GROUP EXTENSIONS OF GIBBS-MARKOV MAPS.

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Abstract. Let ϕ be an aperiodic cocycles with values in a locally compact abelian second countable group $\mathbb{G}$ defined on an exact Gibbs-Markov map $T : X \to X$. We show that the group extension $T_\phi(x, g) = (T(x), g + \phi(x))$ $(x \in X; g \in \mathbb{G})$ is exact. Equivalent conditions for exactness are found.

§1 Introduction

Let $(X, \mathcal{B}, m, T, \alpha)$ be an exact probability preserving Markov map as in [Aar97], where $(X, \mathcal{B}, m)$ denotes a probability space, $T : X \to X$ is a probability preserving transformation and α a generating Markov partition (possibly countable). We can and do assume that X is a topological Markov shift:

$$X = \{x = x_1, x_2, ... \in \alpha^{\mathbb{N}} : \ m(x_n \cap T^{-1}x_{n+1}) > 0 \ \forall \ n \geq 1\}$$

endowed with the Polish topology inherited from the product topology on $\alpha^{\mathbb{N}}$.

It follows that T is *locally invertible* with respect to α in the sense that for each $n \geq 1$, $a \in \alpha_0^{n-1}$ the map $T^n : a \to T^n a$ is nonsingular and invertible. The inverse of this map is denoted $v_a : T^n a \to a$ and given by $v_a(x_1, x_2, \dots) = a, x_1, x_2, \dots$, where a is identified with an element of $\alpha^{\{1,\dots,n\}}$. We let v_a' denote the Radon-Nikodym derivative of $m \circ v_a$ with respect to m.

The partition α enables the definition of a Hölder class of metrics $\{d_r : \ 0 < r < 1\}$ on X:
For $n \geq 1$, define $a_n : X \to \alpha_0^{n-1}$ by $x \in a_n(x) \in \alpha_0^{n-1}$.
For $x, y \in X$ define $t(x, y) := \min\{n \geq 1 : \ a_n(x) \neq a_n(y)\}$ $(\leq \infty)$.
For $r \in (0, 1)$ define $d_r : X \times X \to \mathbb{R}$ by $d_r(x, y) := r^{t(x,y)}$.

1991 *Mathematics Subject Classification.* 28D05, 60B15; (58F15, 58F19, 58F30).

[1] Research supported by Eurandom.
[2] Research supported by Eurandom and the Deutsche Forschungsgemeinschaft, Schwerpunkt Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme.

It is easily seen that the identity : $(X, d_r) \to (X, d_s)$ is Hölder continuous $\forall\ r, s \in (0,1)$.

Accordingly, we define the Hölder constants of a function $h : A \to M$ ($A \subset X$) with values in a metric space (M, ρ) by

$$D_{r,A}(h) := \sup_{x,y \in A} \frac{\rho(h(x), h(y))}{r^{t(x,y)}}.$$

Let $\mathrm{Lip}_r(M) := \{h : X \to M :\ \sup_{a\in\alpha} D_{r,a}(h) < \infty\}$. In case $M = \mathbb{R}$ we simply write $\mathrm{Lip}_r := \mathrm{Lip}_r(M)$ instead. A function $h : X \to M$ is called uniformly Hölder continuous on states if $h \in \mathrm{Lip}_r(M)$ for some $0 < r < 1$.

Recall (see e.g. [AD96]) that $(X, \mathcal{B}, m, T, \alpha)$ has

the *Gibbs property* if $\exists\ C > 1,\ 0 < r < 1$ such that $\forall\ n \geq 1,\ a \in \alpha_0^{n-1},\ m(a) > 0$:
$\left|\frac{v_a'(x)}{v_a'(y)} - 1\right| \leq Cr^{t(x,y)}$ for $m \times m$-a.e. $(x,y) \in T^n a \times T^n a$.
It is called a *Gibbs-Markov map* if it has in addition the property

$$\inf_{a\in\alpha} m(Ta) > 0.$$

Recall that any topologically mixing probability preserving Markov map with the Gibbs property is exact (see for example [ADU93]).

Now let $\mathbb{G}$ be a locally compact, abelian, second countable group, let $\|\cdot\|$ be a Lipschitz norm on $\mathbb{G}$ (i.e. $\gamma : \mathbb{G} \to S^1$ is $\|\cdot\|$-Lipschitz for every $\gamma \in \widehat{\mathbb{G}}$), and let $\phi : X \to \mathbb{G}$ be measurable. Consider the skew product transformation $T_\phi : X \times \mathbb{G} \to X \times \mathbb{G}$ defined by $T_\phi(x,y) := (Tx, y + \phi(x))$ with respect to the (invariant) product measure $m \times m_{\mathbb{G}}$ where $m_{\mathbb{G}}$ denotes Haar measure. We define $\phi_n = \phi + \phi \circ T + \ldots + \phi \circ T^{n-1}$ and for $x \in X$

$$\mathbb{G}_x = \left\{ t \in \mathbb{G} :\ \exists\ k_n \to \infty,\ y_n, z_n \in T^{-k_n}\{x\} :\ \begin{cases} d_r(y_n, z_n) \to 0 \\ \phi_{k_n}(y_n) - \phi_{k_n}(z_n) \to t \end{cases} \right\}.$$

We're interested in the exactness of T_ϕ and prove

Theorem

Let $\mathbb{G}$ be a LCA, second countable group, let $(X, \mathcal{B}, m, T)$ be an exact probability preserving Gibbs-Markov map and let $\phi : X \to \mathbb{G}$ be uniformly Hölder continuous on states.

The following are equivalent:

1. *1.) ϕ is aperiodic in the sense that $\gamma \circ \phi = \frac{zgT}{g}$ has no non-trivial solutions in $\gamma \in \widehat{\mathbb{G}}$, $z \in S^1$ and $g : X \to S^1$ Hölder continuous.*
2. *2.) T_ϕ is weakly mixing.*
3. *3.) T_ϕ is exact.*

(4) *4.) For some $A \in \mathcal{B}$, $m(A) > 0$ and for all $x \in A$, the smallest closed subgroup generated by $\mathbb{G}_x$ is $\mathbb{G}$.*
(5) *5.) For every $x \in X$, $\mathbb{G} = \mathbb{G}_x$.*

Remarks:

1. In case α is a finite Markov partition and m a Gibbs measure as in [Bow08], Guivarc'h ([Gui89]) has obtained exactness of the group extension with respect to aperiodic, Hölder-continuous, $\mathbb{R}^d$-valued cocycles.

2. Let T be as in the theorem and let $\phi : X \to \mathbb{Z}^d$ be aperiodic, locally Lipschitz and in the domain of attraction of a stable distribution of order $0 < p < 2$. For conservative T_ϕ, exactness follows from section 7 in [AD96].

3. The assumptions on the cocycle and the dynamics in these results have been weakened in [AD]:
For an exact Markov map T with the Renyi property and a cocycle $\phi : X \to \mathbb{R}^d$ which is locally constant (on cylinders in α_0^N for some $N \geq 0$), topological mixing of T_ϕ implies its exactness.

4. Let T be a locally invertible, exact endomorphisms with quasicompact Frobenius-Perron operators whose perturbations have a spectral representation à la Nagaev ([Nag57]). If $\phi : X \to \mathbb{R}^d$ is aperiodic and there is a subsequence n_k such that $\phi + \ldots + \phi \circ T^{n_k}$ $(k = 1, 2, \ldots)$ increases at most exponentially, then T_ϕ is exact.

The proof of the theorem is given in the subsequent sections. The only non-trivial implications are 4.) $\Longrightarrow$ 3.) and 1.) $\Longrightarrow$ 5.). Our proof follows general concepts, like [LRW94] and [Fog75] for the first implication and [Sto66] for the second. In particular the last section contains a ratio limit theorem of independent interest.

The *Frobenius-Perron operators* $\widehat{R}^n : L_1(m) \to L_1(m)$ of a nonsingular transformation $(X, \mathcal{B}, m, R)$ are defined by

$$\int_X \widehat{R}^n f \cdot g dm = \int_X f \cdot g \circ R^n dm$$

where $f \in L_1(m)$ and $g \in L_\infty(m)$. For a Gibbs-Markov map T these operators have the form

$$\widehat{T}^n f(x) = \sum_{a \in \alpha_0^{n-1}} 1_{T^n a}(x) \cdot v_a'(x) \cdot f(v_a(x)) = \sum_{T^n(z) = x} p_n(x, z) f(z),$$

where $p_n(x, z) = v'_{a_n(z)}(x) 1_{\{T^n(z)\}}(x)$, and for the group extension T_ϕ

$$\widehat{T}_\phi^n f(x, g) = \widehat{T}^n [f(\cdot, g - \phi_n(\cdot))](x).$$

Fix some $r \in (0,1)$. We define the Banach space L of all L_∞-functions $f : X \to \mathbb{R}$ with

$$D_{r,X}(f) < \infty.$$

We may assume that r is chosen so large that $D_\phi = \sup_{a\in\alpha} D_{r,a}(\phi) < \infty$. It is shown in [AD96] that $\widehat{T}^n : L \to L$ $(n \geq 1)$ has a representation

$$\widehat{T}^n f(x) = \int f dm + O(\rho^n \|f\|_L)$$

for some $0 < \rho < 1$ independent of $f \in L$.

Proof of 4.) $\implies$ 3.)

We begin with the following easy observation: For $\Psi \in L_1(m)$ and $\Gamma \in L_1(\mathbb{G})$ we obtain

$$\begin{aligned}
&\int_X \int_{\mathbb{G}} \left|\hat{T}_\phi^{n+1}(\Psi\otimes\Gamma)(x,g)\right| dg\ m(dx)\\
&\leq \int_X \int_{\mathbb{G}} \sum_{T(z)=x} \left|\hat{T}_\phi^{n}[\Psi\otimes\Gamma](z,g-\phi(z))\right| p_1(x,z) dg\ m(dx)\\
&= \int_{\mathbb{G}} \int_X \hat{T}\left[\left|\hat{T}_\phi^{n}[\Psi\otimes\Gamma](\cdot,g-\phi(\cdot))\right|\right](x) m(dx) dg\\
&= \int_X \int_{\mathbb{G}} \left|\hat{T}_\phi^{n}[\Psi\otimes\Gamma](x,g-\phi(x))\right| dg\ m(dx)\\
&= \int_X \int_{\mathbb{G}} \left|\hat{T}_\phi^{n}[\Psi\otimes\Gamma](x,g)\right| dg\ m(dx) =: U_n(\Psi\otimes\Gamma).
\end{aligned}$$

Therefore $C(\Psi\otimes\Gamma)$ is well defined by

$$U_n(\Psi\otimes\Gamma) \downarrow C(\Psi\otimes\Gamma) \geq 0. \tag{1}$$

We define the operators $M_t : L_1(\mathbb{G}) \to L_1(\mathbb{G})$ by $M_t\Gamma(g) = \Gamma(g+t)$. Let $\Psi \in L_1(X)$ be fixed and let the measures $\{\mu_{n,x} : n \geq 1\}$ on $\mathbb{G}$ be defined by

$$\mu_{n,x} = \sum_{T^n(z)=x} \Psi(z) p_n(x,z) \delta_{\phi_n(z)}.$$

Note that

$$\mu_{n,x} \star \Gamma(g) = \widehat{T}_\phi^n(\Psi\otimes\Gamma)(x,g)$$

hence $\|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} \leq \widehat{T}^n|\Psi|(x)\ \|\Gamma\|_{L_1(\mathbb{G})}$ and $t \mapsto \|\mu_{n,x} \star M_t\Gamma\|_{L_1(\mathbb{G})}$ is continuous with modulus of continuity bounded by $\widehat{T}^n|\Psi|(x)\|\Gamma - M_\delta\Gamma\|_{L_1(\mathbb{G})}$.

Remark: Following [LRW94], p.287, a family of signed random measures $\{\mu_{n,x} : n \geq 1, x \in X\}$ on $\mathbb{G}$ is called *completely mixing in* $L_1(m)$ if for every $\Gamma \in L_1(\mathbb{G})$ with integral $\int_{\mathbb{G}} \Gamma(g)dg = 0$ we have

$$\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})} \to 0$$

in $L_1(m)$. We'll show in Proposition 1 and Lemma 2 below that the random signed measures $\{\mu_{n,x} : n \geq 1\}$ are completely mixing in $L_1(m)$.

Proposition 1: *For every* $\Gamma \in L_1(\mathbb{G})$ *the random sequence*

$$\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})}$$

converges in $L_1(m)$ *to* $C(\Psi \otimes \Gamma)$. *In addition,*

$$C(\Psi \otimes \Gamma) \leq \|\Psi\|_{L_1(m)} \|\Gamma\|_{L_1(\mathbb{G})}.$$

Proof. Since $\hat{T}_\phi^n(\Psi \otimes \Gamma)(x,g) = \hat{T}^n[\Psi(\cdot)\Gamma(g - \phi_n(\cdot))](x)$ for $\Psi \in L_1(X)$ and $\Gamma \in L_1(\mathbb{G})$, it suffices to show the theorem for a subclass of pairs (Ψ, Γ) which generates a dense subspace in $L_1(X) \times L_1(\mathbb{G})$. Here we take the class of all functions $\Psi \otimes \Gamma$ where Ψ belongs to the space L and Γ is an integrable and Lipschitz continuous function on $\mathbb{G}$.

By definition

$$\begin{aligned}
\mu_{n+1,x} \star \Gamma(g) &= \int_{\mathbb{G}} \Gamma(g-h)\mu_{n+1,x}(dh) \\
&= \sum_{T^{n+1}(z)=x} \Psi(z)p_{n+1}(x,z)\Gamma(g - \phi_{n+1}(z)) \\
&= \sum_{T(z)=x} p_1(x,z)\hat{T}_\phi^n[\Psi \otimes \Gamma](z, g - \phi(z))
\end{aligned}$$

whence as before,

$$\begin{aligned}
&\|\mu_{n+1,x} \star \Gamma\|_{L_1(\mathbb{G})} \\
&\leq \int_{\mathbb{G}} \sum_{T(z)=x} p_1(x,z) \left|\hat{T}_\phi^n[\Psi \otimes \Gamma](z, g - \phi(z))\right| dg \\
&= \sum_{T(z)=x} p_1(z,x) \int_{\mathbb{G}} \left|\hat{T}_\phi^n[\Psi \otimes \Gamma](z,g)\right| dg \\
&= \hat{T}\left[\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})}\right](x).
\end{aligned}$$

By induction it is easily seen that for n fixed and $k \geq 1$

$$\|\mu_{n+k,x} \star \Gamma\|_{L_1(\mathbb{G})} \leq \hat{T}^k \left[\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})}\right](x).$$

Since the function

$$x \to \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})}$$

is of class L it follows from the spectral representation of $\widehat{T}$ that for $k \to \infty$

$$\hat{T}^k \left[\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})} \right] \to \int_X \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} m(dx) = U_n(\Psi \otimes \Gamma) \downarrow C(\Psi \otimes \Gamma),$$

whence

$$\limsup_{n\to\infty} \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} \le C(\Psi \otimes \Gamma). \tag{2}$$

By (1) and (2), given $\epsilon > 0$, we can choose n_0 so large that for $n \ge n_0$

$$\int_{\{x : \|\mu_{n,x}\star\Gamma\|_{L_1(\mathbb{G})} - C(\Psi\otimes\Gamma) > 0\}} \left[\|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} - C(\Psi \otimes \Gamma) \right] m(dx) \le \epsilon$$

and

$$\int_X \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} m(dx) - C(\Psi \otimes \Gamma) \ge 0.$$

It follows that

$$\begin{aligned}
&\int_X \Big| \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} - C(\Psi \otimes \Gamma) \Big| m(dx) \\
&= 2 \int_{\{x : \|\mu_{n,x}\star\Gamma\|_{L_1(\mathbb{G})} - C(\Psi\otimes\Gamma) > 0\}} \left[\|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} - C(\Psi \otimes \Gamma) \right] m(dx) \\
&- \int_X \left[\|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} - C(\Psi \otimes \Gamma) \right] m(dx) \\
&\le 2\epsilon.
\end{aligned}$$

The additional claim follows from

$$C(\Psi\otimes\Gamma) \leftarrow_{L_1(m)} \|\mu_{n,x}\star\Gamma\|_{L_1(\mathbb{G})} \le \widehat{T}^n|\Psi|(x)\|\Gamma\|_{L_1(\mathbb{G})} \to \|\Psi\|_{L_1(m)}\|\Gamma\|_{L_1(\mathbb{G})}. \qquad \square$$

Let $(Y, \mathcal{A}, \mu, R)$ and $(Z, \mathcal{C}, \nu, S)$ be nonsingular transformations of probability spaces, where the second transformation is a factor of the first one. The factor map $\pi : Y \to Z$ is called *relatively exact* if for $f \in L_1(\mu)$

$$E(f|\pi^{-1}\mathcal{C}) = 0 \implies \widehat{R}^n f \to 0$$

in $L_1(\mu)$. By [Gui89], see alternatively [AD], R is exact if the factor map $\pi : Y \to Z$ is relatively exact and the factor S is exact. In the present situation T_ϕ is exact if the factor map $(x, g) \mapsto x =: \Pi(x, g)$ $(X \times \mathbb{G} \to X)$ is relatively exact. To establish relative exactness of Π, it suffices to show

$$\int_X \int_{\mathbb{G}} \left| \hat{T}_\phi^n [\Psi \otimes \Gamma](x, g) \right| m_{\mathbb{G}}(dg) m(dx) \to 0$$

for all $\Psi \in L_1(m)$ and $\Gamma \in L_1(\mathbb{G})$ satisfying $\int_{\mathbb{G}} \Gamma dg = 0$.

It is left to prove the following

Lemma 2: *If* $\int_{\mathbb{G}} \Gamma(g)dg = 0$*, then*

$$C(\Psi \otimes \Gamma) = 0.$$

Proof. The proof of this statement follows from a series of claims. For the first 4 claims we assume that $\Gamma \in L_1(\mathbb{G})$ is Lipschitz continuous and has compact support. These claims are needed for the proof of the statement of the lemma in claim 5.

Define the measures $\nu_{n,x} = \sum_{T^n(z)=x} p_n(x,z)\delta_z$ on X.

Claim 1: *Let* $k \geq 0$ *be fixed. For any subsequence* $\{n_l : l \in \mathbb{N}\} \subset \mathbb{N}$ *there exists a further subsequence* $\{m_j : j \geq 1\}$ *such that for a.e.* $x \in X$ *and for every* $B \in \mathcal{B}$

$$\lim_{j\to\infty} \frac{1}{\nu_{k,x}(B)} \int_{\mathbb{G}} \left| \int_B \left(\mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\right)(g)\nu_{k,x}(dy)\right| dg = C(\Psi \otimes \Gamma). \quad (3)$$

In order to see this claim, let n_l be any subsequence and choose m_j so that

$$\|\mu_{m_j,x} \star \Gamma\|_{L_1(\mathbb{G})}, \|\mu_{m_j+k,x} \star \Gamma\|_{L_1(\mathbb{G})} \to C(\Psi \otimes \Gamma) \quad (4)$$

for $x \in \Omega$ where Ω is a T-invariant set of full measure (cf. Proposition 1). On the one hand it follows from this that for every B fixed

$$\begin{aligned} &\frac{1}{\nu_{k,x}(B)} \int_{\mathbb{G}} \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma \ \nu_{k,x}(dy)\right| dg \\ (5) \qquad &\leq \frac{1}{\nu_{k,x}(B)} \int_B \|\mu_{m_j,y} \star \Gamma\|_{L_1(\mathbb{G})}\nu_{k,x}(dy) \to C(\Psi \otimes \Gamma), \end{aligned}$$

because the integrand is uniformly bounded and pointwise convergent by (4).

On the other hand, for $x \in \Omega$,

$$\begin{aligned} C(\Psi \otimes \Gamma) &= \lim_{j\to\infty} \|\mu_{m_j+k,x} \star \Gamma\|_{L_1(\mathbb{G})} \\ &= \lim_{j\to\infty} \int_{\mathbb{G}} \left| \sum_{T^k(y)=x} p_k(x,y)\hat{T}_\phi^{m_j}[\Psi \otimes \Gamma](y, g - \phi_k(y))\right| dg \\ &\leq \lim_{j\to\infty} \int_{\mathbb{G}} \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy)\right| + \left| \int_{B^c} \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy)\right| dg \\ &\leq C(\Psi \otimes \Gamma) \end{aligned}$$

by (5), hence for $x \in \Omega$

$$\lim_{j\to\infty} \frac{1}{\nu_{k,x}(B)} \int_{\mathbb{G}} \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\nu_{k,x}(dy)\right| dg = C(\Psi \otimes \Gamma),$$

proving claim 1.

Claim 2: *Let $k \geq 0$ be fixed. For any subsequence $\{n_l : l \in \mathbb{N}\} \subset \mathbb{N}$ there exists a further subsequence $\{m_j : j \geq 1\}$ such that for a.e. $x \in X$ and for every disjoint sets $A, B \in \mathcal{B}$*

$$\lim_{j\to\infty} \int_{\mathbb{G}} \Bigg| \frac{1}{\nu_{k,x}(A)} \int_A \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \\ + \frac{1}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \Bigg| dg = 2C(\Psi \otimes \Gamma) \tag{6}$$

Choose the subsequence and Ω as in (4), then for $x \in \Omega$ by (3)

$$\begin{aligned} \int_{\mathbb{G}} \Bigg| \frac{1}{\nu_{k,x}(A)} \int_A \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\nu_{k,x}(dy) + \frac{1}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\nu_{k,x}(dy) \Bigg| dg \\ \leq \frac{1}{\nu_{k,x}(A)} \int_{\mathbb{G}} \Bigg| \int_A \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \Bigg| dg \\ + \frac{1}{\nu_{k,x}(B)} \int_{\mathbb{G}} \Bigg| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \Bigg| dg \\ \to 2C(\Psi \otimes \Gamma) \end{aligned} \tag{7}$$

and, since $A \cap B = \varnothing$ (and w.l.o.g. assume that $\nu_{k,x}(A) \leq \nu_{k,x}(B)$),

$$\begin{aligned} \frac{1}{\nu_{k,x}(A)} \int_{\mathbb{G}} \Bigg| \int_A \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\nu_{k,x}(dy) + \frac{\nu_{k,x}(A)}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\nu_{k,x}(dy) \Bigg| dg \\ \geq \frac{1}{\nu_{k,x}(A)} \Bigg(\int_{\mathbb{G}} \Bigg| \int_{A\cup B} \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \Bigg| dg \\ - \left(1 - \frac{\nu_{k,x}(A)}{\nu_{k,x}(B)}\right) \int_{\mathbb{G}} \Bigg| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \Bigg| dg \Bigg) \\ \to 2C(\Psi \otimes \Gamma). \end{aligned} \tag{8}$$

Claim 2 follows from (7) and (8).

Claim 3: *Let $A, B \in \alpha_0^{k-1}$ be images of inverse branches v_A and v_B of T^k, where k is still fixed. Let $\epsilon = d_r(A,B)$ and let Γ be Lipschitz continuous with compact support K; then there exist constants $C_0, C_1 > 0$ such that for every $n \geq 1$*

$$\int_{\mathbb{G}} \left| \mu_{n,v_A(x)} \star M_{\phi_k(v_A(x))}\Gamma(g) - \mu_{n,v_B(x)} \star M_{\phi_k(v_A(x))}\Gamma(g) \right| dg \\ \leq \left[C_1 \|\Gamma\|_{L_1(\mathbb{G})} + D_\Gamma C_0 D_\phi m_{\mathbb{G}}(B(K, C_0 D_\phi \epsilon)) \right] \epsilon, \tag{9}$$

where D_Γ denotes the Lipschitz constant of Γ.

Let $x \in X$, $v = v_A(x)$ and $w = v_B(x)$. We may assume that $d_r(A,B) < r$ so that $A \cup B$ is contained in some atom from α. By the Lipschitz property of ϕ and by the expanding property of T, we have for any inverse branch $v_a : A \cup B \to a \in (\alpha)_0^{n-1}$ of T^n that

$$\begin{aligned}|\phi_n(v_a(v)) - \phi_n(v_a(w))| &\le D_\phi \sum_{l=0}^{n-1} d_r(T^l(v_a(v)), T^l(v_a(w)))\\ &\le C_0' D_\phi d_r(v,w) \le C_0' D_\phi \epsilon,\end{aligned}$$

where C_0' denotes some constant. Since Γ has compact support

$$\|\Gamma(g) - \Gamma(g + \phi_n(v_a(v)) - \phi_n(v_a(w)))\| \le D_\Gamma C_0' D_\phi \epsilon 1_{B(K, C_0' D_\phi \epsilon)}(g).$$

Similarly, there exists a constant C_1' (also depending on the Lipschitz constant of Ψ) so that (see [AD96])

$$|p_n(v, v_a(v))\Psi(v_a(v)) - p_n(w, v_a(w))\Psi(v_a(w))| \le C_1' p_n(v, v_a(v)) d_r(v,w).$$

Therefore

$$\begin{aligned}&\int_{\mathbb{G}} \left|\mu_{n,v_A(x)} \star M_{\phi_k(v_A(x))}\Gamma(g) - \mu_{n,v_B(x)} \star M_{\phi_k(v_A(x))}\Gamma(g)\right| dg\\
&= \int_{\mathbb{G}} \left| \sum_a p_n(v, v_a(v))\Psi(v_a(v))\Gamma(g - \phi_k(v) - \phi_n(v_a(v))) \right.\\
&\left. - \sum_a p_n(w, v_a(w))\Psi(v_a(w))\Gamma(g - \phi_k(v) - \phi_n(v_a(w))) \right| dg\\
&\le \int_{\mathbb{G}} \left| \sum_a [p_n(v, v_a(v))\Psi(v_a(v)) - p_n(w, v_a(w))\Psi(v_a(w))] \right.\\
&\qquad \left. \Gamma(g - \phi_k(v) - \phi_n(v_a(v))) \right| dg\\
&+ \int_{\mathbb{G}} \left| \sum_a p_n(w, v_a(w))\Psi(v_a(w)) \right.\\
&\qquad \left. [\Gamma(g - \phi_k(v) - \phi_n(v_a(v))) - \Gamma(g - \phi_k(v) - \phi_n(v_a(w)))] \right| dg\\
&\le \left(C_1' \|\Gamma\|_{L_1(\mathbb{G})} + D_\Gamma C_0' D_\phi \|\Psi\|_\infty m_{\mathbb{G}}(B(K, C_0' D_\phi \epsilon))\right) \|\hat{T}^n 1\|_\infty \epsilon,\end{aligned}$$

where $\sum_a$ extends over all $a \in \alpha_0^{n-1}$ satisfying $T^n a \supset A \cup B$. The claim follows setting $C_i = 1 \vee C_i' \sup_{n \ge 1} \|\widehat{T}^n\|_\infty$ for $i = 0, 1$.

Claim 4: *There exists a set Ω of measure 1 and a constant $C > 0$ with the following property:*
If $x \in \Omega$, $k \ge 1$ and $v, w \in T^{-k}(\{x\})$, then

$$\left|2C(\Psi \otimes \Gamma) - C(\Psi \otimes (I + M_{\phi_k(v) - \phi_k(w)})\Gamma)\right| < C d_r(v,w). \qquad (1)0$$

By claims 1–3 there exists a subsequence $\{m_j : j \geq 1\} \subset \mathbb{N}$ and a subset Ω so that (6) and (9) hold for any $x \in \Omega$, $k \geq 1$ and $v, w \in T^{-k}(\{x\})$, $A = a_k(v)$, $B = a_k(w)$. Therefore

$$\begin{aligned}
&\int_{\mathbb{G}} \left| \frac{1}{\nu_{k,x}(A)} \int_A \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \right. \\
&\quad \left. + \frac{1}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \right| dg \\
&= \int_{\mathbb{G}} \left|\mu_{m_j,v} \star M_{\phi_k(v)}\Gamma(g) + \mu_{m_j,w} \star M_{\phi_k(w)}\Gamma(g)\right| dg \\
&\leq \int_{\mathbb{G}} \left|\mu_{m_j,v} \star M_{\phi_k(v)}\Gamma(g) - \mu_{m_j,w} \star M_{\phi_k(v)}\Gamma(g)\right| dg \\
&\quad + \int_{\mathbb{G}} \left|\mu_{m_j,w} \star M_{\phi_k(w)}\Gamma(g) + \mu_{m_j,w} \star M_{\phi_k(v)}\Gamma(g)\right| dg \\
&\leq \int_{\mathbb{G}} \left|\mu_{m_j,w} \star (I + M_{\phi_k(v)-\phi_k(w)})\Gamma(g)\right| dg + C d_r(v,w),
\end{aligned}$$

where $C = C_1\|\Gamma\|_{L_1(\mathbb{G})} + D_\Gamma C_0 D_\phi \|\Psi\|_\infty m_{\mathbb{G}}(B(K, C_0 D_\phi))$. The lower bound is shown similarly, proving claim 4.

Claim 5: *Let $\Psi \in L$, then for all $\Gamma \in L_1(\mathbb{G})$,*

$$C(\Psi \otimes (\Gamma - M_t\Gamma)) = 0.$$

First observe that by Proposition 1 the set of $t \in \mathbb{G}$ satisfying the claim is a group.

Hence it suffices to prove the claim for t in a generating set G_0. Moreover, it suffices to prove the claim for Γ Lipschitz continuous with compact support, since $\Gamma \mapsto C(\Psi \otimes \Gamma)$ is $L_1(\mathbb{G})$-norm continuous.

Fix such a Γ. By assumption, and by claim 4 there is a measurable set $A \in \mathcal{B}$ of positive measure and a constant $C > 0$ satisfying:
For $x \in A$ there is a subset $G_0 \subset \mathbb{G}$ generating a dense subgroup of $\mathbb{G}$ such that for all $v, w \in T^{-k}(x)$

$$\left|2C(\Psi \otimes \Gamma) - C(\Psi \otimes (I + M_{\phi_k(v)-\phi_k(w)})\Gamma)\right| < C d_r(v,w), \tag{1)0}$$

and

$$\begin{aligned}
(1) \qquad &\forall t \in G_0 \ \exists k_n \geq 1, v_n, w_n \in T^{-k_n}(x)1 \\
&\text{such that } \phi_{k_n}(v_n) - \phi_{k_n}(w_n) \to t \ \& \ d_r(v_n, w_n) \to 0.
\end{aligned}$$

Since $t \to C(\Psi \otimes M_t\Gamma)$ is continuous, it follows from properties (10) and (11) that

$$2C(\Psi \otimes \Gamma) = C(\Psi \otimes (I + M_t)\Gamma) \quad (t \in G_0). \tag{1)2}$$

It follows that (12) holds for all Lipschitz continuous Γ with compact support. Because of continuity, this equation holds for any $\Gamma \in L_1(\mathbb{G})$. Hence, replacing Γ by $(I - M_t)\Gamma$ and repeating this argument for each $(I + M_t)^k(I - M_t)\Gamma$, $k \geq 0$, we obtain

$$C(\Psi \otimes (I - M_t)\Gamma) = 2^{-k}C(\Psi \otimes (I + M_t)^k(I - M_t)\Gamma)$$

for every $k \geq 0$ and $t \in G_0$. From this we deduce $C(\Psi \otimes \Gamma) = 0$ as in [Fog75].

The lemma follows now from the well known fact (see [Lin71], [**?**]) that

$$\overline{\bigcup_{t \in \mathbb{G}} (I - M_t)L_1(\mathbb{G})} = \{f \in L_1(\mathbb{G}) : \int f(g)dg = 0\}. \quad \square$$

Proof of 1.) $\implies$ 5.)

Ratio limit theorem for symmetric cocycles

Suppose that $\phi : X \to \mathbb{G}$ is Hölder continuous, aperiodic and symmetric in the sense that there exists a probability preserving invertible transformation $S : X \to X$ such that $ST = TS$ and $\phi \circ S = -\phi$, then there exists $u_n > 0$ such that

$$\frac{\widehat{T}_\phi^n(h \otimes f)(x, y)}{u_n} \to \int_{X \times \mathbb{G}} h \otimes f dm \times m_{\mathbb{G}}$$

for all $h \in L,\ f \in C_c(\mathbb{G}),\ x \in X,\ y \in \mathbb{G}$.

Proof.

First let (as in[AD96]) $P_\gamma : L \to L \quad (\gamma \in \widehat{\mathbb{G}})$ be defined by

$$P_\gamma h := \widehat{T}(\gamma \circ \phi \cdot h).$$

As shown in [AD96], $\gamma \mapsto P_\gamma$ is continuous ($\widehat{\mathbb{G}} \to \mathrm{Hom}\,(L, L)$), and $\exists\ \epsilon > 0,\ 0 \leq \theta < 1$ and continuous functions

$$\lambda : B_{\widehat{\mathbb{G}}}(0, \epsilon) \to B_{\mathbb{C}}(0, 1), \text{ and } g : B_{\widehat{\mathbb{G}}}(0, \epsilon) \to L,$$

such that

$$\begin{aligned}
&\lambda(0) = 1,\ g(0) \equiv 1,\ \int_X g(\gamma)dm \equiv 1,\\
&|\lambda(\gamma)| \le 1 \text{ with equality iff } \gamma = 0,\\
&P_\gamma h = \lambda h \implies |\lambda| \le |\lambda(\gamma)| \quad (\gamma \in B_{\widehat{\mathbb{G}}}(0,\epsilon)),\\
&P_\gamma h = \lambda(\gamma)h \iff h \in \mathbb{R}\cdot g(\gamma) \quad (\gamma \in B_{\widehat{\mathbb{G}}}(0,\epsilon)),\\
&P_\gamma^n h = \lambda(\gamma)^n \int hdm\ g(\gamma) + O(\theta^n) \quad (\gamma \in B_{\widehat{\mathbb{G}}}(0,\epsilon))
\end{aligned}$$

and (as is easily shown)

$$g(-\gamma) = \overline{g(\gamma)},\ \lambda(-\gamma) = \overline{\lambda(\gamma)}.$$

Since $TS = ST$ and $\Phi \circ S = -\Phi$, $P_\gamma h(x) = [P_\gamma h \circ S^{-1}](Sx)$. It follows that $P_{-\gamma}[g(\gamma)\circ S](x) = \lambda(\gamma)g(\gamma)\circ S(x)$ whence

$$g(-\gamma) = g(\gamma)\circ S, \text{ and } \lambda(\gamma) \in \mathbb{R}.$$

Next, for $0 < \eta \le \epsilon$ set $u_n(\eta) := \int_{B(0,\eta)} \lambda(\gamma)^n d\gamma$. For η small enough (so that $\lambda > 0$ on $B(0,\eta)$), $u_n(\eta) > 0$. Choose one such $\eta_0 > 0$ and define $u_n := u_n(\eta_0)$. Note that $\rho^n = o(u_n)\ \forall\ \rho < 1$ since $\exists\ \eta < \eta_0$ such that $\min_{|\gamma|<\eta} |\lambda(\gamma)| = r > \rho$ whence

$$\frac{u_n}{\rho^n} \ge \frac{u_n(\eta)}{\rho^n} \ge \frac{r^n}{\rho^n}\cdot m(B(0,\eta)) \to \infty.$$

Also, for $0 < \eta < \eta'$,

$$u_n(\eta) = u_n(\eta') \pm O(\rho(\eta)^n)$$

where $\rho(\eta) := \sup_{\eta\le|\gamma|\le\epsilon} |\lambda(\gamma)| < 1$. Thus

$$u_n(\eta) \sim u_n \text{ as } n \to \infty\ \ \forall\ 0 < \eta \le \epsilon.$$

Now fix $h \in L$ and $f \in L^1(\mathbb{G})$ with $\hat{f} \in C_c(\widehat{\mathbb{G}})$, then $\forall\ x \in X,\ y \in \mathbb{G}$,

$$\begin{aligned}
\widehat{T}_\phi^n(h\otimes f)(x,y) &= \int_{\widehat{\mathbb{G}}} \hat{f}(\gamma)\overline{\gamma(y)}P_\gamma^n h(x)d\gamma\\
&= \int_X hdm \int_{B(0,\eta_0)} \lambda(\gamma)^n \Re(\overline{\gamma(y)}\hat{f}(\gamma)g(\gamma)(x))d\gamma + O(\theta^n)
\end{aligned}$$

(by reality of $\lambda(\gamma)$, for some $0 < \theta < 1$). Since $\Re(\hat{f}(\gamma)\overline{\gamma(y)}g(\gamma)(x)) \to \int_{\mathbb{G}} fdm_{\mathbb{G}}$ as $\gamma \to 0$, it follows that

$$\widehat{T}_\phi^n(h\otimes f)(x,y) \sim u_n \int_X hdm \int_{\mathbb{G}} fdm_{\mathbb{G}}.$$

By the method of Breiman ([Bre68], Theorem 10.7),

$$\widehat{T}_\phi^n(h\otimes f)(x,y) \sim u_n \int_X hdm \int_{\mathbb{G}} fdm_{\mathbb{G}} \quad \forall\ h \in L, f \in C_c(\mathbb{G}). \quad \square$$

Corollary

Under the same assumptions, $\forall\ x, y \in X,\ t \in \mathbb{G},\ \epsilon > 0,\ \exists n_0$ *such that* $\forall\ n \ge n_0\ \exists\ z \in T^{-n}\{x\}$ *such that* $d(y, z) < \epsilon$ *and* $\|t - \phi_n(z)\| < \epsilon$.

Proof.

Let $a = [a_1, \ldots, a_N] = B(y, \epsilon),\ h = 1_a \in L$ and let $f \in C(\mathbb{G}),\ f \ge 0,\ [f > 0] \subset B(0, \epsilon)$, then

$$\frac{\widehat{T}_\phi^n(h \otimes f)(x,t)}{u_n} \to \int_{X \times \mathbb{G}} h \otimes f dm \times m_{\mathbb{G}}$$

and $\exists\ n_0$ such that $\forall\ n \ge n_0$,

$$0 < \widehat{T}_\phi^n(h \otimes f)(x,t) = \sum_{T^n z = x,\ d(y,z) < \epsilon} p_n(x,z) f(t - \phi_n(z))$$

and in particular $\exists\ \ z \in T^{-n}\{x\}$ such that $d(y, z) < \epsilon$ and $\|t - \phi_n(z)\| < \epsilon$. $\square$

Exactness lemma

Suppose that $\phi : X \to \mathbb{G}$ *is Hölder continuous, aperiodic, then* $\forall\ x \in X,\ t \in \mathbb{G},\ \epsilon > 0,\ \exists n_0$ *such that* $\forall\ n \ge n_0\ \exists\ y, z \in T^{-n}\{x\}$ *such that* $d(y, z) < \epsilon$ *and* $\|t + \phi_n(y) - \phi_n(z)\| < \epsilon$.

Proof.

Consider the mixing Gibbs-Markov map $(X \times X, \mathcal{B}(X \times X), T \times T, m \times m, \alpha \times \alpha)$ equipped with the cocycle $\tilde{\phi} : X \times X \to \mathbb{G}$ defined by $\tilde{\phi}(x, x') := \phi(x) - \phi(x')$.

The cocycle $\tilde{\phi} : X \times X \to \mathbb{G}$ is also Hölder continuous, aperiodic, but also symmetric: $\tilde{\phi} \circ S = -\tilde{\phi}$ where $S(x, x') := (x', x)$ (evidently $S(T \times T) = (T \times T)S$). Thus the conclusion of the corollary holds and this is the lemma. $\square$

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