \times B) = \mu(A) \int_B e^{rt} dr \quad (A \in \mathcal{B}(\Sigma \times \mathbb{Z}^d), \ B \in \mathcal{B}(\mathbb{R}), \ A \times B \subset X).
\]

Moreover $(\Sigma \times \mathbb{Z}^d, B(\Sigma \times \mathbb{Z}^d), T_f, \mu)$ is ergodic.

**Proof.** Fix $t_0 > 0$. We prove first that \(\frac{dm \circ G_{t_0}}{dm}\) is $T(G)$-invariant and hence constant. Suppose that $A \subset X$ is Borel, and that $K : A \to KA$ is a $T(G)$-holonomy. Without loss of generality, $G_{t_0}|_A$, $G_{t_0}|_{KA}$ are 1-1. It follows that $K_1 := G_{t_0} \circ K \circ G_{t_0}^{-1} : G_{t_0} A \to G_{t_0} KA$ is a well-defined $T(G)$-holonomy. By the $T(G)$-invariance of $m$,

\[
m(G_{t_0} KA) = m(K_1 G_{t_0} A) = m(G_{t_0} A).
\]

This shows that \(\frac{dm \circ G_{t_0}}{dm}\) is indeed $T(G)$-invariant and hence constant. Disintegrating the measure $m$ over $\Sigma \times \mathbb{Z}^d$, we see that $\exists \lambda \in \mathcal{M}(\Sigma \times \mathbb{Z}^d)$ locally finite, and $m_x \in \mathcal{M}(\mathbb{R}_+)$ such that $x \mapsto m_x$ is measurable, and such that

\[
m(A \times B) = \int_A m_x(B) d\lambda(x).
\]
It follows that \( m_\varepsilon(J + t) = e^{\varepsilon t}m_\varepsilon \) for open \( J \subset (0, h(x)) \) and \( t \in \mathbb{R} \) small, whence \( dm_\varepsilon(y) = c(x)e^{\varepsilon t}dy \) and \( d\mu(x) := c(x)d\lambda(x) \). The equation \( \frac{d\mu \circ T_f}{d\mu} = e^{\varepsilon t} \) now follows from \( \frac{d\mu \circ \phi}{d\mu} = e^{\varepsilon t} \), and the ergodicity of \((\Sigma, T_f, \mu)\) is standard. \(\square\)

**Claim 2:** \( \exists \) a homomorphism \( \alpha : \mathbb{Z}^d \to \mathbb{R} \) and \( c > 0 \) such that \( \mu(A \times \{n\}) = ce^{-\alpha(n)}\nu(A) \) where \( \nu \in \mathcal{P}(\Sigma) \) is \((e^{\alpha f} + \varepsilon, T)\)-conformal.

**Proof.** We first claim it suffices to show that \( H := \{ n \in \mathbb{Z}^d : \mu \circ Q_n \sim \mu \} = \mathbb{Z}^d \) where \( Q_n(x, k) := (x, k + n) \). To see this, note that

\[
\frac{d\mu \circ Q_n \circ T_f}{d\mu} = \frac{d\mu \circ T_f}{d\mu} \circ Q_n = e^{\varepsilon h} \quad \forall n \in \mathbb{Z}^d.
\]

The ergodicity of \((\Sigma, T_f, \mu)\) ensures that \( \forall n \in \mathbb{Z}^d \), either \( \mu \circ Q_n \perp \mu \) or \( \mu \circ Q_n = c_n \mu \) for some \( c_n > 0 \). The condition \( H = \mathbb{Z}^d \) ensures that \( \mu \circ Q_n = e^{-\alpha(n)}\mu \) where \( \alpha : \mathbb{Z}^d \to \mathbb{R} \) is a homomorphism. Thus, \( (A \times \{n\}) = ce^{-\alpha(n)}\nu(A) \) where \( c > 0 \) and \( \nu \in \mathcal{P}(\Sigma) \). The \((e^{\alpha f} + \varepsilon, T)\)-conformality of \( \nu \) follows from the \((e^{\varepsilon h}, T_f)\)-conformality of \( \mu \).

We now prove that \( H = \mathbb{Z}^d \). Suppose otherwise that \( H \neq \mathbb{Z}^d \), then \( \exists \gamma \in \mathbb{Z}^d \) non-constant, such that \( \gamma|_H \equiv 1 \). Using non-arithmeticity and lemma \( 2 \) we fix \( n \geq 1 \) so that \( \forall a \in W_n \) and \( c \in S \), \( a \cdot c \in W_{n+1} \), \( \exists k = k(a) \leq n \) and \( b = b(a, c) \in W_k \) such that \( a_1 = b_1, a_n = b_k \) and \( \gamma \circ f_n(a, c) \neq \gamma \circ f_k(b, c) \).

By choice of \( a \), this means that \( f_n(a, c) \neq f_k(b, c) \). \(\square\)

Set \( J := \{ f_n(a, c) \neq f_k(b, c) : a \in W_n, c \in S, a \cdot c \in W_{n+1} \} \), then \( J \subset \mathbb{Z}^d \backslash H \) and \( J \) is finite. Set \( \pi := \sum_{j \in J} \mu \circ Q_j \), then \( \pi \perp \mu \) and \( \exists \ K \subset \Sigma \) compact and \( g \in \mathbb{Z}^d \) such that \( \mu(K \times \{g\}) > 0 \), \( \pi(K \times \{g\}) = 0 \).

Set \( I := \cup_{[g]} \{ h_j(x) - h_j(y) : j \geq 1, x^j_1 = y_j^1 \} \), \( L := 2 \max_{k \in S} \sup |h_k| \) and \( M := |W_{n+1}|e^{T(I+L)} \). Approximating \( K \) by larger open sets, we see that \( \exists \ U \subset \Sigma \) open, such that \( K \subset U \) and \( \mu(U \times \{g\}) < \frac{\mu(K \times \{g\})}{2M} \). It follows that \( \exists \) a cylinder set \( d = \{d_1, \ldots, d_N\} \) such that \( \mu(d \times \{g\}) > 0 \) and \( \mu(d \times \{g\}) < \frac{\mu(d \times \{g\})}{2M} \).

Since \( d \times \{g\} = \bigcup_{[g]} \{ h_j(x) - h_j(y) \} \), \( \exists \ a \in W_n, c \in S \) with \( a \cdot c \in W_{n+1} \) such that \( \mu([d, a, c] \times \{g\}) \geq \frac{\mu(d \times \{g\})}{|W_{n+1}|} \). Next, \( \exists \ b = (b_1, \ldots, b_k) \in W_k \) such that \( a_1 = b_1, a_n = b_k \) and \( f_n(a, c) \neq f_k(b, c) \). Define \( \kappa : [d, a, c] \times \{g\} \to d \times \mathbb{Z}^d \) by \( \kappa((d, a, x), g) := ((d, b, x), g + f_k(b, c) - f_n(a, c)) \). Since \( \frac{d\mu \circ T_f}{d\mu} = e^{\varepsilon h} \), we have that

\[
\frac{d\mu \circ \kappa}{d\mu}(x, v) = e^{\varepsilon h_{N+k}(d, b, x) - h_{N+n}(d, a, x)} \in [e^{-\varepsilon(I+L)}, e^{\varepsilon(I+L)}],
\]

where the last estimate follows from

\[
|h_{N+k}(d, b, x) - h_{N+n}(d, a, x)| \leq |h_N(d, b, x) - h_N(d, a, x)| + |h_k(b, x)| + |h_n(a, x)| \leq I + L.
\]

\(^1\)We are using here the assumption \( f(x) = f(x_0, x_1) \) to note that lemma \( 2 \) can be used with \( \ell = 1 \) and that \( f_n \) (resp. \( f_k \)) is constant on \( (a, c) \in W_{n+1} \) (resp. \( (b, c) \in W_{k+1} \)) so that the notation \( f_n(a, c), f_k(b, c) \) makes sense.
Thus
\[
(\mu \circ \kappa)([d,a,c] \times \{g\}) = \int_{[d,a,c] \times \{g\}} d\mu \circ \kappa = e^{-\tau(I+L)} \mu([d,a,c] \times \{g\})
\]
\[
\geq e^{-\tau(I+L)} \mu([d \times \{g\}]) \leq \frac{\mu([d \times \{g\}])}{M}
\]
On the other hand, \(\kappa([d,a,c] \times \{g\}) \subset Q_{f_b(a,c)}(d \times \{g\})\) whence
\[
\frac{\mu([d \times \{g\}])}{M} \leq \mu([d,a,c] \times \{g\}) \leq \mu(Q_{f_b(a,c)}(d \times \{g\})) \leq \frac{\mu([d \times \{g\}])}{2M}
\]
and \(1 < \frac{1}{2}\). This contradiction establishes claim 2. \(\square\)

Since the \((e^{\alpha f_T}, T)\)-conformal probability is unique, it follows from claim 2 that \(m\) is proportional to the corresponding B-L measure. \(\square\)

References


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