ON THE STRUCTURE OF STABLE RANDOM WALKS

JON. AARONSON

ABSTRACT. We show that the Cauchy random walk on the line, and the Gaussian random walk on the plane are similar as infinite measure preserving transformations. (As in Proc. Indian Acad. Sci. (Math.Sci.) 104, 1993, 413-419).

§1 Introduction

A measure preserving transformation T is considered acting on a *standard* measure space $(X_T, \mathcal{B}_T, m_T)$ (a complete, separable metric space equipped with its Borel sets and a σ-finite, non-atomic measure). It is known that standardness is unaffected by replacing X_T with a T-invariant subset $X'_T \in \mathcal{B}_T$ of full measure, and we shall consider T acting on $(X_T, \mathcal{B}_T, m_T)$ to be the same as T acting on $(X'_T, \mathcal{B}_T \cap X'_T, m_T)$.

Let S and T be measure preserving transformations. A *factor map* from S to T is a map $\pi : X_S \to X_T$ such that

$$\pi S = T\pi, \ \pi^{-1}\mathcal{B}_T \subset \mathcal{B}_S, \ \text{and } m_S \circ \pi^{-1} = c m_T$$

where $0 < c < \infty$.

In this situation (denoted by $\pi : S \to T$), one says that T is a *factor* of S and that S is an *extension* of T (both denoted $S \to T$).

It is necessary to consider factor maps with $c \neq 1$ as our measure spaces are not normalised. The constant c can be thought of as a relative normalisation of the transformations concerned.

Two measure preserving transformations are said to be *similar* if they have a common extension, that is: if there is another measure preserving transformation of which they are both factors; and they are said to be *strongly disjoint* if they have no common extension. We denote the statement that S and T are similar by $S \sim T$.

1991 *Mathematics Subject Classification.* 28D05.
The author would like to thank the University of Paris XIII for providing hospitality while this paper was written.

Any two transformations preserving finite measures are similar, their Cartesian product being a common extension. Invariants for similarity are given in [Aar77, Aar87], where it is shown that similarity is an equivalence relation. Examples of conservative, ergodic, measure preserving transformations which are strongly disjoint from their inverses are given in [Aar87].

In this paper, we consider random walks on $\mathbb{R}$ and $\mathbb{R}^2$. For f a probability on G, a locally compact, second countable abelian group, the *random walk* on G with *jump distribution* f can be defined as follows: Let S_f be the shift on $G^{\mathbb{Z}}$ considered with the S_f-invariant product measure $m_f = \prod f$. The *random walk* on G with *jump distribution* f is the measure preserving transformation T_f defined on $(G^{\mathbb{Z}} \times G, m_f \times m_G)$ (where m_G is Haar measure on G) by

$$T_f(\underline{x}, y) = (S_f \underline{x}, y + x_0).$$

The structure of random walks on $\mathbb{Z}$ and $\mathbb{Z}^2$ has been considered in [AK94] and [ALP94] where conditions for isomorphism are given.

For $\alpha \in [1,2]$ let f_α denote the symmetric α-stable law on $\mathbb{R}$ with characteristic function

$$\int_{\mathbb{R}} e^{itx} df_\alpha(x) = e^{-|t|^\alpha}.$$

It is well known that f_1 has a Cauchy density, f_2 has a Gaussian density, and indeed, f_α is absolutely continuous with strictly positive, continuous density $\forall \alpha \in [1,2]$. Let $T_\alpha = T_{f_\alpha}$, the random walk on $\mathbb{R}$ with α-stable jump distribution.

Theorem.

$$T_1 \sim T_2 \times T_2.$$

Remark. For $\alpha, \alpha' \in [1,2]$ not both 2, we have that $\frac{1}{\alpha} + \frac{1}{\alpha'} > 1$, and $T_\alpha \times T_{\alpha'}$ is dissipative, and isomorphic to $z \mapsto z+1$ on $\mathbb{R}^2$ equipped with Lebesgue measure. This will be established below.

The method of proof of the theorem is by renewal theory. In §2 we formulate, and deduce the theorem from the main lemma, (also proving the remark). The main lemma also shows (via [Aar77]) that the transformations $\{T_\alpha : \alpha \in [1,2]\}$ are pairwise strongly disjoint.

§2 Renewal theory: the main lemma.

Recall from [Fel68] that a bounded sequence of non-negative real numbers $u = (u_0, u_1, \dots)$ is called a *renewal sequence* if $u_0 = 1$ and there is a sequence of non-negative real numbers $g = g(u) = (g_1, g_2, \dots)$ satisfying the *renewal equation*

$$u_n = \sum_{k=1}^{n} g_k u_{n-k} \ \forall \ n \in \mathbb{N}.$$

The renewal sequence u is called *recurrent* if $g(u)$ is a probability on $\mathbb{N}$ (that is $\sum_{n=1}^{\infty} g_n = 1$).

Let T be a conservative ergodic measure preserving transformation. Recall from [Aar77] that a set $A \in \mathcal{B}_T$, $0 < m_T(A) < \infty$ is called a *recurrent event* for T if, for every $0 = n_0 < n_1 < \dots < n_K$,

$$m_T(\bigcup_{k=0}^{K} T^{-n_k} A | A) = \prod_{k=1}^{K} u_{n_k - n_{k-1}}$$

where $u_n = m_T(T^{-n} A | A)$.

If A is a recurrent event for T, then the sequence

$$u = u(A) := (u_0, u_1, \dots)$$

defined by $u_n = m_T(T^{-n} A | A)$ is a recurrent renewal sequence. Conversely, every recurrent renewal sequence corresponds in the above manner to a recurrent event of some conservative ergodic measure preserving transformation. Let u be a recurrent renewal sequence, and let $g = g(u)$ be the associated probability on $\mathbb{N}$ satisfying the renewal equation.

One can define ([Chu67]) a stochastic matrix $P = P_u$ with state space $\mathbb{N}$ by

$$p_{j,k} = \begin{cases} g_k \text{ if } j = 1, \\ 1 \text{ if } j - k = 1, \\ 0 \text{ else.} \end{cases}$$

This matrix is irreducible, recurrent, and has the stationary distribution

$$m_k = \sum_{j=k}^{\infty} g_j.$$

Let T_u denote the Markov shift of (P_u, m), that is the shift on $\mathbb{N}^{\mathbb{Z}}$ equipped with the T_u-invariant measure μ defined by

$$\mu([s_1, \dots, s_n]_k) = m_{s_1} p_{s_1,s_2} \dots p_{s_{n-1},s_n}$$

where $[s_1, \dots, s_n]_k = \{x \in I\!N^{\mathbb{Z}} : x_{k+j} = s_j \ \forall\ 1 \le j \le n\}$, then T_u is a conservative ergodic measure preserving transformation, and the set $[1]_0$ is a recurrent event for T_u with renewal sequence u.

It was shown in [Aar77] that if A is a recurrent event for (the conservative ergodic measure preserving transformation) T, then $T \to T_{u(A)}$.

If u, u' are recurrent renewal sequences, then uu' (defined by $(uu')_n = u_n u'_n$) is a renewal sequence, and if uu' is recurrent, then

$$T_u \times T_{u'} \to T_{uu'}.$$

This is because $T_u \times T_{u'}$ is the Markov shift of $P_u \times P_{u'}$.

For $\beta > 0$, define $u(\beta)$ by

$$u_n(\beta) = \left(\frac{1}{n+1}\right)^{\beta} \quad (n \ge 0),$$

then $\frac{u_{n+1}}{u_n} \uparrow$ as $n \uparrow$, so by Kaluza's theorem ([Kal28, Kin72]), $u(\beta)$ is a renewal sequence, which is recurrent iff $\beta \le 1$. Note that $u(\beta)u(\beta') = u(\beta + \beta')$. We are now in a position to state the

Main Lemma.

$$T_\alpha \sim T_{u(\frac{1}{\alpha})} \quad \forall\ \alpha \in [1, 2].$$

Given the main lemma, the theorem follows easily:

$$T_1 \sim T_{u(1)} = T_{u(\frac{1}{2})u(\frac{1}{2})} \leftarrow T_{u(\frac{1}{2})} \times T_{u(\frac{1}{2})} \sim T_2 \times T_2.$$

The truth of the remark is also established easily. Write $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1 + 2\epsilon$ where $\epsilon > 0$, then

$$T_\alpha \times T_{\alpha'} \sim T_{u(\frac{1}{\alpha})} \times T_{u(\frac{1}{\alpha'})} \leftarrow T_{u(\frac{1}{\alpha})} \times T_{u(\frac{1}{\alpha'} - \epsilon)} \times T_{u(\epsilon)} := S.$$

The transformation $T_{u(\frac{1}{\alpha})} \times T_{u(\frac{1}{\alpha'} - \epsilon)}$ is dissipative. If W is a generating wandering set for it, then $W \times X_{T_{u(\epsilon)}}$ is an infinite generating wandering set for S. It follows that $T_\alpha \times T_{\alpha'}$ (being similar to S) also has an infinite generating wandering set, and is therefore isomorphic to $z \mapsto z + 1$ on $I\!R^2$ considered with respect to Lebesgue measure.

The rest of this paper is therefore devoted to the proof of the main lemma, which is in two stages.

The first stage is to show that there is a recurrent renewal sequence $w(\alpha)$ such that $T_\alpha \sim T_{w(\alpha)}$. This is a mild restatement of [AN78].

The second stage is to show that $T_{w(\alpha)} \sim T_{u(\frac{1}{\alpha})}$. This uses the notion of equivalence of renewal sequences introduced in [AK94]. Recall from [AK94] that two renewal sequences u, and u' are *equivalent* (denoted

$u \sim u'$) if there are positively recurrent, aperiodic renewal sequences v, and v' such that $uv = u'v' := w$. In this situation,

$$T_u \leftarrow T_u \times T_v \to T_w$$

whence $T_u \sim T_{u'}$ when $u \sim u'$.

§3 Proof of the the main lemma.

Fix $\alpha \in [1,2]$ and consider the positive operator P defined on $L^1(\mathbb{R})$ by

$$Pg(x) := \int_{\mathbb{R}} g(x+y)df_\alpha(y).$$

Lemma 1.([AN78]) *There exists* $q \in (0,1)$ *such that if*

$$w_0 = 1,\ w_n = q\int_I P^n 1_I dm \quad (n \geq 1),$$

then w *is a renewal sequence, and* $T_\alpha \sim T_w$. *Here,* $m = m_{\mathbb{R}}$ *denotes Lebesgue measure on* $\mathbb{R}$*, and* $I = [0,1]$.
Lemma 2. Let w *be as in lemma 1, then*

$$w \sim u(1/\alpha).$$

Clearly, the main lemma follows from lemmas 1 and 2.
Proof of Lemma 1. The transformation T_α is isomorphic to T_P, the *shift* of P, which is the shift on $\mathbb{R}^{\mathbb{Z}}$ equipped with the T_P-invariant measure μ_P defined by

$$\mu_P([A_1, A_2, \ldots, A_n]_k) = \int_{\mathbb{R}} \tau(A_1, A_2, \ldots, A_n)dm$$

where, for $A_1, A_2, \ldots, A_n \in \mathcal{B}$,

$$[A_1, A_2, \ldots, A_n]_k = \{\underline{x} \in \mathbb{R}^{\mathbb{Z}} : x_{k+j} \in A_j,\ 1 \leq j \leq n\},$$

and $\tau = \tau_P$ is defined by

$$\tau(A_1) := 1_{A_1},\ \tau(A_1, A_2, \ldots, A_n) := 1_{A_1} P\tau(A_2, \ldots, A_n).$$

Note that the "consistency" and T_P-invariance of μ_P follow from $P1 = 1$ and $\int_{\mathbb{R}} Pgdm = \int_{\mathbb{R}} gdm$.

Since f_α has a strictly positive, continuous density, it follows that

$$\exists\ q \in (0,1)\ \ni\ Pg \geq q1_I \int_I gdm\ \forall\ g \in L^1_+(\mathbb{R}).$$

As in [AN78], let $X = \mathbb{R} \times \{0,1\}$, $\overline{m} = m \times (1-q, q)$, and define $R : L^1(X) \to L^1(X)$ by

$$Rg := 1_{I^c \times \{0,1\}}(PEg)\circ\psi + 1_{I\times\{0\}}\frac{1}{1-q}\left((PEg)\circ\psi - q\int_I Egdm\right) + 1_{I\times\{1\}}(x,\delta)\int_I Egdm$$

where $E : L^1(X) \to L^1(I\!R)$ is defined by $Eg(x) := (1-q)g(x,0) + qg(x,1)$, and $\psi : X \to I\!R$ is defined by $\psi(x,\delta) = x$. The choice of q ensures that R is a positive operator. It may be checked that $R1 = 1$ and $\int_{I\!R} Rg d\overline{m} = \int_{I\!R} g d\overline{m}$, and so T_R, the shift of R may be defined as above.

It follows from $ERg = PEg$ that $\pi : T_R \to T_P$, where $\pi : X^{\mathbb{Z}} = I\!R^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}} \to I\!R^{\mathbb{Z}}$ is the projection $\pi(\underline{x}, \underline{\epsilon}) = \underline{x}$.

We now show that w is a renewal sequence (for the chosen value of q), and that $T_R \to T_w$. Set $A = I \times \{1\}$, and $B = [A]_0$. We claim that B is a recurrent event for T_R with renewal sequence w, whence lemma 1. It may be checked that

$$1_A Rg = 1_A \int_I Eg dm,$$

whence

$$1_A R^{n+1} 1_A = 1_A \int_I ER^n 1_A dm = q 1_A \int_I P^n 1_I dm.$$

In particular, we have that

$$\mu_R(T_R^{-(n+1)} B | B) = \frac{1}{q} \int_X 1_A R^{n+1} 1_A d\overline{m} = w_{n+1}.$$

To show that B is a recurrent event for T_R, note that for $0 = n_0 < n_1 < \cdots < n_k$,

$$\bigcap_{j=0}^{k} T_R^{-n_j} B = [A, X^{m_1}, A, X^{m_2}, A, ..., A, X^{m_k}, A]_0$$

where $m_j = n_j - n_{j-1} - 1$. Now,

$$\tau_R(A, X^{m_1}, A, X^{m_2}, A, ..., A, X^{m_k}, A) = \tau_R(A, X^{m_1}, A, X^{m_2}, A, ..., A, X^{m_{k-1}}, A) w_{m_k}$$

$$= \cdots \prod_{j=1}^{k} w_{m_j} 1_A.$$

This shows that B is a recurrent event for T_R.

Proof of Lemma 2. By lemma 5.2 of [AK94], it is sufficient to show

$$w_n = \frac{1}{n^{\frac{1}{\alpha}}} \left(a + \frac{b}{n^{\frac{2}{\alpha}}} + \frac{c}{n^{\frac{4}{\alpha}}} + O\left(\frac{1}{n^{\frac{6}{\alpha}}} \right) \right)$$

as $n \to \infty$ where $a > 0$, $b, c \in I\!R$.

To see this, let f_α^{n*} denote the n-th convolution power of the probability f_α, and note that

$$\begin{aligned} w_n &= \int_I \int_{I\!R} 1_I(x+y) df_\alpha^{n*}(y) dx \\ &= \int_{[-1,1]} (1-|y|) df_\alpha^{n*}(y) \\ &= \int_{I\!R} \int_{[-1,1]} (1-|y|) \cos ty dy e^{-n|t|^\alpha} dt \\ &= 4 \int_0^\infty \phi(t) e^{-nt^\alpha} dt \end{aligned}$$

where

$$\phi(t) = \frac{1-\cos t}{t^2}.$$

Changing variables,

$$w_n = \frac{4}{\alpha n^{\frac{1}{\alpha}}} \int_0^\infty \phi\left(\left(\frac{x}{n}\right)^{\frac{1}{\alpha}}\right) x^{\frac{1}{\alpha}-1} e^{-x} dx,$$

and we analyse the integral.

There exists $r \in (0,1)$ such that

$$\int_n^\infty \phi\left(\left(\frac{x}{n}\right)^{\frac{1}{\alpha}}\right) x^{\frac{1}{\alpha}-1} e^{-x} dx = O(r^n),$$

and

$$\int_n^\infty x^{\frac{k}{\alpha}-1} e^{-x} dx = O(r^n),$$

as $n \to \infty$ for $k \geq 1$.

By Taylor's theorem,

$$\phi(t) = a + bt^2 + ct^4 + \kappa(t) t^6$$

where $\sup_{-1 \leq t \leq 1} |\kappa(t)| := M < \infty$. It follows that

$$\begin{aligned} \int_0^n \phi\left(\left(\frac{x}{n}\right)^{\frac{1}{\alpha}}\right) x^{\frac{1}{\alpha}-1} e^{-x} dx &= \int_0^n \left(\sum_{k=0}^2 b_k \left(\frac{x}{n}\right)^{\frac{2k}{\alpha}} + \kappa(\frac{x}{n}) \left(\frac{x}{n}\right)^{\frac{6}{\alpha}}\right) x^{\frac{1}{\alpha}-1} e^{-x} dx \\ &= \sum_{k=0}^2 \frac{b'_k}{n^{\frac{2k}{\alpha}}} + O\left(\frac{1}{n^{\frac{6}{\alpha}}}\right) \end{aligned}$$

where $\{b_k,\ b'_k : 0 \leq k \leq 2\}$ are constants and $b'_0 > 0$. This proves lemma 2.

References

[Aar77] Jon Aaronson. Rational ergodicity and a metric invariant for Markov shifts. *Israel J. Math.*, 27(2):93–123, 1977.

[Aar87] Jon Aaronson. The intrinsic normalising constants of transformations preserving infinite measures. *J. Analyse Math.*, 49:239–270, 1987.

[AK94] Jon Aaronson and Michael Keane. Isomorphism of random walks. *Israel J. Math.*, 87(1-3):37–63, 1994.

[ALP94] Jon Aaronson, Thomas Liggett, and Pierre Picco. Equivalence of renewal sequences and isomorphism of random walks. *Israel J. Math.*, 87(1-3):65–76, 1994.

[AN78] K. B. Athreya and P. Ney. A new approach to the limit theory of recurrent Markov chains. *Trans. Amer. Math. Soc.*, 245:493–501, 1978.

[Chu67] Kai Lai Chung. *Markov chains with stationary transition probabilities.* Second edition. Die Grundlehren der mathematischen Wissenschaften, Band 104. Springer-Verlag New York, Inc., New York, 1967.

[Fel68] William Feller. *An introduction to probability theory and its applications. Vol. I.* Third edition. John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[Kal28] Th. Kaluza. Über die Koeffizienten reziproker Potenzreihen. *Math. Z.*, 28:161–170, 1928.

[Kin72] J. F. C. Kingman. *Regenerative phenomena.* John Wiley & Sons Ltd., London-New York-Sydney, 1972. Wiley Series in Probability and Mathematical Statistics.

School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel.

Email address: aaromath.tau.ac.il