ISOMORPHISM OF RANDOM WALKS

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ABSTRACT. We show that any two aperiodic, recurrent random walks on the integers whose jump distributions have finite seventh moment, are isomorphic as infinite measure preserving transformations. The method of proof involved uses a notion of equivalence of renewal sequences, and the "relative" isomorphism of Bernoulli shifts respecting a common state lumping with the same conditional entropy. We also prove an analogous result for random walks on the two dimensional integer lattice.

§0 Introduction: Markov shifts and random walks

The *Markov shift* $T = T_P$ of an irreducible, recurrent stochastic matrix P on the countable state space S, with stationary distribution $\{m_s : s \in S\}$ is the left shift on $X_T = S^{\mathbb{Z}}$ equipped with the T-invariant σ-algebra generated by *cylinder* sets of form

$$[s_1, \ldots, s_n] = \{x \in X_T : x_k = s_k \ (1 \le k \le n)\},$$

and the T-invariant measure m_T defined by

$$m_T([s_1, \ldots, s_n]) = m_{s_1} p_{s_1,s_2} \cdots p_{s_{n-1},s_n}.$$

It is known that T is a conservative, ergodic, measure preserving transformation.

The isomorphism theory of positively recurrent Markov shifts is well understood. All aperiodic, positively recurrent Markov shifts with the same given entropy are isomorphic ([O1,O2,FO], see also [O3]). All Bernoulli shifts with the same given entropy are finitarily isomorphic ([KS,Pet]).

This paper is the first establishing isomorphism theorems for a class of null recurrent Markov shifts, and indeed for a class of conservative infinite measure preserving transformations.

Invariants for isomorphism of Markov shifts are given in [A1] where it was shown that the *asymptotic type*, the asymptotic proportionality class of the *return sequence*

$$a_n(T_P) \sim \frac{1}{m_s}\sum_{k=0}^{n-1} p_{s,s}^{(k)} \ \forall \ s \in S$$

is a similarity invariant: if T_P and $T_{P'}$ are similar (that is, have a common extension, for example are isomorphic, see §1), then

$$\exists \lim_{n\to\infty} \frac{a_n(T_P)}{a_n(T_{P'})} \in \mathbb{R}_+.$$

In this paper, we consider random walks on $\mathbb{Z}^d$ $(d = 1, 2))$. Let f be a probability on G, a countable Abelian group, and define a stochastic matrix $P = P_f$ on $S = G$ by $p_{s,t} := f_{t-s}$. It is evident that $m_s = 1$ is a stationary distribution for P_f, and the shift T_f of (P_f, m) is known as the *random walk* (on G) with *jump distribution* f. The stochastic matrix P_f is irreducible iff $S_f := \{t \in G : f_t > 0\}$ is contained in no proper subgroup of G, and P_f is aperiodic if S_f is contained in no coset of any proper subgroup of G. It is known that a random walk on $\mathbb{Z}$ is conservative if, for example its jump distribution has first moment, and is centred. A random walk on $\mathbb{Z}^2$ is conservative if its jump distribution has second moment, and is centred. In [A1], an uncountable, pairwise dissimilar collection of aperiodic, recurrent random walks on $\mathbb{Z}$ was presented. The jump distributions of this collection are in the domains of attraction of different stable laws, and have return sequences, no two of which are asymptotically proportional.

For irreducible, random walks on $\mathbb{Z}^d$ $(d = 1, 2)$ with centred jump distributions f with finite second moment, return sequences are of form

$$a_n(T_f) \sim \begin{cases} c_f^{(1)}\sqrt{n} \text{ on } \mathbb{Z}, \\ c_f^{(2)} \log n \text{ on } \mathbb{Z}^2. \end{cases}$$

where

$$c_f^{(1)} = \sqrt{\frac{2}{\pi\sigma_f^2}}, \quad \sigma_f^2 = \sum_{n\in\mathbb{Z}} n^2 f_n,$$

and

$$c_f^{(2)} = \frac{1}{2\pi \det \Gamma} \quad \Gamma_{i,j} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} n_i n_j f_{(n_1,n_2)} \quad (i, j = 1, 2).$$

Here, we consider random walks on $\mathbb{Z}^d$ ($d = 1, 2$) with jump distributions f satisfying

$$\sum_{n \in \mathbb{Z}^d} |n|^7 f_n < \infty, \qquad \sum_{n \in \mathbb{Z}^d} n f_n = 0, \tag{1}$$

for example

$$\pi_k^{(d)} = \begin{cases} \left(\frac{1}{3}\right)^d & k \in \{0, \pm 1\}^d \\ 0 & \text{else.} \end{cases}$$

Main Theorem If f is an aperiodic probability on $\mathbb{Z}^d$ ($d = 1, 2$), and satisfies (1), then T_f is isomorphic with $T_{\pi^{(d)}}$.

The structure of the proof is as follows. A Markov shift is viewed as a *Markov tower* (§3) that is, a (Kakutani) tower (§1) whose height function is measurable with respect to an independent generator for the base transformation. The distribution of the height function of a Markov tower is determined by a canonically associated renewal sequence (§3). The finitary isomorphism of Markov towers with the same renewal sequence (proposition 3.1) is contingent on the finitary "relative" isomorphism of Bernoulli shifts with respect to a common state lumping (theorem 2.1). A non-finitary relative isomorphism theorem was established in [Th].

The notion of equivalence of renewal sequences is introduced in §3, and the isomorphism of ergodic random walks with equivalent renewal sequences is established (theorem 3.6). The proof that the random walks appearing in the main theorem have equivalent renewal sequences (theorem 5.1) is given in §§4,5. It uses the structure of Kaluza sequences (§4), and a refined local limit theorem.

Subsequently, theorem 5.1 has been improved in [ALP].

§1 Isomorphism, Factors, and Similarity

In this paper, a measure preserving transformation T is considered acting on a *standard* measure space $(X_T, \mathcal{B}_T, m_T)$ (a complete, separable metric space equipped with its Borel sets and a σ-finite, non-atomic measure). It is known that standardness is unaffected by replacing X_T with a T-invariant subset $X'_T \in \mathcal{B}_T$ of full measure, and we shall consider T acting on $(X_T, \mathcal{B}_T, m_T)$ to be the same as T acting on $(X'_T, \mathcal{B}_T \cap X'_T, m_T)$.

Let S and T be measure preserving transformations, and let $c \in \mathbb{R}_+$. A *c-factor map* from S to T is a map $\pi : X_S \to X_T$ such that

$$\pi S = T\pi,\ \pi^{-1}\mathcal{B}_T \subset \mathcal{B}_S,\ \text{and } m_S \circ \pi^{-1} = cm_T.$$

In this situation (denoted by $\pi : S \xrightarrow{c} T$), one says that T is a *c-factor* of S and that S is a *c-extension* of T (both denoted $S \xrightarrow{c} T$).

It is necessary to consider c-factor maps with $c \neq 1$ as our measure spaces are not normalised. The constant c can be thought of as a relative normalisation of the transformations concerned. It was shown in [A1] that if T_P and T_Q are Markov shifts and S is a measure preserving transformation such that

$$T_P \xleftarrow{1} S \xrightarrow{c} T_Q,$$

then

$$\frac{a_n(T_Q)}{a_n(T_P)} \to c \text{ as } n \to \infty.$$

In case we wish to suppress the relative normalisation, we say that a measure preserving transformation T is a *factor* of a measure preserving transformation S (written $S \to T$) if it is a c-factor for some $c \in \mathbb{R}_+$. In this case, we also say that S is an *extension* of T.

Two measure preserving transformations are said to be *similar* if they have a common extension, that is: if there is another measure preserving transformation of which they are both factors; and they are said to be *strongly disjoint* if they have no common extension.

Any two transformations preserving finite measures are similar, their Cartesian product being a common extension. Invariants for similarity of transformations preserving infinite measures are studied in [A1,A2], where it is shown that similarity is an equivalence relation. As mentioned in §0, an uncountable family of pairwise strongly disjoint aperiodic, recurrent random walks is presented in [A1]. Examples of conservative, ergodic, measure preserving transformations which are strongly disjoint from their inverses are given in [A2]. We note here that it follows from proposition 3.1 of this paper that any conservative, ergodic Markov shift is finitarily isomorphic to its inverse.

An invertible c-factor map from S to T is known as a *c-isomorphism* (from S to T), and if one exists, S is said to be *c-isomorphic* to T. Measure preserving transformations S and T are said to be *isomorphic* if S is c-isomorphic to T for some $c \in \mathbb{R}_+$. For example, if T_f and $T_{f'}$ are isomorphic random walks on $\mathbb{Z}$ as in the main theorem, then T_f is $\frac{\sigma_f}{\sigma_{f'}}$-isomorphic to $T_{f'}$ (because $\frac{a_n(T_{f'})}{a_n(T_f)} \to \frac{\sigma_f}{\sigma_{f'}}$ as $n \to \infty$).

In this paper, we establish isomorphism between infinite measure preserving transformations by means of an isomorphism of (Kakutani) towers ([Kak]). A tower T is built using a finite measure preserving transformation S, called the *base transformation*, and a measurable function $\varphi : X_S \to \mathbb{N}$, called the *height-*, or *return time function*. One sets

$$X_T = \{(x,n) : x \in X_S,\ 1 \le n \le \varphi(x)\},$$
$$\mathcal{B}_T = \sigma\{A \times \{n\} : n \in \mathbb{N},\ A \in \mathcal{B}_S \cap [\varphi \ge n]\},\ m_T(A \times \{n\}) = m_S(A),$$
$$\text{and } T(x,n) = \begin{cases} (Sx, \varphi(Sx)) \text{ if } n = 1, \\ (x, n-1) \text{ if } n \ge 2. \end{cases}$$

The tower is called *finite*, or *infinite* in accordance with the value of

$$m_T(X_T) = \int_{X_S} \varphi dm_S.$$

It was shown in [Kak] that a tower is a conservative, measure preserving transformation, and is ergodic iff the base is ergodic. It was also shown there that any invertible, conservative, ergodic, measure preserving transformation is isomorphic to a tower over any of its induced transformations. In view of this, we shall sometimes specify this tower representation for T by specifying the *base set* $A \in \mathcal{B}_T$.

Proposition 1.1 Suppose that T and T' are towers over the base transformations S and S', with height functions φ and φ' respectively. Suppose that $\pi : S \to S'$ is an isomorphism of S to S' such that

$$\varphi' \circ \pi = \varphi \ m_S\text{-a.e. on } X_S,$$

then T and T' are isomorphic.

proof We define $\phi : X_T \to X_{S'} \times \mathbb{N}$ by

$$\phi(x,n) = (\pi(x), n).$$

The conditions imposed on π ensure that ϕ is an isomorphism from T to T'.

We shall say that a tower is *normalised* if its base is a probability preserving transformation. If the towers in proposition 1.1 are both normalised, then ϕ turns out to be a 1-isomorphism.

§2 Isomorphisms of Bernoulli shifts respecting state lumpings

The kind of isomorphism of bases assumed in proposition 1.1 respects the return time function, and has been studied in [Th]. In this section, we establish isomorphisms of Bernoulli shifts which respect state lumpings. Let $B(\beta)$ denote the Bernoulli shift with independent generating partition β. Now let α be a *lumping* of β, that is $\alpha \prec \beta$ (every $a \in \alpha$ can be written as a union of elements of β), then the Bernoulli shift $B(\alpha)$ is a factor of $B(\beta)$ by the canonical factor map $\pi = \pi_\alpha : B(\beta) \to B(\alpha)$ defined by $\pi(x)_n = \alpha(x_n)$ where $x \in \alpha(x) \in \alpha$. This kind of factor map is called a *state lumping*.

Recall that in this situation, the Kolmogorov-Sinai entropies of the measure preserving transformations $B(\alpha)$, and $B(\beta)$, $h(B(\beta))$, and $h(B(\alpha))$ are related in the following way:

$$h(B(\beta)) = H(\beta) = H(\alpha) + H(\beta|\alpha) = h(B(\alpha)) + H(\beta|\alpha),$$

where $H(\beta)$ denotes the entropy of the partition β,

$$H(\beta) = \sum_{b\in\beta} m(b) \log\left(\frac{1}{m(b)}\right),$$

and the quantity $H(\beta|\alpha)$, (the conditional entropy of β given α) is given by

$$H(\beta|\alpha) = \sum_{a\in\alpha} m(a) H(\beta(a))$$

where for a a union of atoms of β,

$$\beta(a) = \{b \in \beta : b \subseteq a\} \text{ and } H(\beta(a)) = \sum_{b\in\beta(a)} m(b|a) \log\left(\frac{1}{m(b|a)}\right).$$

Note that there may be many ways in which α occurs as a lumping of β. In case $H(\alpha) < \infty$, the conditional entropy is always the same (being $H(\beta) - H(\alpha)$).

Suppose now that $B(\beta)$, $B(\beta')$, and $B(\alpha)$ are Bernoulli shifts, and that α appears as a lumping both of β, and β'. Let $\pi_\alpha : B(\beta) \to B(\alpha)$, and $\pi'_\alpha : B(\beta') \to B(\alpha)$ be state lumpings. We'll say that an isomorphism $\pi : B(\beta) \to B(\beta')$ *respects* the state lumpings π_α and π'_α if

$$\pi'_\alpha \circ \pi = \pi_\alpha.$$

We shall sometimes abbreviate this to "π respects α".

Theorem 2.1 Suppose that $B(\beta)$, $B(\beta')$, and $B(\alpha)$ are Bernoulli shifts, α appearing as a partition both of β, and β'. Let $\pi_\alpha : B(\beta) \to B(\alpha)$, and $\pi'_\alpha : B(\beta') \to B(\alpha)$ be state lumpings. If

$$H(\beta|\alpha) = H(\beta'|\alpha),$$

then there is a finitary isomorphism $\pi : B(\beta) \to B(\beta')$ respecting α.

Remarks

1. The assumptions of the theorem hold if, in particular, $H(\alpha) < \infty$ and $H(\beta) = H(\beta')$.

2. A non-finitary isomorphism respecting the state lumpings (at least when β, β' are finite) can be obtained using the methods of [Th]: the transformation $B(\beta)$ is "relatively weak Bernoulli" with respect to the factor $(B(\beta), \alpha)$ which therefore "splits off".

3. Theorem 2.1 in case β, β' are finite can be obtained by a modification of the methods of [KS]. The modification made is in the definition of filler measures. Choose a "marker" in $a \in \alpha$, and use this as in [KS] to define skeletons. Now consider a "sinew" to be a word of α-symbols whose projection onto the marker process is a skeleton. Given a sinew s of length m, corresponding to a skeleton which concludes in a run of ℓ a-s. Consider the filler space $\mathcal{F}(s) = \beta^{m-\ell}$ equipped with the filler measure $\mu(\cdot|s)$. The last ℓ coordinates are left free in order to ensure independence of fillers. Now continue as in [KS] obtaining "assignments" which respect sinews, and which yield an isomorphism, as by the ergodic theorem,

$$\frac{1}{n}\log\left(\frac{1}{\mu(b_1,\ldots,b_n|[\alpha(b_1),\ldots,\alpha(b_n)])}\right) \to H(\beta|\alpha)$$

in measure as $n \to \infty$. This isomorphism respects α. An analogous modification of the methods of [Pet] works when β, β' are both infinite.

We outline a proof of theorem 2.1 assuming only the non-relative

Finitary Isomorphism Theorem ([KS,Pet])

Bernoulli shifts with the same entropy are finitarily isomorphic.

Remark

This theorem is established in [KS] for Bernoulli shifts on finite state spaces, and in [Pet] for Bernoulli shifts on infinite state spaces. In order to complete the proof of this theorem, it is necessary to show that a Bernoulli shift $B(\beta)$ on a finite state space is isomorphic to a Bernoulli shift $B(\beta')$ on an infinite state space with the same entropy.

This can be done as follows. Firstly, the Bernoulli shift $B(\varepsilon)$ on four equiprobable symbols is finitarily isomorphic to $B(\varepsilon')$ where $\varepsilon' =$

$\{E_1, E_2, \dots\}$ with $m(E_k) = \frac{1}{2^{k+1}}$. This follows from the Blum-Hanson theorem (see [BH] or [Jac; p.289]), and can be proved by the methods of the proof of step 2 (below).

Next, by [KS,Pet], we may assume that there is some Bernoulli shift $B(\eta)$ such that $B(\beta) = B(\eta) \times B(\delta)$ and $B(\beta') = B(\eta) \times B(\delta')$, where δ and δ' both have a lumping $\alpha = \{0, 1\}$ and

$$\delta(0) = \delta'(0) = \{0\},\ \delta(1) = \varepsilon,\ \delta'(1) = \varepsilon'.$$

It follows from step 1 (below) that $B(\delta)$ and $B(\delta')$ are finitarily isomorphic, whence, $B(\beta)$ and $B(\beta')$ are finitarily isomorphic.

proof of theorem 2.1 For the rest of this section, all isomorphisms are finitary, α is fixed, and it will be convenient to refer to an α-respecting isomorphism $\pi : B(\beta) \to B(\beta')$ as being *over* (states) $D \subset \alpha$ if $\pi(x)_n = x_n$ whenever $\alpha(x_n) \notin D$. Clearly, if $B(\beta)$ and $B(\beta')$ are isomorphic over D, then $\beta(a) = \beta'(a)\ \forall\ a \in \alpha \setminus D$, and any isomorphism of $B(\beta)$ and $B(\beta')$ over D respects

$$\alpha' := D \cup \bigcup_{a \in \alpha \setminus D} \beta(a)$$

of which α is a lumping.

Step 1 (Isomorphism over a single state) Suppose that there is a state $a \in \alpha$ such that

$$H(\beta(a)) = H(\beta'(a)),\quad \beta(a') = \beta'(a')\ \forall\ a' \neq a,\ a' \in \alpha,$$

then $B(\beta)$ and $B(\beta')$ are isomorphic over a.

proof Let $\psi : B(\beta(a)) \to B(\beta'(a))$ be an isomorphism as in the finitary isomorphism theorem. Consider $B(\beta)$ and $B(\beta')$ as towers over $B(\beta(a))$ and $B(\beta'(a))$ with height functions h and h' respectively. Writing $H_n = [h = n]$, it is clear that

$$\gamma = \{H_n \cap [b, c_1, \dots, c_{n-1}] : n \geq 1, b \in \beta(a), c_k \in \beta(a_k),\ a_k \neq a,\ 1 \leq k \leq n-1\}$$

is a generator for $B(\beta(a))$, as is γ' (defined analogously) a generator for $B(\beta'(a))$. Using the notation

$$x \in \gamma(x) = H_{h(x)} \cap [b(x), c_1(x), \dots, c_{h(x)-1}(x)] \in \gamma,$$

an isomorphism $\phi : B(\beta(a)) \to B(\beta'(a))$ is given by

$$\gamma'(\phi(x)) = H'_{h(x)} \cap [(\psi(x))_0, c_1(x), \dots, c_{h(x)-1}(x)].$$

An isomorphism of the towers $\varphi : B(\beta) \to B(\beta')$ is obtained by proposition 1.1 as, evidently, $h' \circ \psi = h$. It is directly verifiable that π has the required properties.

<u>**Step 2 (Meshalkin, Blum-Hanson Isomorphism)**</u> Suppose that $0, 1 \in \alpha$, and

$$mq_0 = nq_1, \beta(0) = \beta'(0)\times C^n, \beta'(1) = \beta(1)\times C^m,\ \beta'(a) = \beta(a)\ \forall\ a \neq 0, 1,\ a \in \alpha,$$

then $B(\beta)$ and $B(\beta')$ are isomorphic over $\{0, 1\}$.
<u>**proof**</u> See also [Mesh,BH,Jac p.289]. We establish a matching of 0-s and 1-s in generic printouts of the α-process. Let $x \in \alpha^{\mathbb{Z}}$ and let $K_i = K_i(x) = \{n \in \mathbb{Z} : x_n = i\}$. A *matching* will be a map

$$M : \alpha^{\mathbb{Z}} \to 2^{(\mathbb{Z}\times\{1,\dots,m\})\times(\mathbb{Z}\times\{1,\dots,n\})}$$

such that for a.e. $x \in \alpha^{\mathbb{Z}}$ there is a bijection $\varphi_x : K_0(x) \times \{1, \dots, m\} \to K_1(x) \times \{1, \dots, n\}$ satisfying

$$M(x) = \{((k, j), \varphi_x(k, j)) : k \in K_0(x),\ 1 \le j \le m\},$$

and $\varphi_{Tx}(k - 1, j) = \varphi_x(k, j) - (1, 0)$. The matching M is measurable if

$$\{x \in \alpha^{\mathbb{Z}} : ((k, j), (\ell, i)) \in M(x)\} \in \mathcal{B}(\alpha^{\mathbb{Z}})\ \forall\ k, j, \ell, i \in \mathbb{Z},$$

and finitary if the above sets are open mod 0.

Given a measurable matching, write for $\ell \in K_1(x),\ 1 \le j \le m$

$$\varphi_x^{-1}(\ell, j) = (k(\ell, j), \kappa(\ell, j)) \in K_1(x) \times \{1, \dots, n\}.$$

Writing, for $b \in B(0)$, $b = (y(b), c_1(b), \dots, c_n(b)) \in \beta'(0) \times C^n$, we define the isomorphism $\psi : B \to B'$ by

$$\psi(x)_u = \begin{cases} x_u & u \notin K_0 \cup K_1 \\ y(x_u) & u \in K_0 \\ (x_u, c_{\kappa(u,1)}(x_{k(u,1)}), \dots, c_{\kappa(u,m)}(x_{k(u,m)})) \in \beta(1) \times C^m = \beta'(1) & u \in K_1 \end{cases}$$

If the matching M is finitary, then so is the isomorphism ψ.

The matching M is established as in [Mesh,BH] as follows. Consider a printout of the α process. Define an (initial) *valency* function $V : \alpha^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{Z}_+$ by

$$V(u) = \begin{cases} n & u \in K_0, \\ m & u \in K_1, \\ 0 & \text{else.} \end{cases}$$

Set

$$M_V = \{((u, V(u)), (v, V(v))) : (u, v) \in K_0\times K_1,\ u < v, V(w) = 0\ \forall\ u < w < v\}$$

and define the modified valency function

$$V'(u) = \begin{cases} V(u) - 1 & \exists\ v \ni (u, v) \text{ or } (v, u) \in M_1 \\ V(u) & \text{else .} \end{cases}$$

Note that $0 \le V' \le V$. Continue this process, obtaining

$$M_k \subset (K_0 \times \{1,\dots,m\}) \times (K_1 \times \{1,\dots,n\}) \text{ and } V_k : \mathbb{Z} \to \mathbb{Z}_+$$

such that $V_1 = V$, $M_k = M_{V_k}$, and $V_{k+1} = V_k'$. It follows from $mq_0 = nq_1$ by random walk theory, that $V_k(u) \downarrow 0$ a.e. $\forall\ u \in \mathbb{Z}$, and this means that

$$M := \bigcup_{k=1}^{\infty} M_k$$

is the required matching, which is easily checked to be finitary.

Step 3 (Isomorphism over two states) Suppose that $\exists\ 0,1 \in \alpha$, such that

$$q_0H(\beta(0))+q_1H(\beta(1)) = q_0H(\beta'(0))+q_1H(\beta'(1)),\ \beta'(a) = \beta(a)\ \forall\ a \neq 0,1,\ a \in \alpha,$$

then $B(\beta)$ and $B(\beta')$ are isomorphic over $\{0,1\}$.
proof A combination of steps 1 and 2 demonstrates this in case q_0 and q_1 are rationally dependent. Without loss of generality, $H(\beta(0)) > H(\beta'(0))$. The first case we consider is that $H(\beta(0)) > H(\beta'(0)) > 0$. Using step 1, we may assume that $\beta(0)$ refines $\beta'(0)$, and that $\exists\ b \in \beta'(0)$ such that $p(0 \setminus b)$ and $p(1)$ are rationally dependent. Now let $\alpha' = \{a \in \alpha : a \neq 0\} \cup \{b, 0 \setminus b\}$, and apply step 2, respecting α', and over the states $0 \setminus b$ and 1. This yields an isomorphism of $B(\beta)$ and $B(\beta')$ respecting α', and hence over α (which is a lumping of α'). To obtain the case $H(\beta'(0)) = 0$, consider an intermediary Bernoulli shift $B(\beta'')$ also with α as a state lumping, also satisfying

$$q_0H(\beta(0))+q_1H(\beta(1)) = q_0H(\beta''(0))+q_1H(\beta''(1)),\ \beta''(a) = \beta(a)\ \forall\ a \neq 0,1,$$

and in addition, $H(\beta(0)) > H(\beta''(0)) > 0$. By the above, $B(\beta)$ and $B(\beta'')$ are isomorphic over α. It follows from $H(\beta'(0)) = 0$ and $0 < H(\beta''(0)) < H(\beta(0))$ that $H(\beta'(1)) > H(\beta''(1)) > 0$, whence, again by the above, $B(\beta'')$ and $B(\beta')$ are isomorphic.

We establish the theorem in the case of a finite lumping α by concatenating finitely many isomorphisms as in step 3.

Step 4

There is a Bernoulli shift $B(\gamma)$ such that $H(\gamma) = H(\beta|\alpha)$, and such that $B(\beta)$ is isomorphic to $B(\alpha \times \gamma)$ respecting α.
proof We assume first that $H(\beta(a)) < \infty\ \forall\ a \in \alpha$. We write $\alpha = \{a_k : k \ge 1\}$, and for $\gamma = \{g_n\}_{n \in \mathbb{N}}$, we write $\overline{\gamma}_n = \{g_1, \dots, g_n, \overline{g}_n\}$ where $\overline{g}_n = \bigcup_{k>n} g_k$.

There is a partition γ, and a sequence $n_k \to \infty$ such that

$$\sum_{j=1}^{n_k} m(a_j | \bigcup_{i=1}^{n_k} a_i) H(\beta(a_j)) = H(\overline{\gamma}_k) \ \forall \ k \geq 1.$$

Now set $\beta = \beta_0$, and consider Bernoulli shifts $B(\beta_k)$ with α as a state lumping for each $k \geq 1$, such that

$$\beta_k(a_j) = \begin{cases} \overline{\gamma}_k & 1 \leq j \leq n_k, \\ \beta(a_j) & j > n_k. \end{cases}$$

Use step 2 to get isomorphisms $\pi_k : B(\beta_k) \to B(\beta_{k+1})$ respecting the refinement α_k of α defined by

$$\alpha_k(a_j) = \begin{cases} \overline{\gamma}_j & 1 \leq j \leq n_k, \\ \{a_j\} & j > n_k. \end{cases}$$

It now follows that each coordinate of

$$\pi_k \circ \cdots \circ \pi_0$$

changes only finitely many times a.e., whence

$$\pi_k \circ \cdots \circ \pi_0$$

converges a.e. The limit is clearly an isomorphism

$$\overline{\pi} : B(\beta) \to B(\alpha \times \gamma)$$

which respects α.

In case our assumption is not satisfied, we can write $\alpha = \{a_k : k \geq 1\}$ where $H(\beta(a_1)) = \infty$. Let $H(\gamma) = \infty$ and define a sequence β_k $(k \geq 1)$ of refinements of α by

$$\beta_k(a_j) = \begin{cases} \gamma & 1 \leq j \leq k, \\ \beta(a_j) & j \geq k+1. \end{cases}$$

By step 1, $B(\beta)$ and $B(\beta_1)$ are isomorphic over a_1, and by step 3, $B(\beta_k)$ and $B(\beta_{k+1})$ are isomorphic over $\{a_k, a_{k+1}\}$ $\forall$ $k \geq 1$. As above, the concatenations of these isomorphisms converge to yield an α-respecting isomorphism of $B(\beta)$ with $B(\alpha \times \gamma)$.

We claim that theorem 2.1 is now established, for by step 4, there are Bernoulli shifts $B(\gamma)$, $B(\gamma')$ such that $B(\beta)$ is isomorphic to $B(\alpha \times \gamma)$ respecting α and $B(\beta')$ is isomorphic to $B(\alpha \times \gamma')$ respecting α. Since $H(\gamma) = H(\beta|\alpha) = H(\beta'|\alpha) = H(\gamma')$, $B(\gamma)$ is isomorphic to $B(\gamma')$ by [Pet], whence $B(\alpha \times \gamma) = B(\alpha) \times B(\gamma)$ is isomorphic to $B(\alpha \times \gamma') = B(\alpha) \times B(\gamma')$ respecting α.

§3 Markov towers and their renewal sequences

A *Markov tower* is a tower equipped with an independent generator for the base transformation, with respect to which, the height function is measurable (i.e. the independent generator refines the partition generated by the height function).

Let T be a normalised Markov tower over the base S with height function φ. The *relative entropy* of the Markov tower is given by $H(\beta|\alpha(\varphi))$ where $\alpha(\varphi) = \{[\varphi = k]\}_{k\in\mathbb{N}}$, and β is the independent generator for S which refines $\alpha(\varphi)$.

Proposition 3.1

Suppose that T, and T' are two infinite, normalised Markov towers with the same height function distribution, and the same relative entropy,

then T and T' are finitarily isomorphic.

proof The base transformations have a common state lumping in the return time process, and there are, by assumption, independent generators for these base transformations refining the height function partititions, and having the same relative entropy. By theorem 2.1, there is an isomorphism of the base transformations which respects the return time state lumping, and hence by proposition 1.1, the towers are isomorphic.

Remarks about relative entropy. Let T be a conservative, ergodic, measure preserving transformation. It follows from Abramov's theorem [Ab] that

$$\underline{h}(T) = m_T(A)h(T_A)$$

is the same for any set $A \in \mathcal{B}_T$ of positive, finite measure, where $h(T_A)$ is the Kolmogorov-Sinai entropy of T_A, the induced transformation on A with respect to the probability $m_T(\cdot|A)$. The number $\underline{h}(T)$ is called the *entropy* of T. This generalisation of Kolmogorov-Sinai entropy was introduced in [Kre].

Suppose that T is a normalised Markov tower over the base set $A \in \mathcal{B}_T$, considered with respect to the independent generator β. Clearly,

$$\underline{h}(T) = h(T_A) = H(\beta) = H(\beta|\alpha(\varphi_A)) + H(\alpha(\varphi_A)),$$

whence, in case $H(\alpha(\varphi_A)) < \infty$, the relative entropy of the Markov tower over A is given by $\underline{h}(T) - H(\alpha(\varphi_A))$. It now follows from [A1] that two normalised Markov towers with the same finite entropy height function distribution, are isomorphic only if they have the same relative entropy.

We do not know if it is possible for the same conservative, ergodic, measure preserving transformation to be isomorphic to different normalised Markov towers with isomorphic bases, identical (infinite entropy) height function distributions, but different relative entropies.

If T is a Markov tower over the base set $A \in \mathcal{B}_T$, then $\{\varphi_A \circ T_A^n : n \in \mathbb{Z}\}$ are independent, identically distributed random variables on the probability space $(A, \mathcal{B}_T \cap A, m_T(\cdot|A))$, and hence the sequence $u = u(A)$ defined by

$$u_n = m_T(T^{-n}A|A) \quad (n \geq 0)$$

is a recurrent renewal sequence. It characterises the distribution of the height function φ_A by means of the renewal equation

$$u_n = \sum_{k=1}^{n} g_k u_{n-k} \ \forall \ n \in \mathbb{N}$$

(where $g_k = m([\varphi_A = k]|A)$). We call u the *renewal sequence of the Markov tower* T over A.

By proposition 3.1, two infinite, normalised Markov towers with the same renewal sequence, and the same relative entropy are finitarily isomorphic.

A Markov tower T with base A is called *simple* if the partition $\{[\varphi_A = k] : k \in \mathbb{N}\}$ is a generator for T_A. Any Markov tower is a (canonical) extension of a simple Markov tower with the same base, height function and renewal sequence. Simple Markov towers can be constructed as Markov shifts from their renewal sequences in the following way (due to [Ch]). Let u be a recurrent renewal sequence. There is a probability $g = g(u)$ on $\mathbb{N}$ satisfying the renewal equation

$$u_n = \sum_{k=1}^{n} g_k u_{n-k} \ \forall \ n \in \mathbb{N}.$$

One can define a stochastic matrix $P = P_u$ with state space $\mathbb{N}$ by

$$p_{j,k} = \begin{cases} g_k \text{ if } j = 1, \\ 1 \text{ if } j - k = 1, \\ 0 \text{ else.} \end{cases}$$

This matrix is irreducible, recurrent, and has the stationary distribution

$$m_k = \sum_{j=k}^{\infty} g_j.$$

The shift T_u of (P_u, m) is a normalised, simple Markov tower over the base $[1]$. The distribution of the height function is given by $m(\varphi_{[1]} = k|[1]) = g_k$, and the renewal sequence of the tower is

$$m(T_u^{-n}[1]|[1]) = p_{1,1}^{(n)} = u_n.$$

Clearly any simple Markov tower can be constructed in this way, and so if T is a Markov tower over the base set $A \in \mathcal{B}_T$, then $T \to T_{u(A)}$.

Proposition 3.2

If P is an irreducible, recurrent stochastic matrix on S, and $s \in S$, $p_{s,s}^{(n)} = u_n$, then T_P is a Markov tower over $[s]$ with renewal sequence u.

proof The partition

$$\beta = \{[s, t_1, \dots, t_n, s] : n \geq 0,\ t_1, \dots, t_n \in S \setminus \{s\}\}$$

is an independent generator for $T_{[s]}$ with respect to which $\varphi_{[s]}$ is measurable.

Proposition 3.3

Let B be a Bernoulli shift. Suppose that T is a normalised Markov tower over A, then $T \times B$ is a normalised Markov tower over $A \times X_B$ with relative entropy at least $m_T(X_T)h(B)$.

proof Again, we produce a suitable independent generator for the base transformetion $(T \times B)_{A \times X_B}$.

Let β be the independent generator for T_A with respect to which φ_A is measurable. If

$$\tilde{\beta} = \{b \times [a_1, \dots, a_k] : b \in \beta,\ k \in \mathbb{N},\ b \subset [\varphi_A = k],\ a_1, \dots a_k \in S\}$$

where S is the alphabet of B, then $\tilde{\beta}$ is an independent generator for $(T \times B)_{A \times X_B}$, and evidently, $\varphi_{A \times X_B} = \varphi_A$ is $\tilde{\beta}$-measurable. The relative entropy of this Markov tower is given by

$$\begin{aligned} H(\tilde{\beta}|\alpha(\varphi_{A \times X_B}) &= \sum_{k=1}^{\infty} g_k \sum_{b \in \tilde{\beta},\ b \subset [\varphi_A = k]} m(b|[\varphi_A = k]) \log\left(\frac{1}{m(b|[\varphi_A = k])}\right) \\ &\geq \sum_{k=1}^{\infty} g_k \sum_{a_1, \dots a_k \in S} m_B([a_1, \dots, a_k]) \log\left(\frac{1}{m_B([a_1, \dots, a_k])}\right) \\ &= \sum_{n=1}^{\infty} n h(B) g_n = m_T(X_T)h(B). \end{aligned}$$

Proposition 3.4 Suppose that G is a discrete Abelian group, and that $f \in \mathcal{P}(G)$, gives rise to an irreducible, recurrent random walk. Then T_f is a Markov tower over $[0]$ with infinite relative entropy.
proof Let $g_n = g_n(u) = m_{T_f}([\varphi_{[0]} = n]),\ n \in I\!N$. By ergodicity of the random walk
$$\sum_{n\in I\!N} n g_n = \infty.$$
Let
$$\beta = \{[0,\underline{b},0] : k \in I\!N,\ \underline{b} \in (G \setminus \{0\})^{k-1}\},$$
then, as in proposition 3.2, β is an independent generator for $(T_f)_{[0]}$ with respect to which $\varphi_{[0]}$ is measurable. Now

$$\begin{aligned}
&H(\beta|\alpha)\\
&= \sum_{n=1}^{\infty} g_{n+1} \sum_{\underline{b}\in(G\setminus\{0\})^n} \log\left(\frac{1}{\mu([0,\underline{b},0]|[\varphi = n+1])}\right)\mu([0,\underline{b},0]|[\varphi = n+1])\\
&= \lim_{N\to\infty} \sum_{n=1}^{N} g_{n+1} \sum_{\underline{b}\in(G\setminus\{0\})^n} \log\left(\frac{1}{\mu([0,\underline{b},0]|[\varphi = n+1])}\right)\mu([0,\underline{b},0]|[\varphi = n+1]).
\end{aligned}$$

Since $\exists\ p < 1$ such that $f_g \le p\ \forall\ g \in G$, we have that

$$\begin{aligned}
&\sum_{n=1}^{N} g_{n+1} \sum_{\underline{b}\in(G\setminus\{0\})^n} \log\left(\frac{1}{\mu([0,\underline{b},0]|[\varphi = n+1])}\right)\mu([0,\underline{b},0]|[\varphi = n+1])\\
&\ge \sum_{n=1}^{N} g_{n+1} \sum_{\underline{b}\in(G\setminus\{0\})^n} \left((n+1)\log\left(\frac{1}{p}\right) - \log\left(\frac{1}{g_{n+1}}\right)\right)\mu([0,\underline{b},0]|[\varphi = n+1])\\
&= \sum_{n=1}^{N} g_{n+1}\left((n+1)\log\left(\frac{1}{p}\right) - \log\left(\frac{1}{g_{n+1}}\right)\right)\\
&= \sum_{1\le n\le N,\ g_{n+1}>\frac{1}{n^3}} + \sum_{1\le n\le N,\ g_{n+1}\le\frac{1}{n^3}}\\
&\ge \sum_{1\le n\le N,\ g_{n+1}>\frac{1}{n^3}} \left((n+1)\log\left(\frac{1}{p}\right) - 3\log n\right) g_n + O(1)\\
&= \sum_{n=1}^{N} \left((n+1)\log\left(\frac{1}{p}\right) - 3\log n\right) g_n + O(1)\\
&\to \infty.
\end{aligned}$$

Remark In the case of an ergodic random walk on $\mathbb{Z}$ with jump distribution having finite second moment, $H(\alpha(\varphi_{[0]})) < \infty$ and proposition 3.4 could have been deduced from the remarks above about relative entropy. However, in the case of the simple ergodic random walk on $\mathbb{Z}^2$, $H(\alpha(\varphi_{[0]})) = \infty$.

Proposition 3.1 tells us when we can obtain isomorphism of Markov towers with the same renewal sequence. Now, we collect some information about Markov towers with different renewal sequences.

Two renewal sequences u, and u' are *equivalent* (denoted $u \sim u'$) if there are positively recurrent, aperiodic renewal sequences v, and v' such that $uv = u'v'$.

Theorem 3.5

Suppose that T, and T' are infinite Markov towers with infinite relative entropy, and equivalent renewal sequences. Then T and T' are isomorphic.

proof Let T and T' be infinite Markov towers over the sets $A \in \mathcal{B}_T$ and $A' \in \mathcal{B}_{T'}$ respectively. By assumption, there are positively recurrent, aperiodic renewal sequences v and v' such that

$$u(A)v = u(A')v' := w.$$

Let B be a Bernoulli shift with infinite entropy. Then

– T is a Markov tower over A with infinite relative entropy (by assumption).

– $T_{u(A)} \times B$ is a Markov tower over $A \times X_B$ with the same height function distribution, and (by propositions 3.2 and 3.3) with infinite relative entropy.

Therefore T and $T_{u(A)} \times B$ are isomorphic (by proposition 3.1).

Next, B and $T_v \times B$ are isomorphic (see [FO] and [O3]), whence $T_{u(A)} \times B$ and $T_{u(A)} \times T_v \times B$ are isomorphic.

Thus far, we have shown that T and $T_{u(A)} \times T_v \times B$ are isomorphic. By symmetry, T' and $T_{u(A')} \times T_{v'} \times B$ are also isomorphic.

To continue,

– $T_{u(A)} \times T_v$ is the Markov shift of the stochastic matrix $P_{u(A)} \times P_v$ on $\mathbb{N} \times \mathbb{N}$.

By proposition 3.2, $T_{u(A)} \times T_v$ is a Markov tower over $[1] \times [1]$ with renewal sequence w, and therefore $T_{u(A)} \times T_v \times B$ is a Markov tower over $[1] \times [1] \times X_B$ with renewal sequence w, and with infinite relative entropy (by proposition 3.3).

Similarly, $T_{u(A')} \times T_{v'} \times B$ is also a Markov tower over $[1] \times [1] \times X_B$ with renewal sequence w, and with infinite relative entropy.

Therefore $T_{u(A)} \times T_v \times B$ is isomorphic with $T_{u(A')} \times T_{v'} \times B$ (again by proposition 3.1), and the conclusion is that T and T' are isomorphic.

Theorem 3.6 Suppose that G and G' are discrete Abelian groups, and that $f \in \mathcal{P}(G)$, $f' \in \mathcal{P}(G')$ give rise to irreducible, recurrent random walks. If the renewal sequences $u(f) := (f_0^{*n})_{n\geq 0}$ and $u(f')$ are equivalent, then the random walks T_f and $T_{f'}$ are isomorphic.
proof By proposition 3.4, and theorem 3.5.

In view of theorem 3.6, in order to establish isomorphism between two random walks, it is sufficient to show that their renewal sequences are equivalent. Accordingly, the rest of this paper is devoted to studying the equivalence of renewal sequences.

§4 Kaluza sequences

A bounded sequence $u = \{u_n : n \geq 0\}$ is called a *Kaluza sequence* if $u_n > 0$, $u_0 = 1$, and

$$v_n := \frac{u_{n+1}}{u_n} \uparrow q \leq 1 \text{ as } n \uparrow \infty.$$

If $\{u_n : n \geq 0\}$ is a Kaluza sequence, then, [Ken,Kin],

$$u_n = q^n \prod_{k=1}^{\infty} p_k^{k\wedge n} \text{ where } p_k = \frac{v_{k-1}}{v_k} \in (0,1].$$

Clearly, any sequence of this form is a Kaluza sequence. We'll denote $p_n = p_n(u)$, and sometimes specify Kaluza sequences by giving $p_n \in (0,1]$.

Note that if u is a Kaluza sequence, then, since $u_1 > 0$, necessarily

$$\sum_{k=1}^{\infty} \log\left(\frac{1}{p_k(u)}\right) < \infty.$$

It was shown in [Kal], that Kaluza sequences are aperiodic renewal sequences (see also [Ken,Kin]), and it was shown in [Ken] that the Kaluza sequences are precisely the infinitely divisible elements in the semigroup of aperiodic renewal sequences (under pointwise multiplication).

If u is a recurrent Kaluza sequence, then $q(u) = 1$. Accordingly, we restrict attention henceforth to Kaluza sequences with $q(u) = 1$. The Kaluza sequence u is positively recurrent iff $u_n \downarrow u_\infty > 0$ as $n \uparrow \infty$, or equivalently

$$\sum_{k=1}^{\infty} k \log\left(\frac{1}{p_k(u)}\right) < \infty.$$

This will be the main tool for constructing the positively recurrent renewal sequences appearing in renewal sequence equivalence.

Two questions arise. When are two Kaluza sequences equivalent? When is a renewal sequence equivalent to a Kaluza sequence?

Let u be any non-negative sequence, with $u_n > 0$ for all n large (for example an aperiodic renewal sequence), then

$$\upsilon_n(u) = \frac{u_{n+1}}{u_n}, \text{ and } p_n(u) = \frac{\upsilon_{n-1}}{\upsilon_n}$$

are defined for large enough n. Indeed, if

$$n_u := \min\{n \geq 1,\ u_k > 0\ \forall\ k \geq n-1\}$$

then $p_n(u)$ is defined $\forall\ n \geq n_u - 1$.

Suppose $u_n > 0$ for every $n \geq 0$, then

$$\frac{\upsilon_n}{\upsilon_{n+k+1}} = \prod_{j=1}^{k} p_{n+j}.$$

Thus, if $u_n \sim u_{n+1}$, and $p_n \leq 1\ \forall\ n \geq 1$, then

$$0 < \upsilon_n \underset{k\to\infty}{\longleftarrow} \frac{\upsilon_n}{\upsilon_{n+k+1}} = \prod_{j=1}^{k} p_{n+j} \underset{k\to\infty}{\longrightarrow} = \prod_{j=1}^{\infty} p_{n+j},$$

whence u is a Kaluza sequence, and

$$\sum_{k=1}^{\infty} \log\left(\frac{1}{p_k(u)}\right) < \infty.$$

On the other hand, if

$$\sum_{n=1}^{\infty} |\log p_n(u)| < \infty,$$

we have

$$u_n = \prod_{k=1}^{\infty} p_k^{k\wedge n}.$$

Proposition 4.1 Let u be an aperiodic renewal sequence satisfying $u_n \sim u_{n+1}$ as $n \to \infty$. If

$$\sum_{n=n_u}^{\infty} n(\log p_n(u))_+ < \infty,$$

then u is equivalent to a Kaluza sequence w such that

$$\sum_{n=n_u}^{\infty} n|\log p_n(u) - \log p_n(w)| < \infty.$$

proof By aperiodicity, there exists $n_0 \geq 0$ such that the semigroup in $\mathbb{N}$ generated by $\{1 \leq k < n_0 : u_k > 0\}$ contains every $n \in \mathbb{N} \cap [n_0, \infty)$. If $\{g_n : n \in \mathbb{N}\}$ is the probability distribution on $\mathbb{N}$ satisfying

$$u_n = \sum_{k=1}^{n} g_k u_{n-k},$$

set

$$h_n = \begin{cases} g_n \text{ for } 1 \leq n < n_0 \\ \sum_{k=n_0}^{\infty} g_k \text{ for } n = n_0 \\ 0 \text{ else }, \end{cases}$$

and let v be the renewal sequence defined by

$$v_n = \sum_{k=1}^{n} h_k v_{n-k}.$$

Since $v_n = u_n$ for $1 \leq n < n_0$, we have that $v_n > 0$ for $n \geq n_0$. Since P_v is defined on finitely many states, we have that v is positively recurrent, and, moreover

$$\exists v_\infty > 0,\ r \in (0,1) \ \ni\ v_n = v_\infty(1 + O(r^n)).$$

Define a by

$$a_n = \begin{cases} 1 \text{ for } 1 \leq n < n_0 \\ \frac{u_n}{v_n} \text{ for } n \geq n_0, \end{cases}$$

then $a_n > 0$ for every $n \geq 1$, $a_n \sim a_{n+1}$, and (as $u_n = v_n\ \forall\ 1 \leq n < n_0$)

$$u_n = a_n v_n\ \forall\ n \geq 1.$$

Moreover,

$$\log p_n(a) - \log p_n(u) = O(r^n) \text{ as } n \to \infty,$$

whence

$$\sum_{n=1}^{\infty} n(\log p_n(a))_+ < \infty.$$

Define sequences v', and w by

$$v'_n := \prod_{k\ni\ p_k(a)>1} \left(\frac{1}{p_k(a)}\right)^{k\wedge n}, \quad w_n = a_n v'_n.$$

This product converges, and v' is a positively recurrent Kaluza sequence, since

$$\sum_{k=1}^{\infty} k \log\left(\frac{1}{p_k(v')}\right) < \infty.$$

Since $av' = w$, we have $w_n \sim w_{n+1}$, and by inspection,

$$1 \ge p_n(w) = \begin{cases} p_n(a) \text{ for } p_n(a) \le 1, \\ 1 \text{ for } p_n(a) > 1, \end{cases}$$

whence, by the remarks preceding this proposition, w is a Kaluza sequence.

Moreover,

$$uv' = avv' = wv$$

and

$$\sum_{n=n_u}^{\infty} n|\log p_n(u) - \log p_n(w)| < \infty.$$

Proposition 4.2 If u, u' are Kaluza sequences, and

$$\sum_{n=1}^{\infty} n|\log p_n(u) - \log p_n(u')| < \infty,$$

then u and u' are equivalent.

proof Define another Kaluza sequence w by

$$p_n(w) := p_n(u) \vee p_n(u').$$

It follows easily from the definitions that

$$u = vw, \ u' = v'w$$

where v, v' are Kaluza sequences defined by

$$p_n(v) = \frac{p_n(u)}{p_n(w)}, \quad p_n(v') = \frac{p_n(u')}{p_n(w)},$$

indeed, it follows that v, and v' are positively recurrent, whence u and u' are equivalent.

Proposition 4.3 Let u be an aperiodic renewal sequence, and let w be a Kaluza sequence such that

$$\sum_{n=n_u}^{\infty} n|\log p_n(u) - \log p_n(w)| < \infty,$$

then

$$u \sim w.$$

proof This follows from propositions 4.1 and 4.2.

For $\beta \geq 0$, let $u^{(\beta)}$ be defined by

$$u_n^{(\beta)} = \frac{1}{(n+1)^{\beta}}.$$

It can be calculated that

$$\log\left(\frac{1}{p_n(u^{(\beta)})}\right) = \frac{\beta}{n^2} + \delta_n \text{ where } \sum_{n=1}^{\infty} n|\delta_n| < \infty.$$

The renewal sequence $u^{(\beta)}$ is recurrent for $0 < \beta \leq 1$. By [1], $T_{u^{(\beta)}}$ and $T_{u^{(\beta')}}$ are dissimilar when $0 < \beta \neq \beta' \leq 1$.

Corollary 4.4 Let u be an aperiodic renewal sequence such that

$$\sum_{n=n_u}^{\infty} n\left|\log\left(\frac{1}{p_n(u)}\right) - \frac{\beta}{n^2}\right| < \infty,$$

then

$$u \sim u^{(\beta)}.$$

§5 Random walks

In this section, we prove

Theorem 5.1

If f is an aperiodic probability on $\mathbb{Z}^d$ satisfying (1), and $u_n(f) = p_{0,0}^{(n)}(f)$, then

$$u \sim u^{(\frac{d}{2})}.$$

Lemma 5.2 Suppose that u is a bounded, non-negative sequence, $\beta > 0$, $\Gamma \subset (0,2]$ is a finite set, $\varepsilon > 0$, and that

$$u_n = K u_n^{(\beta)}\left(1 + \sum_{\gamma \in \Gamma} \frac{a_\gamma}{n^\gamma} + O\left(\frac{1}{n^{2+\varepsilon}}\right)\right) \text{ as } n \to \infty$$

where $K > 0$, and $a_\gamma \in \mathbb{R}$ $(\gamma \in \Gamma)$, then

$$\log\left(\frac{1}{p_n(u)}\right) = \frac{\beta}{n^2} + \delta_n \text{ where } \sum_{n=1}^{\infty} n|\delta_n| < \infty.$$

proof We'll show that

$$\log\left(\frac{1}{p_n}\right) = \log\left(\frac{1}{p_n(u^{(\beta)})}\right) + O\left(\frac{1}{n^{2+\varepsilon'}}\right) \text{ as } n \to \infty$$

where $\varepsilon' > 0$. Set

$$A_n := \frac{u_n}{u_n^{(\beta)}} = K\left(1 + \sum_{\gamma\in\Gamma} \frac{a_\gamma}{n^\gamma} + O\left(\frac{1}{n^{2+\varepsilon}}\right)\right) \text{ as } n \to \infty.$$

Using the Taylor expansion of $\log(1 + x)$ near 0, it follows that

$$B_n := \log\left(\frac{1}{A_n}\right) = -\log K + \sum_{\gamma\in\Gamma'} \frac{a'_\gamma}{n^\gamma} + O\left(\frac{1}{n^{2+\varepsilon'}}\right) \text{ as } n \to \infty,$$

where $\Gamma' \subset (0, 2]$ is a finite set, $a'_\gamma \in \mathbb{R}$ $(\gamma \in \Gamma')$, and $\varepsilon' > 0$. If $B'_n = B_{n+1} - B_n$, and $B''_n = B'_{n+1} - B'_n$, then

$$\log\left(\frac{1}{p_n}\right) = \log\left(\frac{1}{p_n(u^{(\beta)})}\right) + B''_{n-1}.$$

Using the Taylor expansion of $(1 + x)^{-\gamma}$, $(\gamma > 0)$ near 0, we see that

$$\frac{1}{n^\gamma} - \frac{1}{(n+1)^\gamma} = \frac{\gamma}{n^{\gamma+1}} + O\left(\frac{1}{n^{\gamma+2}}\right),$$

whence

$$B''_n = O\left(\frac{1}{n^{2+\varepsilon''}}\right) \text{ as } n \to \infty$$

where $\varepsilon'' > 0$.

proof of theorem 5.1 We'll establish the precondition of lemma 5.2. Let X be a random variable on $\mathbb{Z}^d$ distributed as f, and let

$$\varphi(x) = E(e^{ix\cdot X}) = \sum_{n\in\mathbb{Z}^d} f_n e^{ix\cdot n}, \quad (x \in \mathbb{R}^d).$$

Choose $\delta > 0$ such that $|1 - \varphi(x)| \le r < 1$ for $|x| \le \delta$, and set

$$\psi(x) := \log\varphi(x) + \frac{\sigma(x)}{2},$$

where $\sigma(x) = E((x\cdot X)^2)$, and log is defined in a small neighbourhood of 1 as that holomorphic inverse of the exponential function with $\log 1 = 0$. Since f is aperiodic, we have that

$$|\varphi(x)| \le r' < 1 \ \forall \ \delta \le |x| \le \pi.$$

It follows that

$$\begin{aligned} u_n &= \frac{1}{(2\pi)^d}\int_{N(0,\delta)}(\varphi(t))^n dt + O(r'^n)\\ &= \frac{1}{(2\pi)^d n^{\frac{d}{2}}}\int_{N(0,\delta\sqrt{n})}\left(\varphi\left(\frac{t}{\sqrt{n}}\right)\right)^n dt + O(r'^n)\\ &= \frac{A_n}{(2\pi)^d n^{\frac{d}{2}}} + O(r'^n)\end{aligned}$$

where $N(0,r) = \{x \in \mathbb{R}^d : |x| < r\}$, and

$$\begin{aligned} A_n &= \int_{N(0,\delta\sqrt{n})}\left(\varphi\left(\frac{t}{\sqrt{n}}\right)\right)^n dt\\ &= \int_{N(0,\delta\sqrt{n})}\left(\varphi\left(\frac{t}{\sqrt{n}}\right)\right)^n dt\\ &= \int_{N(0,\delta\sqrt{n})} e^{-\frac{\sigma(t)}{2}}\exp\left(n\psi\left(\frac{t}{\sqrt{n}}\right)\right)dt.\end{aligned}$$

Clearly

$$A_n \to A = \int_{\mathbb{R}^d} e^{-\frac{\sigma(t)}{2}}dt,$$

and

$$A - A_n = \int_{N(0,\delta\sqrt{n})} e^{-\frac{\sigma(t)}{2}}\left(1 - \exp\left(n\psi\left(\frac{t}{\sqrt{n}}\right)\right)\right)dt + O(r''^n)$$

where $r'' \in (0,1)$.

Using (1) and Taylor's theorem, for $|x| \le \delta$,

$$\psi(x) = \sum_{k=3}^{6}\sum_{1\le i_1,\ldots,i_k\le d} a(i_1,\ldots,i_k)\prod_{\nu=1}^{k} x_{i_\nu} + O(|x|^7)$$

as $x \to 0$, whence for $|x| \le \delta\sqrt{n}$,

$$n\psi(\frac{x}{\sqrt{n}}) = \sum_{k=3}^{6}\frac{1}{n^{\frac{k}{2}-1}}\sum_{1\le i_1,\ldots,i_k\le d} a(i_1,\ldots,i_k)\prod_{\nu=1}^{k} x_{i_\nu} + O\left(\frac{|x|^7}{n^{\frac{5}{2}}}\right),$$

and

$$e^{n\psi(\frac{x}{\sqrt{n}})} - 1 = \sum_{k=1}^{4}\frac{1}{n^{\frac{k}{2}}}\sum_{1\le i_1,\ldots,i_{k+2}\le d} b(i_1,\ldots,i_{k+2})\prod_{\nu=1}^{k+2} x_{i_\nu} + O\left(\frac{|x|^7}{n^{\frac{5}{2}}}\right)$$

as $x \to 0$. Thus

$$A_n - A = \int_{\mathbb{R}^d} e^{-\frac{\sigma(x)}{2}} \left(\sum_{k=1}^{4} \frac{1}{n^{\frac{k}{2}}} \sum_{1 \le i_1, \dots, i_{k+2} \le d} b(i_1, \dots, i_{k+2}) \prod_{\nu=1}^{k+2} x_{i_\nu} \right) dx + O\left(\frac{1}{n^{\frac{5}{2}}} \right)$$

$$= \sum_{k=1}^{4} \frac{c_k}{n^{\frac{k}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}} \right).$$

It follows that

$$u_n = u_n(1/2) \left(A + \sum_{k=1}^{4} \frac{c_k}{n^{\frac{k}{2}}} + O\left(\frac{1}{n^{5/2}} \right) \right)$$

as $n \to \infty$.

Theorem 5.1 is now established by lemma 5.2, and corollary 4.4.

§6 Proof of the main theorem

Suppose $f \in \mathcal{P}(\mathbb{Z}^d)$ satisfies (1), then, by theorem 5.1, $u(f) \sim u(\frac{d}{2})$, and hence $u(f) \sim u(\pi^{(d)})$ (as, in particular, $\pi^{(d)}$ satisfies (1), and $u(\pi^{(d)}) \sim u(\frac{d}{2})$).

Thus, by theorem 3.6, (for $d = 1, 2$), T_f and $T_{\pi^{(d)}}$ are isomorphic.

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