EQUIVALENCE OF RENEWAL SEQUENCES, AND ISOMORPHISM OF RANDOM WALKS.

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Let X be a $\mathbb{Z}^d$-valued random variable (where $d \in \mathbb{N}$). The *random walk* with *jump random variable* X is a measure preserving transformation T_X of the σ-finite, infinite measure space $(\mathbb{Z}^d)^{\mathbb{Z}} \times \mathbb{Z}^d$ equipped with the σ-algebra generated by cylinder sets, and the measure $f^{\mathbb{Z}} \times$ counting measure (where $f_n =$Prob$(X = n)$). It is defined by

$$T_X\Big((\dots, x_{-1}, x_0, x_1, \dots), n\Big) = \Big((\dots, x_0, x_1, x_2, \dots), n + x_1\Big).$$

The random walk is conservative and ergodic in case $d = 1, 2$, and the jump random variable X:

has finite second moment ($E(|X|^2) < \infty$), is *centred* ($E(X) = 0$), and is *strictly aperiodic* in the sense that if $\varphi(s) = E(e^{is\cdot X})$ ($s \in \mathbb{R}^d$), then $|\varphi(s)| = 1$ if and only if $s \in 2\pi\mathbb{Z}^d$.

Let $Y^{(d)} = \epsilon \in \{0, \pm 1\}^d$ with probability $\frac{1}{3^d}$. It is shown in [1] that if $d = 1, 2$, and X is a $\mathbb{Z}^d$-valued random variable with $E(|X|^7) < \infty$, which is centred, and strictly aperiodic, then T_X and $T_{Y^{(d)}}$ are isomorphic as measure preserving transformations.

We prove here the

Main Theorem If $d = 1, 2$, and X is a centred, strictly aperiodic $\mathbb{Z}^d$-valued random variable with $E\Big(|X|^2\sqrt{\log^+|X|}\Big) < \infty$, then T_X and $T_{Y^{(d)}}$ are isomorphic.

The method of proof, using results in [1], involves a study of the *equivalence* of renewal sequences of *jump random variables*, that is, renewal sequences of form

$$u(n) = u_n(X) = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\varphi(s)^n ds = \text{ Prob}\Big(\sum_{k=1}^{n} X_k = 0\Big)$$

where $X_1, X_2, \dots$ are i.i.d.r.v.'s, each distributed as X.

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Recall from [1] that the renewal sequences $u,\ u'$ are said to be *equivalent* if there are positively recurrent, aperiodic renewal sequences $v,\ v'$ such that $u(n)v(n) = u'(n)v'(n)$, and that [1, theorem 3.6] conservative, ergodic random walks whose jump random variables have equivalent renewal sequences are isomorphic.

Let X be a $\mathbb{Z}^d$-valued random variable. A necessary condition for the equivalence of the renewal sequences $u(X)$ and $u(Y^{(d)})$, is that X be centred, strictly aperiodic, and with finite second moment. This is because otherwise, $\liminf_{n\to\infty} \frac{u_n(X)}{u_n(Y^{(d)})} = 0$.

A sufficient condition for equivalence of renewal sequences is given by [1, corollary 4.4], which says that the renewal sequences $u = (u(n))_{n\geq 0}$ and $u(Y^{(d)})$ are equivalent if

$$(*) \qquad \sum_{n\in A} n\left|\log\left(\frac{1}{p_n}\right) - \frac{d}{2n^2}\right| < \infty$$

where $p_n = \frac{u(n)^2}{u(n-1)u(n+1)}$ for $n \in A = \{n \in \mathbb{N} : u(k) > 0\ \forall\ k \geq n-1\}$.

Thus, given a centred, strictly aperiodic, $\mathbb{Z}^d$-valued jump random variable X, $(d = 1, 2)$, in order to prove that T_X and $T_{Y^{(d)}}$ are isomorphic, it is sufficient to establish that $u(X)$ satisfies $(*)$.

The isomorphism theorem of [1] is proved in this way by means of [1, theorem 5.1] which establishes $(*)$ when $E(|X|^7) < \infty$ and X is centred, strictly aperiodic.

Similarly, we prove our main theorem by establishing the

Theorem Suppose that $E(|X|^2\sqrt{\log^+|X|}) < \infty$, and that X is centred, and strictly aperiodic,

then $(*)$ holds.

Remark We do not know of any centred, strictly aperiodic random variable X on $\mathbb{Z}^d$ with finite second moment for which $u(X)$ and $u(Y^{(d)})$ are not equivalent.

Proof of the Theorem

It follows from the local limit theorem that

$$\exists \lim_{n\to\infty} n^{\frac{d}{2}} u(n) \in \mathbb{R}_+,$$

and hence that $(*)$ holds if and only if

$$\sum_{n=1}^{\infty} n^{d+1}\left|u(n-1)u(n+1) - u(n)^2 - \frac{du(n)^2}{2n^2}\right| < \infty.$$

Set

$$D_n := n^{d+1}(u(n-1)u(n+1) - u(n)^2 - \frac{du(n)^2}{2n^2}),$$

and let $a_n \sim a'_n$ mean that $\sum_n |a_n - ca'_n| < \infty$ for some $0 < c < \infty$. In particular, if $a_n \sim a'_n$, then $\sum_n |a_n| < \infty$ iff $\sum_n |a'_n| < \infty$.

We'll show, under the assumption $E(|X|^2\sqrt{\log^+|X|}) < \infty$, that $D_n \sim 0$.

Using Taylor's theorem for φ around 0, we find that,

$$\exists\ \delta > 0,\ \epsilon \in (0, \frac{1}{2}]\ \text{ such that } 0 < |\varphi(x)| \le e^{-\epsilon\Gamma(x)}\ \forall\ |x| \le \delta \qquad (\ddagger)$$

where $\Gamma(x) := E((X\cdot x)^2)\ \ (x \in \mathbb{R}^d)$.

By aperiodicity of X, $\sup_{|t|\ge\delta} |\varphi(t)| < 1$, and we have that

$$\begin{aligned}
D_n &:= n^{d+1}(u(n-1)u(n+1) - u(n)^2 - \frac{du(n)^2}{2n^2})\\
&= \frac{n^{d+1}}{(2\pi)^{2d}} \iint_{\mathbb{T}^d\times\mathbb{T}^d} \Big(\varphi(x)^{n-1}\varphi(y)^{n+1} - \varphi(x)^n\varphi(y)^n(1+\frac{d}{2n^2})\Big)dxdy\\
&\sim \frac{n^{d+1}}{(2\pi)^{2d}} \iint_{N(0,\delta)^2} \Big(\varphi(x)^{n-1}\varphi(y)^{n+1} - \varphi(x)^n\varphi(y)^n(1+\frac{d}{2n^2})\Big)dxdy\\
&= \frac{n^{d+1}}{(2\pi)^{2d}} \iint_{N(0,\delta)^2} \varphi(x)^n\varphi(y)^n\Big(\frac{\varphi(x)-\varphi(y)}{\varphi(y)} - \frac{d}{2n^2}\Big)dxdy\\
&= \frac{n^{d+1}}{(2\pi)^{2d}} \iint_{N(0,\delta)^2} \varphi(x)^n\varphi(y)^n\Big(\frac{\varphi(x)-\varphi(y)}{\varphi(y)} - (\varphi(x)-\varphi(y)) - \frac{d}{2n^2}\Big)dxdy
\end{aligned}$$

since

$$\iint_{N(0,\delta)^2} \varphi(x)^n\varphi(y)^n(\varphi(x)-\varphi(y))dxdy = 0.$$

Here, $N(0,\delta) := \{x \in \mathbb{T}^d = [-\pi,\pi]^d : |x| < \delta\}$.

Changing variables, we have

$$\begin{aligned}
D_n &\sim \frac{n}{(2\pi)^{2d}} \iint_{N(0,\delta\sqrt{n})^2} \varphi(\frac{s}{\sqrt{n}})^n\varphi(\frac{t}{\sqrt{n}})^n\Bigg(\frac{[\varphi(\frac{s}{\sqrt{n}}) - \varphi(\frac{t}{\sqrt{n}})](1-\varphi(\frac{t}{\sqrt{n}}))}{\varphi(\frac{t}{\sqrt{n}})} - \frac{d}{2n^2}\Bigg)dsdt\\
&= \frac{1}{n(2\pi)^{2d}} \iint_{N(0,\delta\sqrt{n})^2} \varphi(\frac{s}{\sqrt{n}})^n\varphi(\frac{t}{\sqrt{n}})^n\Bigg(\frac{(\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}}) - \frac{\Gamma(s)}{2}\phi(\frac{s}{\sqrt{n}}))\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})}{\varphi(\frac{t}{\sqrt{n}})} - \frac{d}{2}\Bigg)dsdt.
\end{aligned}$$

where $\Gamma(x) := E((X\cdot x)^2) = x^*\Gamma x\ \ (x \in \mathbb{R}^d)$, and $1-\varphi(s) = \frac{\Gamma(s)}{2}\phi(s)$.

Using (‡), and $1-\varphi(\frac{t}{\sqrt{n}}) = O(\frac{|t|^2}{n})$, we have, by Lebesgue's dominated convergence theorem, that

$$D_n \sim \frac{1}{n}\iint_{N(0,\delta\sqrt{n})^2} \varphi(\frac{s}{\sqrt{n}})^n\varphi(\frac{t}{\sqrt{n}})^n\Bigg([\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{\Gamma(s)}{2}\phi(\frac{s}{\sqrt{n}})]\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{d}{2}\Bigg)dsdt.$$

Write

$$a_n(s) = \frac{\Gamma(s)}{2}(1-\phi(\frac{s}{\sqrt{n}})) = n(\varphi(\frac{s}{\sqrt{n}})-1+\frac{\Gamma(s)}{2n}).$$

Using again the Taylor expansion of φ around 0, we obtain $K \in I\!R_+$ such that

$$|a_n(s)| \le KE((s\cdot X)^2(\frac{|X||s|}{\sqrt{n}}\wedge 1)) \le K\Gamma(s)\ \forall\ s\in I\!R^d,\ n\in I\!N.$$

To continue, we need to use the moment assumption on X.

Lemma 1 If $E\Bigg(|X|^2(\log^+|X|)^{\frac{1}{k}}\Bigg) < \infty$, then

$$\sum_{n=1}^{\infty}\frac{|a_n(s)|^k}{n} \le M_k|s|^{2k}(1+|s|)^k$$

where $M_k \in I\!R_+$, and hence,

$$\int_{I\!R^d}|s|^\ell\Bigg(\sum_{n=1}^{\infty}\frac{|a_n(s)|^k}{n}\Bigg)e^{-\epsilon\Gamma(s)}ds < \infty \quad \forall\ \ell \ge 0$$

proof

$$|a_n(s)| \le KE((s\cdot X)^2(\frac{|X||s|}{\sqrt{n}}\wedge 1))$$
$$\le \frac{K|s|^2(1+|s|)}{\sqrt{n}}E(|X|^2(|X|\wedge\sqrt{n})),$$
$$\therefore\quad \frac{|a_n(s)|^k}{n} \le \frac{K^k|s|^{2k}(1+|s|)^k}{n^{1+\frac{k}{2}}}\Bigg(E(|X|^2(|X|\wedge\sqrt{n}))\Bigg)^k$$

and it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{k}{2}}}(E(|X|^2(|X|\wedge\sqrt{n})))^k < \infty.$$

Now,

$$E(|X|^2(|X|\wedge\sqrt{n})) = E(|X|^3 1_{[|X|\le\sqrt{n}]}) + \sqrt{n}E(|X|^2 1_{[|X|\ge\sqrt{n}]}).$$

Therefore, by Jensen's inequality,

$$\Bigg(E(|X|^2(|X|\wedge\sqrt{n}))\Bigg)^k \le 2^{k-1}\Bigg(E(|X|^3 1_{[|X|\le\sqrt{n}]})\Bigg)^k + 2^{k-1}n^{\frac{k}{2}}\Bigg(E(|X|^2 1_{[|X|\ge\sqrt{n}]})\Bigg)^k,$$

and it is now sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{k}{2}}} (E(|X|^3 1_{[|X|\leq\sqrt{n}]})\Big)^k < \infty, \ \& \ \ \sum_{n=1}^{\infty} \frac{1}{n}\Big(E(|X|^2 1_{[|X|\geq\sqrt{n}]})\Big)^k < \infty.$$

Letting $X_1, \ldots, X_k$ be independent copies of X, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{k}{2}}} \Big(E(|X|^3 1_{[|X|\leq\sqrt{n}]})\Big)^k &= E\Big(|X_1|^3 \ldots |X_k|^3 \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{k}{2}}} 1_{[|X_1|\leq\sqrt{n}]} \ldots 1_{[|X_k|\leq\sqrt{n}]}\Big) \\ &= E\Big(|X_1|^3 \ldots |X_k|^3 \sum_{n\geq|X_1|^2\vee\cdots\vee|X_k|^2} \frac{1}{n^{1+\frac{k}{2}}}\Big) \\ &\leq 2E\Big(\frac{|X_1|^3 \ldots |X_k|^3}{(|X_1| \vee \cdots \vee |X_k|)^k}\Big) \\ &\leq 2\Big(E(|X|^2)\Big)^k < \infty,\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n}\Big(E(|X|^2 1_{[|X|\geq\sqrt{n}]})\Big)^k &= E\Big(|X_1|^2 \ldots |X_k|^2 \sum_{n=1}^{\infty} \frac{1}{n} 1_{[|X_1|\geq\sqrt{n}]} \ldots 1_{[|X_k|\geq\sqrt{n}]}\Big) \\ &= E\Big(|X_1|^2 \ldots |X_k|^2 \sum_{n=1}^{|X_1|^2\wedge\cdots\wedge|X_k|^2} \frac{1}{n}\Big) \\ &\leq E\Big(|X_1|^2 \ldots |X_k|^2 \log^+(|X_1|^2 \wedge \cdots \wedge |X_k|^2)\Big) \\ &\leq \Big(E(|X|^2(\log^+|X|^2)^{\frac{1}{k}})\Big)^k < \infty.\end{aligned}$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{k}{2}}} (E(|X|^2(|X| \wedge \sqrt{n})))^k < \infty. \ \ \diamondsuit$$

Lemma 2 If $E(|X|^2\sqrt{\log^+|X|}) < \infty$, then

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{N(0,\delta\sqrt{n})} |s|^\ell |\varphi(\frac{s}{\sqrt{n}})^n - e^{-\frac{\Gamma(s)}{2}}(1 + a_n(s))| ds < \infty \quad \forall \ \ell \geq 0.$$

proof

Using Taylor's theorem yet again, we have that $\exists\, M \in \mathbb{R}_+$ such that

$$\varphi(\frac{s}{\sqrt{n}}) - e^{-\frac{\Gamma(s)}{2n}} = \frac{a_n(s)}{n} + \Theta_n(s) \ \ \forall \ s \in \mathbb{R}$$

where $|\Theta_n(s)| \leq \frac{M|s|^4}{n^2}$.

Therefore,

$$\varphi(\frac{s}{\sqrt{n}})^n - e^{-\frac{\Gamma(s)}{2}} = \sum_{k=0}^{n-1} \varphi(\frac{s}{\sqrt{n}})^{n-1-k} e^{-\frac{k\Gamma(s)}{2n}} (\frac{a_n(s)}{n} + \Theta_n(s))$$

$$= \frac{a_n(s)}{n} \sum_{k=0}^{n-1} \varphi(\frac{s}{\sqrt{n}})^{n-1-k} e^{-\frac{k\Gamma(s)}{2n}} + \widetilde{\Theta}_n(s),$$

where

$$|\widetilde{\Theta}_n(s)| \le n|\Theta_n(s)| e^{-\epsilon(1-\frac{1}{n})\Gamma(s)} \le \frac{M|s|^4}{n} e^{-\frac{\epsilon}{2}\Gamma(s)} \quad \forall\ |s| \le \delta\sqrt{n}.$$

Proceeding further,

$$(\varphi(\frac{s}{\sqrt{n}})^n - e^{-\frac{\Gamma(s)}{2}}(1+e^{\frac{\Gamma(s)}{2n}} a_n(s))) = \frac{a_n(s)}{n} \sum_{k=0}^{n-1} e^{-\frac{(n-1-k)\Gamma(s)}{2n}} (\varphi(\frac{s}{\sqrt{n}})^k - e^{-\frac{k\Gamma(s)}{2n}}) + \widetilde{\Theta}_n(s).$$

Now

$$(\varphi(\frac{s}{\sqrt{n}})^k - e^{-\frac{k\Gamma(s)}{2n}}) = \sum_{\nu=0}^{k-1} \varphi(\frac{s}{\sqrt{n}})^\nu e^{-\frac{(k-\nu-1)\Gamma(s)}{2n}} (\frac{a_n(s)}{n} + \Theta_n(s)),$$

and hence

$$|\varphi(\frac{s}{\sqrt{n}})^k - e^{-\frac{k\Gamma(s)}{2n}}| \le k e^{-\frac{\epsilon(k-1)\Gamma(s)}{n}} (\frac{|a_n(s)|}{n} + \Theta_n(s)).$$

Substituting back in the above, we obtain for $|s| \le \delta\sqrt{n},\ n \ge 4$,

$$\begin{aligned}
&|\varphi(\frac{s}{\sqrt{n}})^n - e^{-\frac{\Gamma(s)}{2}}(1 + e^{\frac{\Gamma(s)}{2n}} a_n(s))| \\
&\le \frac{|a_n(s)|}{n} \sum_{k=0}^{n-1} e^{-\frac{(n-1-k)\Gamma(s)}{2n}} \left(k e^{-\frac{(k-1)\epsilon\Gamma(s)}{n}} (\frac{|a_n(s)|}{n} + |\Theta_n(s)|) \right) + |\widetilde{\Theta}_n(s)| \\
&\le e^{-\frac{\epsilon\Gamma(s)}{2}} \left(|a_n(s)|^2 + n|a_n(s)||\Theta_n(s)| \right) + |\widetilde{\Theta}_n(s)| \\
&\le e^{-\frac{\epsilon\Gamma(s)}{2}} \left(|a_n(s)|^2 + KME(|X|^2)| \frac{|s|^6}{n} + \frac{M|s|^4}{n} \right).
\end{aligned}$$

Lastly, by Taylor's theorem, $\exists\ \overline{M} \in I\!\!R_+$ such that

$$0 < e^{\frac{\Gamma(s)}{2n}} - 1 \le \frac{\overline{M}}{n} \quad \forall\ n \ge 1,\ |s| < \delta\sqrt{n},$$

whence for $|s|<\delta\sqrt{n}$,

$$\Big|\varphi(\frac{s}{\sqrt{n}})^n - e^{-\frac{\Gamma(s)}{2}}(1+a_n(s))\Big| \le e^{-\frac{\epsilon\Gamma(s)}{2}}\Big(|a_n(s)|^2 + KME(|X|^2)|\frac{|s|^6}{n} + \frac{M|s|^4}{n} + \overline{M}K\frac{\Gamma(s)}{n}\Big),$$

and lemma 2 is established by lemma 1. $\diamondsuit$

As a corollary of lemma 2, we get that
$D_n \sim$

$$n^{-1}\iint_{N^2} e^{-\frac{\Gamma(s)+\Gamma(t)}{2}}(1+a_n(s))(1+a_n(t))\Big((\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{\Gamma(s)}{2}\phi(\frac{s}{\sqrt{n}}))\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{d}{2}\Big)dsdt$$

where $N=N(0,\delta\sqrt{n})$. This is because

$$|(\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{\Gamma(s)}{2}\phi(\frac{s}{\sqrt{n}}))\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{d}{2}| \le f(s,t)\;\forall\, n\ge 1,\ |s|,|t|<\delta\sqrt{n},$$

where f is a polynomial in $|s|$ and $|t|$.

Using the existence of $0<r_k<1$, $(k\ge 1$ fixed), such that

$$\int_{\mathbb{R}^d\setminus N(0,\delta\sqrt{n})}|s|^k e^{-\frac{\Gamma(s)}{2}} = O(r_k^n) \text{ as } n\to\infty$$

we obtain that $D_n \sim$

$$n^{-1}\iint e^{-\frac{\Gamma(s)+\Gamma(t)}{2}}(1+a_n(s))(1+a_n(t))\Big((\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{\Gamma(s)}{2}\phi(\frac{s}{\sqrt{n}}))\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{d}{2}\Big)dsdt$$

where here, and henceforth, we supress the domain of integration where this is maximal.

Our next task is to simplify our expression $D'_n \sim D_n$ on the basis of the identity

$$\iint e^{-\frac{\Gamma(s)+\Gamma(t)}{2}}\Big((\frac{\Gamma(t)}{2}-\frac{\Gamma(s)}{2})\frac{\Gamma(t)}{2}-\frac{d}{2}\Big)dsdt = 0. \qquad (\dagger)$$

Lemma 3 If $E(|X|^2\sqrt{\log^+|X|})<\infty$, then

$$D_n \sim \frac{1}{n}\int e^{-\frac{\Gamma(s)}{2}}a_n(s)\Big(\Gamma(s)^2 - 2(d+2)\Gamma(s) + d(d+2)\Big)ds. \qquad (2)$$

proof

It will be convenient to use the following notation:

$$f_n(s,t)\approx g_n(s,t) \text{ if } \frac{1}{n}\iint e^{-\frac{\Gamma(s)+\Gamma(t)}{2}}f_n(s,t)dsdt \sim \frac{1}{n}\iint e^{-\frac{\Gamma(s)+\Gamma(t)}{2}}g_n(s,t)dsdt.$$

We'll use the following consequence of lemma 1,

$$a_n(s)^2 \approx a_n(s)a_n(t) \approx 0. \qquad (i)$$

Also, by symmetry,

$$f_n(s,t) \approx g_n(s,t) := f_n(t,s) \qquad (ii)$$

We'll also use the following formulae

$$\int_{\mathbb{R}^d} \Phi(\Gamma(s)) e^{-\frac{\Gamma(s)}{2}} ds = \frac{1}{\sqrt{\det \Gamma}} \int_{\mathbb{R}^d} \Phi(|s|^2) e^{-\frac{|s|^2}{2}} ds. \qquad (iii)$$

$$\int_{\mathbb{R}^d} e^{-\frac{|s|^2}{2}} |s|^{2k} ds = (2\pi)^{\frac{d}{2}} r_{2k} \text{ where } r_0 = 1,\ r_2 = d,\ r_4 = d(d+2),\ \&\ r_6 = d(d+2)(d+4). \qquad (iv)$$

$$(1 + a_n(s))(1 + a_n(t))\left((\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}}) - \frac{\Gamma(s)}{2}\phi(\frac{s}{\sqrt{n}}))\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}}) - \frac{d}{2}\right)$$

$$= (1 + a_n(s))(1 + a_n(t))\left([\frac{\Gamma(t)}{2} - \frac{\Gamma(s)}{2} - (a_n(t) - a_n(s))][\frac{\Gamma(t)}{2} - a_n(t)] - \frac{d}{2}\right) \overset{\dagger}{\to}\approx$$

$$\left(\{\frac{\Gamma(t)}{2} - \frac{\Gamma(s)}{2} - [a_n(t) - a_n(s)]\}\{\frac{\Gamma(t)}{2} - a_n(t)\} - \{\frac{\Gamma(t)}{2} - \frac{\Gamma(s)}{2}\}\frac{\Gamma(t)}{2}\right)$$

$$+ [a_n(s) + a_n(t)]\left((\frac{\Gamma(t)}{2} - \frac{\Gamma(s)}{2})\frac{\Gamma(t)}{2} - \frac{d}{2}\right) \overset{(i)}{\to}\approx$$

$$- a_n(t)(\frac{\Gamma(t)}{2} - \frac{\Gamma(s)}{2}) - [a_n(t) - a_n(s)]\frac{\Gamma(t)}{2} + [a_n(s) + a_n(t)]\left((\frac{\Gamma(t)}{2} - \frac{\Gamma(s)}{2})\frac{\Gamma(t)}{2} - \frac{d}{2}\right)$$

$$= a_n(s)\left(\frac{\Gamma(t)^2}{4} - \frac{\Gamma(s)\Gamma(t)}{4} + \frac{\Gamma(t)}{2} - \frac{d}{2}\right) + a_n(t)\left(\frac{\Gamma(t)^2}{4} - \frac{\Gamma(s)\Gamma(t)}{4} + \frac{\Gamma(s)}{2} - \Gamma(t) - \frac{d}{2}\right)$$

$$\overset{(ii)}{\to}\approx a_n(s)\left(\frac{\Gamma(s)^2}{4} + \frac{\Gamma(t)^2}{4} - \frac{\Gamma(s)\Gamma(t)}{2} - \Gamma(s) + \Gamma(t) - d\right)$$

$$\overset{(iii)}{\to}\approx a_n(s)\left(\frac{\Gamma(s)^2}{4} r_0 + \frac{r_4}{4} - \frac{r_2\Gamma(s)}{2} - \Gamma(s) r_0 + r_2 - d r_0\right)$$

$$\overset{(iv)}{\to} = a_n(s)\left(\frac{\Gamma(s)^2}{4} + \frac{d(d+2)}{4} - \frac{d\Gamma(s)}{2} - \Gamma(s)\right)$$

$$= \frac{1}{4} a_n(s)(\Gamma(s)^2 - 2(d+2)\Gamma(s) + d(d+2)). \quad \diamondsuit$$

Remark It now follows from lemmas 1 and 3 that (*) holds if, in addition, $E(|X|^2 \log^+ |X|) < \infty$.

To continue, we calculate

$$\frac{1}{n}\int e^{-\frac{\Gamma(s)}{2}}a_n(s)\Big(\Gamma(s)^2-2(d+2)\Gamma(s)+d(d+2)\Big)ds$$

more precisely using the formulae

$$\int_{I\!R^d} e^{-\frac{|s|^2}{2}}|s|^{2k}e^{i\gamma\cdot s}ds=(2\pi)^{\frac{d}{2}}q_{2k}(\gamma)e^{-\frac{|\gamma|^2}{2}} \qquad (v)$$

where $q_0=1,\ q_2(\gamma)=d-|\gamma|^2,\ q_4(\gamma)=|\gamma|^4-2(d+2)|\gamma|^2+d(d+2).$ (vi)

Lemma 4

$$\frac{1}{n}\int e^{-\frac{\Gamma(s)}{2}}a_n(s)\Big(\Gamma(s)^2-2(d+2)\Gamma(s)+d(d+2)\Big)ds=\frac{(2\pi)^{\frac{d}{2}}}{\sqrt{|\det\Gamma|}}E\Bigg(e^{-\frac{|A^{-1*}X|^2}{2n}}\frac{|A^{-1*}X|^4}{n^2}\Bigg) \qquad (3)$$

proof

$$\frac{1}{n}\int e^{-\frac{\Gamma(s)}{2}}a_n(s)\Big(\Gamma(s)^2-2(d+2)\Gamma(s)+d(d+2)\Big)ds$$
$$=\int e^{-\frac{\Gamma(s)}{2}}\Big(\Gamma(s)^2-2(d+2)\Gamma(s)+d(d+2)\Big)\Big(\varphi(\frac{s}{\sqrt{n}})-1+\frac{\Gamma(s)}{2n}\Big)ds$$
$$\overset{(iii)}{\to}=\frac{1}{\sqrt{|\det\Gamma|}}\int e^{-\frac{|x|^2}{2}}(|x|^4-2(d+2)|x|^2+d(d+2))(\varphi(\frac{A^{-1}x}{\sqrt{n}})-1+\frac{|x|^2}{2n})dx$$

where $\Gamma=A^*A$.

Next,

$$\int e^{-\frac{|x|^2}{2}}(|x|^4-2(d+2)|x|^2+d(d+2))\varphi(\frac{A^{-1}x}{\sqrt{n}})dx$$
$$=E\Bigg(\int e^{-\frac{|x|^2}{2}}e^{i\frac{A^{-1}x\cdot X}{\sqrt{n}}}(|x|^4-2(d+2)|x|^2+d(d+2))dx\Bigg)$$
$$\overset{(v)}{\to}=(2\pi)^{\frac{d}{2}}E\Bigg(e^{-\frac{|A^{-1*}X|^2}{2n}}(q_4(\frac{A^{-1*}X}{\sqrt{n}})-2(d+2)q_2(\frac{A^{-1*}X}{\sqrt{n}})+d(d+2)q_0(\frac{A^{-1*}X}{\sqrt{n}}))\Bigg)$$
$$\overset{(vi)}{\to}=(2\pi)^{\frac{d}{2}}E\Bigg(e^{-\frac{|A^{-1*}X|^2}{2n}}\frac{|A^{-1*}X|^4}{n^2}\Bigg).$$

$$\int e^{-\frac{|x|^2}{2}}(|x|^4-2(d+2)|x|^2+d(d+2))(-1+\frac{|x|^2}{2n})dx$$
$$=\int e^{-\frac{|x|^2}{2}}\left[\frac{|x|^6}{2n}-(1+\frac{d+2}{n})|x|^4+(2(d+2)+\frac{d(d+2)}{2n})|x|^2-d(d+2)\right]dx$$
$$\overset{(iv)}{\to}=0. \qquad \diamondsuit$$

We have, by (2), and (3) that

$$D_n \sim E\left(e^{-\frac{|A^{-1*}X|^2}{2n}}\frac{|A^{-1*}X|^4}{n^2}\right).$$

To conclude the proof of the theorem,

$$\begin{aligned}\sum_{n=1}^{\infty}E\left(\frac{|A^{-1*}X|^4}{n^2}e^{-\frac{|A^{-1*}X|^2}{2n}}\right) &= \sum_{n=1}^{\infty}E\left(\frac{|A^{-1*}X|^4}{n^2}\int_{\frac{|A^{-1*}X|}{\sqrt{n}}}^{\infty}se^{-\frac{s^2}{2}}ds\right)\\ &= \int_0^{\infty}se^{-\frac{s^2}{2}}E\left(\sum_{n=1}^{\infty}\frac{|A^{-1*}X|^4}{n^2}1_{[|A^{-1*}X|\le s\sqrt{n}]}\right)ds\\ &\le 2\int_0^{\infty}se^{-\frac{s^2}{2}}E\left(|A^{-1*}X|^2(s^2\wedge|A^{-1*}X|^2)\right)ds\\ &\le 2E(|A^{-1*}X|^2)\int_0^{\infty}s^3e^{-\frac{s^2}{2}}ds\\ &< \infty. \qquad \heartsuit\end{aligned}$$

Bibliography

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