A Generalization of Meyniel's Conjecture on the Cops and Robber Game

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Abstract

We consider a variant of Cops and Robbers game where the robber can move s edges at a time, and show that in this variant, the cop number of an n-vertex graph can be as large as $\Omega(n^{\frac{s}{s+1}})$. This improves the $\Omega(n^{\frac{s-3}{s-2}})$ lower bound of Frieze et al. [4]. We also conjecture a general upper bound of $O(n^{\frac{s}{s+1}})$ for the cop number in this variant, generalizing Meyniel's conjecture.

1 Introduction

The game of Cops and Robbers, introduced by Nowakowski and Winkler [7] and independently by Quilliot [9], is a perfect information game played on a finite graph G. There are two players, a set of cops and a robber. Initially, the cops are placed onto vertices of their choice in G (where more than one cop can be placed at a vertex). Then the robber, being fully aware of the cops placement, positions himself in one of the vertices of G. Then the cops and the robber move in alternate rounds, with the cops moving first; however, players are permitted to remain stationary on their turn if they wish. The players use the edges of G to move from vertex to vertex. The cops win and the game ends if eventually a cop steps into the vertex currently occupied by the robber; otherwise, i.e., if the robber can elude the cops indefinitely, the robber wins. The parameter of interest is the *cop number of* G, which is defined as the minimum number of cops needed to ensure that the cops can win. We will assume that the graph G is simple and connected, because deleting multiple edges or loops does not affect the set of possible moves of the players, and the cop number of a disconnected graph obviously equals the sum of the cop numbers for each connected component.

For a survey of results on the cop number and related search parameters, see [5]. The most well known open question in this area is Meyniel's conjecture, published by Frankl in [3]. It states that for every graph G on n vertices, $O(\sqrt{n})$ cops are enough to win. This is asymptotically tight, i.e. for every n there exists an n-vertex graph with cop number $\Omega(\sqrt{n})$. The best upper bound found so far is $n2^{-(1-o(1))\sqrt{\log_2 n}}$ ([4, 6, 10]).

Here we consider the variant where in each move, the robber can take any path of length at most s from his current position, but she is not allowed to pass through a vertex occupied by a cop. The parameter s is called the *speed* of the robber. This variant was first considered in [2]. Frieze et al. [4] showed that the cop number of a connected n-vertex graph can be as large as $\Omega(n^{\frac{s-3}{s-2}})$. In this paper, we improve their bound by proving the existence of an infinite family of graphs with cop number $\Omega(n^{s/s+1})$, for every fixed positive integer s. We generalize Meyniel's conjecture by conjecturing that this is also an upper bound for the cop number. In Section 2 the main result is proved, and in Section 3 the conjecture is proposed, predicting the asymptotic value of cop number in this general setting.

2 The main result

Let k be a positive integer. For a vertex u of a graph G, $N_k(u)$ denotes the set of vertices whose distance to u is exactly k. If $k = 1$, then we simply write $N(u)$. If A is a subset of vertices, then $N_k^A(u)$ denotes the set of vertices v such that:

- The distance between u and v is k, and
- for every shortest (u, v)-path $uu_1u_2 \dots u_{k-1}v$, we have $u_1 \notin A$.

Note that for every A and k, $N_k^{A\cap N(u)}(u) = N_k^A(u)$. For vertices u, v, we denote their shortest-path distance by $d(u, v)$. The *diameter* of G is the maximum distance between any two vertices of G.

Lemma 1. *Let s, d, m be positive integers and q be a positive real such that $qd^s/2$ is an integer larger than m. Let G be a d-regular bipartite graph of diameter larger than s with the following properties:*

(1) For every two vertices u, v of G of distance at most $s + 1$, there are at most m distinct shortest (u, v)-paths.

(2) For every vertex u of G and every subset A of size at most m, $|N_s^A(u)| \geq qd^s$.

Then, assuming the robber has speed s, the cop number of G is at least $q^2d^s/24ms$.

Proof. Let us first define a few terms. A cop *controls* a vertex u if the cop is on u or on an adjacent vertex. A cop controls a path if it controls a vertex of the path. The cops control a path if there is a cop controlling it. A vertex r is *safe* if there is a subset $X \subseteq N_s(r)$ of size $qd^s/2$ such that for all $x \in X$, all shortest (r, x)-paths are uncontrolled.

Let the number of cops be c with $c < q^2d^s/24ms$, and we will show that the robber can evade forever. If this many cops can capture the robber, then they can capture the robber from any starting configuration. Thus we may assume that the cops all start in one vertex u, and the robber starts in a vertex r at distance $s+1$ from u. Such two vertices exist as G has diameter larger than s. Property (2) gives $N_s(r) \geq qd^s$, and by property (1), the cops control at most m vertices of $N_s(r)$. Since $qd^s - m > qd^s/2$, the robber is in a safe vertex at the starting configuration. Hence we just need to show that if the robber is in a safe vertex before cops' move, then she can move to a safe vertex after cops' move.

Suppose that the robber is in a safe vertex r, so by definition, there is a subset $X \subseteq N_s(r)$ of size $qd^s/2$ such that for all $x \in X$, all shortest (r,x)-paths are uncontrolled. Denote by A the set of vertices of all shortest (r,x)-paths for all $x \in X$. In particular, $r \in A$ and $X \subseteq A$. Now, the cops move to new positions. At this moment there is no cop in A, so the robber is able to move to any vertex of X in her turn; thus to complete the proof, we need to show that there is a safe vertex in X.

Claim. Every vertex $u \notin A$ has at most m neighbors in X.

Proof. Let $u \notin A$. If u has no neighbor in X, then the claim is true, otherwise let $x \in X$ be adjacent to u. Note that as $d(r,x) = s$, we have $d(r,u) \in \{s-1, s, s+1\}$. The graph G is bipartite, so $d(r,u) \neq s$. If $d(r,u) = s-1$ then u is on a shortest (r,x)-path, which contradicts the assumption $u \notin A$. Therefore, $d(r,u) = s+1$, and x is on a shortest (r,u)-path. Hence by property (1), the number of neighbors of u in X is at most m. □

Remark. It can be shown using a similar argument that every $x \in X$ has at most m neighbors in A.

By an *escaping pair* we mean a pair (x,y) of vertices with $x \in X$ and $y \in N_s^A(x)$. We call x the *head* and y the *tail* of the pair. By the remark, the set $A \cap N(x)$ has at most m elements, and property (2) ensures that $|N_s^A(x)| = |N_s^{A\cap N(x)}(x)| \geq qd^s$. That is, every $x \in X$ is the head of at least qd^s distinct escaping pairs. We say that an escaping pair (x,y) is *free* if all shortest (x,y)-paths are uncontrolled. We just need to prove that there is an $x \in X$ such that x is the head of at least $qd^s/2$ free escaping pairs, because then x would be a safe vertex, and the robber, having speed s, can move to x in her move. If (x,y) is an escaping pair, then every shortest (x,y)-path is called an *escaping path*. By definition, every escaping path can be written as $u_1u_2u_3\dots u_{s+1}$, where $u_1 \in X$ and $u_2 \notin A$.

Claim. Each cop controls at most $3msd^s$ escaping paths.

Proof. We first prove that every vertex v is on at most $d^s + msd^{s-1}$ escaping paths, and if $v \notin X$, then v is on at most msd^{s-1} escaping paths. Let $u_1u_2u_3\dots u_{s+1}$ be an escaping path with $u_1 \in X$ and $u_2 \notin A$, such that v is its i-th vertex, i.e. $v = u_i$.

Assume first that $i \neq 1$. There are at most d choices for each of $u_{i-1}, \dots, u_2$, and for each of $u_{i+1}, u_{i+2}, \dots, u_{s+1}$. By the previous claim, once u_2 is determined, there are at most m choices for u_1. Consequently, for each $2 \leq i \leq s+1$, v is the i-th vertex of at most md^{s-1} escaping paths, so if $v \notin X$ then v is on at most msd^{s-1} escaping paths.

If $i = 1$ then $v \in X$ and there are at most d choices for each of $u_2, u_3, \dots, u_{s+1}$, thus each $v \in X$ is the first vertex of at most d^s escaping paths. This shows that v is on at most $d^s + msd^{s-1}$ escaping paths.

Recall that since the robber was in a safe vertex before the cops' move, no cop is in A at this moment. By the previous claim, each cop can control at most m vertices of X, through which he can control at most $m(d^s + msd^{s-1})$ escaping paths. Through every other neighbor he can control at most msd^{s-1} escaping paths. He controls $d+1$ vertices in total, so he controls no more than

$$m(d^s + msd^{s-1}) + (d+1-m)(msd^{s-1}) \leq 3msd^s$$

escaping paths. □

Since there are c cops in the game, the cops control at most $3msd^s c$ escaping paths. By controlling each escaping path, the cops can decrease the number of free escaping pairs by at most 2 (as each path has two endpoints), hence the number of non-free escaping pairs is at most $6msd^s c$.

Now we prove that there is an $x \in X$ such that x is the head of at least $qd^s/2$ free escaping pairs, completing the proof. Recall that every $x \in X$ is the head of at least qd^s escaping pairs. Hence if there were no $x \in X$ such that x is the head of at least $qd^s/2$ free escaping pairs, then every $x \in X$ would be the head of at least $qd^s/2$ non-free escaping paths. As by definition of safeness, X has size $qd^s/2$, this would imply that the number of non-free escaping pairs is at least $(qd^s/2)^2$, which is larger than $6msd^s c$. This contradiction shows that the robber can evade c cops forever. ∎

Let k, s be positive integers and $d = 2^k - 1$. Let $x_1, x_2, \ldots, x_d$ be the d nonzero elements of $GF(2^k)$ represented as column vectors of length k over $\mathbb{Z}_2$. Let H be the following $1 + k(s+1)$ by d matrix over the field $\mathbb{Z}_2$:

$$H = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_d \\ x_1^3 & x_2^3 & \ldots & x_d^3 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{2s+1} & x_2^{2s+1} & \ldots & x_d^{2s+1} \end{bmatrix}$$

Let $S = \{e_1, e_2, \ldots, e_d\} \subseteq \mathbb{Z}_2^{1+k(s+1)}$ be the set of columns of H. It is known that every set of $2s+3$ columns of H is linearly independent over $\mathbb{Z}_2$ (see e.g. [1]), hence, in particular, every $(2s+2)$-subset of S is linearly independent over the field $\mathbb{Z}_2$. Consider the graph with vertex set $\mathbb{Z}_2^{1+k(s+1)}$, and with vertices u, v adjacent if $u - v \in S$ (the Cayley graph of the additive group $\mathbb{Z}_2^{1+k(s+1)}$ with respect to S). Let G be the connected component containing the all-zero vector.

Lemma 2. *If $d \geq 2(s+1)!$, then the graph G has the following properties.*

(i) G has at most $2^{s+2}d^{s+1}$ vertices.

(ii) G is d-regular.

(iii) G is bipartite.

(iv) For every two vertices u, v of G of distance at most $s+1$, there are at most $(s+1)!$ distinct shortest (u, v)-paths.

(v) For every vertex u of G and every subset A of size at most $(s+1)!$, $|N_s^A(u)| \geq (d/2s)^s$.

(vi) G has diameter larger than s.

Proof. (i) Every vertex of G is a vector in $\mathbb{Z}_2^{1+k(s+1)}$, so $|V(G)| \leq 2^{1+k(s+1)} \leq 2^{s+2}d^{s+1}$.

(ii) This is clear as $|S| = d$.

(iii) This follows from the fact that each member of S has 1 as its first element, hence there is no odd-size subset of S whose sum of members is zero.

(iv) Let u, v be two vertices of G of distance m, where $m \leq s+1$. Each (u,v)-path of distance m corresponds to a unique ordered representation

$$u - v = s_1 + s_2 + \cdots + s_m,$$

with $s_1, \ldots, s_m \in S$. If some $s \in S$ appears more than once in this summation, then we can delete a pair of them (we are in $\mathbb{Z}_2$, so $2s = 0$) and find a shorter representation (and a shortest (u,v)-path), which does not exist. So $s_1, \ldots, s_m$ are distinct. Moreover, if there is another ordered representation

$$u - v = s'_1 + s'_2 + \cdots + s'_m,$$

then $s'_1, \ldots, s'_m$ are distinct by a similar argument, and we have $s_1 + \cdots + s_m + s'_1 + \cdots + s'_m = 0$. By linear independence of every $(2s+2)$-subset of S, $(s'_1, \ldots, s'_m)$ is a permutation of $(s_1, \ldots, s_m)$. Therefore, the number of ordered representation of $u - v$ using members of S is $m!$, so the number of shortest (u,v)-paths in G is also $m!$.

(v) Without loss of generality, we may assume that $A \subseteq N(u)$. Every $a \in A$ can be written as $a = u + e_i$ for some $e_i \in S$. There is a set $B \subseteq S$ of size at least $d - |A|$ such that for every $e \in B$, $u + e \notin A$. For every s-subset $\{e_{i_1}, \ldots, e_{i_s}\}$ of B, we have a vertex $u + e_{i_1} + \cdots + e_{i_s}$ of distance s from u. These vertices are all in $N_s^A(u)$ and are distinct, because of the linear independence of every $(2s+2)$-subset of S. Hence we have

$$|N_s^A(u)| \geq \binom{d - |A|}{s} \geq \left(\frac{d - |A|}{s}\right)^s \geq \frac{d^s}{(2s)^s},$$

where the last inequality follows from $d \geq 2(s+1)! \geq 2|A|$.

(vi) By linear independence of every $2s+2$ members of S, the distance between vertices 0 and $e_1 + \cdots + e_{s+1}$ is at least $s+1$. ■

Theorem 1. *Let s be a fixed positive integer denoting the speed of the robber. For infinitely many n, there exists a connected n-vertex graph with cop number $\Omega(n^{s/s+1})$.*

Proof. Take k large enough so that $d = 2^k - 1$ satisfies $d \geq 2(s+1)!$ and $d^s > 4(s+1)!(2s)^s$. Let G be the graph described above with parameters k, s. Let $m = (s+1)!$ and let q satisfy the equation $qd^s = 2\lfloor \frac{d^s}{2(2s)^s} \rfloor$. By Lemma 2, G is a connected bipartite d-regular graph with $n = O(d^{s+1})$ vertices and diameter larger than s. Also, for every two vertices u, v of G of distance at most $s+1$, there are at most m distinct shortest (u,v)-paths, and for every vertex u of G and every subset B of size at most m,

$$|N_s^B(u)| \geq (d/2s)^s \geq qd^s.$$

Moreover, $qd^s/2$ is an integer and

$$qd^s/2 = \left\lfloor \frac{d^s}{2(2s)^s} \right\rfloor \geq \frac{d^s}{4(2s)^s} > m.$$

Now by Lemma 1, if the robber has speed s, then the cop number of G is $\Omega(d^s) = \Omega(n^{s/s+1})$. ■

3 Concluding remarks

Following the notation of [2], for a graph G and a positive integer s, let us denote the cop number of G when the robber has speed s by $c_s(G)$. It is well-known that there exists n-vertex connected graphs with $c_1(G) = \Omega(\sqrt{n})$ (the standard construction comes from the incidence graph of a projective plane, see [8]). Meyniel conjectured that if G is connected and has n vertices, then $c_1(G) = O(\sqrt{n})$ [3]. Frieze et al. [4] showed that if the robber has speed s, then the cop number of a connected n-vertex graph can be as large as $\Omega(n^{\frac{s-3}{s-2}})$. In this paper, we proved that for every fixed s, for infinitely many n, there exists a connected graph G on n vertices with $c_s(G) = \Omega(n^{s/s+1})$. Generalizing Meyniel's conjecture, we conjecture that this is also an upper bound on $c_s(G)$.

Conjecture. For every fixed positive integer s, every connected n-vertex graph G has $c_s(G) = O(n^{s/s+1})$.

The only general upper bound was proved by Frieze et al. Letting $\alpha = 1 + s^{-1}$, they showed that every connected graph on n vertices has $c_s(G) \le n\alpha^{-(1-o(1)\sqrt{\log_\alpha n}}$.

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