On K_3-decomposition of graphs

Contents

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1 Introduction

1.1 F-decomposition Problem

People are interested in the so-called (hyper-)graph decomposition problem. To be precise, a (hyper-)graph G has an F-decomposition if the edges of G can be partitioned into edge-disjoint copies of F. This essay aims to show the special case from paper [1] by Barber-Glock-Kühn-Lo-Montgomery-Osthus where G is a simple loopless graph and $F = K_3$. In order to do so, we shall first introduce some general terminology and results about graph decompositions. And then in section 1.2, we will focus on the triangle decomposition problem.

Let us begin by giving the formal definition of F-decompositions of a graph.

Definition 1.1. *A simple graph is said to admit an F-decomposition if its edge set can be partitioned into edge-disjoint copies of F.*

We aim to find some sufficient conditions for a graph G to be F-decomposable, so we can start by finding some necessary conditions first. This leads us to come up with the following definition called F-divisible, which is a basic property that every F-decomposable graph holds.

For a graph G, we define $gcd(G)$ to be the greatest common divisor of the degrees of all vertices of G, then we say a graph G is F-divisible if $|E(G)|$ is divisible by $|E(F)|$ and $gcd(G)$ is divisible by $gcd(F)$.

Hence, being F-divisible is a necessary condition for a graph G to be F-decomposable. Now we are led to ask what would be the sufficient conditions? In order to elicit this matter, we shall present and analyze results from various studies. This shall thus help direct the goal of this paper better.

It might be worth mentioning that if G is the complete graph, then this graph decomposition problem is already solved in 1975 by Wilson ([21]), who proves the following theorem.

Theorem 1.2 (Wilson's theorem)**.** *For any given F, there is an integer N such that any complete graph K_n with $n \geq N$ that is F-divisible is also F-decomposable.*

Although this theorem is limited in its usage because it only considers complete graphs, it inspires us to consider decomposition problems of very dense graphs. And we can see that the definitions and results we will be introducing are mostly dealing with dense graphs. So following the spirit of Wilson's theorem, one may ask the following question: given an F, is there a number $\delta \in [0, 1)$ such that any F-divisible graph G with $\delta(G) \geq \delta n$ on n vertices is F-decomposable, where $\delta(G)$ is the minimum degree of graph G? This leads us of the following definition, the so-called decomposition threshold.

Definition 1.3. *For $n \in \mathbb{N}$, define $\delta_F(n)$ to be the minimum of all natural numbers d such that every F-divisible graph G on n vertices with $\delta(G) \geq d$ admits an F-decomposition. Furthermore, the decomposition threshold of F is defined to be $\delta_F := \limsup_{n\to\infty} \frac{\delta_F(n)}{n}$.*

The decomposition threshold is an exact real number between 0 and 1 for any given graph F. Therefore, it would be of great importance if we can find a formula to determine it. However, as expected, this is no easy task. A partial reason is that it can be really difficult to prove the existence of an F-decomposition when the only property we know about the graph is its minimum degree. Consequently, researchers tend to take some detours through some not-so-exact decompositions. We shall introduce three ways: two regarding approximate decompositions and one regarding fractional decompositions.

Let us consider an approximate decomposition of G. How close is it to a complete decomposition, in other words, how large is the leftover graph? There are two ways to quantify it, the total number of edges left or the maximum degree left. These two ways lead to two approaches that we can use as a detour towards the graph decomposition problem.

First, consider the total number of edges left. We say that a μ-approximate F-decomposition of G is a set of edge-disjoint copies of F covering all but at most μn^2 edges of G.

Definition 1.4. *For $n \in \mathbb{N}$ and $\mu > 0$, we define $\delta^\mu_F(n)$ to be the infimum over all δ such that every graph G on n vertices with $\delta(G) \geq \delta n$ has a μ-approximate F-decomposition, and let $\delta^\mu_F := \limsup_{n\to\infty} \delta^\mu_F(n)$ to be the μ-approximate F-decomposition threshold.*

Clearly, δ^μ_F is a decreasing function with respect to the variable μ. Furthermore, note that if we take $\mu = 0$, then we get an F-decomposition. Therefore, it is natural to ask whether the following holds or not:

$$\lim_{\mu\to 0} \delta^\mu_F = \delta_F \tag{1}$$

Interestingly, there are graphs for which equation (1) holds but there are also graphs for which it does not. In [3; 22], it is shown that equation (1) does not hold when $F = K_{r,r}$ for even r. However, this tiny failure does not hinder δ^μ_F to be useful. In fact, the next theorem is the main result in [3] showing us a preview of what can we can get from this approximate threshold.

Theorem 1.5. *Let F be an r-regular graph. Then for each $\epsilon > 0$, there exists an $n_0 =$*

$n_0(\epsilon, F)$ and an $\mu = \mu(\epsilon, F)$ such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + \epsilon)n$, where $\delta := max\{\delta_F^{\mu}, 1 - 1/3r\}$, has an F-decomposition.

This theorem in [3] is very useful and below we present three applications illustrating how one can make use of it.

Application 1: Part of the proof of Theorem (1.5) and the result in [2] together gives a new proof of Wilson's theorem. This new proof is a purely combinatorial one, unlike the original algebraic proof.

Application 2: It can be used to derive the exact decomposition threshold for even cycles and get a good bound for odd cycles as well.

Theorem 1.6. *Let $l \geq 4$ be even, and let $\delta = 1/2$ for $l \geq 6$ and $\delta = 2/3$ for $l = 4$. Then for all $\epsilon > 0$, there is an n_0 such that any C_l-divisible graph on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + \epsilon)$ has a C_l-decomposition.*
When $l \geq 3$ is odd, there is also an $n_0 \in \mathbb{N}$ such that every C_l-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (0.9 + \epsilon)n$ has a C_l-decomposition.

In fact, one can remove the ϵ in the statement of theorem (1.6), but the more exact version is shown using another method in [20]. Nevertheless, theorem (1.6) is just a special case of a more general result in [7], which we shall introduce later in the introduction since that result needs more terminology.

Application 3: Although theorem (1.5) and both of the first two applications are only dealing with regular graphs, we must note that it can still be used to obtain some results for general G, which is our goal, eventually. The next theorem illustrates the aforementioned idea:

Theorem 1.7 ([3])**.** *Let $\chi = \chi(F)$ be the chromatic number of a given graph F, let $C := \min\{9\chi^2(\chi-1)^2/2, 10000\chi^{3/2}\}$, and let $t := \max\{C, 6|E(F)|\}$. Then for each $\epsilon > 0$, there is an $n_0 = n_0(\epsilon, F)$ such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1 - 1/t + \epsilon)n$ has an F-decomposition.*

Although our choice of t is only of size $O(|F|^2)$ which is not the best bound in general, this is a seminal result as it does not require any further property of graph G or F.

The above three applications show the importance of theorem (1.5) and δ_F^{μ}. Now let us introduce the second way of estimating the approximate decomposition.

Definition 1.8. *The approximate decomposition threshold δ_F^{0+} is defined to be the infimum of all $\delta \in [0, 1]$ with the following property:*

for all $\gamma > 0$, there exists $n_0 \in \mathbb{N}$ such that any graph G on $n \geq n_0$ vertices with $\delta(G) \geq \delta n$ contains an F-decomposable subgraph H such that $\Delta(G - H) \leq \gamma n$, where $\Delta(G)$ denotes the maximum vertex degree of the graph G.

We shall see that in [1], the main result we wish to show contains $\delta_{K_3}^{0+} =: \delta^{0+}$. Indeed, in many cases, any improvement on the bound of δ_F^{0+} would lead to an improvement of δ_F as well. But it is not so easy to get an estimation for δ_F^{0+}, which is why we need a third detour through fractional decomposition. We assign each copy of F in G a weight in a way that the total weight for any edge sums up to 1. If there is a way of assigning weights, then we say that the graph G assumes a fractional decomposition of F.

Definition 1.9. *The fractional decomposition threshold δ_F^* is defined to be the infimum of all δ such that for any large enough n, any graph G on n vertices with $\delta(G) \geq \delta n$ has a fractional decomposition.*

Indeed, we do have a few non-trivial bounds on the fractional decomposition thresholds of certain graphs. In [5], it is shown that $\delta_{K_3}^* \leq 0.9$. And in [16], Montgomery shows that for $r \geq 4$, $\delta_{K_r}^* \leq 1 - 1/100r$.

Now that we have all the recipes, we can go on and show another important result on the F-decomposition problem, or more precisely, on δ_F.

Theorem 1.10 (Main theorem in [7]). *Let F be a graph and $\chi = \chi(F)$ be its chromatic number. The followings are bounds for the decomposition threshold δ_F:*
(1) $\delta_F \leq \max\{\delta_F^{0+}, 1 - 1/(\chi + 1)\}$;
(2) if $\chi \geq 5$, then $\delta_F \in \{\delta_F^*, 1 - 1/\chi, 1 - 1/(\chi + 1)\}$.

Similar to theorem (1.5), this theorem also tells us a lot about decomposition thresholds. Several applications are given below.

Corollary 1.10.1. *For $r \geq 3$, $\delta_{K_r} = \delta_{K_r}^{0+}$.*

Proof. By a result from [23] which says that $\delta_{K_r}^{0+} \geq 1-1/(r+1)$ and the fact that $\chi(K_r) = r$, we know $\delta_{K_r} \leq \delta_{K_r}^{0+}$ by part 1 of Theorem (1.10). By definitions 1.3 and 1.6, we know that $\delta_{K_r} \geq \delta_{K_r}^{0+}$, thus the result follows. □

Indeed, another application is that one can use it to determine the exact number of the decomposition threshold for any bipartite graph F. Since even cycles are bipartite graphs, then the first part of theorem 1.6 is immediate. This theorem also has some more corollaries regarding the chromatic number of F (see [7]), but we shall omit them as chromatic number is not what we want to study in this essay.

1.2 K_3-decomposition problem

Now we shall focus on the problem of K_3-decomposition, that is, we take $F = K_3$. As already mentioned in section 1.1, we would like to study the sufficient conditions for a graph G to be K_3-decomposable. We give the definition of K_3-divisible again though it is just a special case of F-divisibility.

Definition 1.11. *A simple graph G is said to be K_3-divisible if the number of edges of G is divisible by 3, and all the vertices of G have even degrees.*

The definitions of $\delta_{K_3}, \delta^{0+} := \delta^{0+}_{K_3}, \delta^* := \delta^*_{K_3}$ are just the same as in section 1.1. So a natural question arises after our introduction of the general picture: what is δ_{K_3}?

A conjecture by Nash-Williams ([17]) suggests that $\delta_{K_3} = 3/4$.

Indeed, this lower bound is tight in the sense that we can construct an explicit example showing that any threshold smaller than 3/4 is not possible.

Example 1.12. *Let $V(G) := V_1 \cup V_2 \cup V_3 \cup V_4$, where $|V_1| = |V_2| = |V_3| = |V_4| = 6k + 3$, for some integer k. Let the edges of G be the union of the complete graphs on V_1, V_2, V_3, V_4 and the complete bipartite graph between $V_1 \cup V_2$ and $V_3 \cup V_4$. Then $\delta(G) = 18k + 8$ which is divisible by 2, and $|E| = 4(6k + 3)(9k + 4)$ which is divisible by 3. Thus, G is K_3-divisible. And furthermore, $\delta(G)/|V| = (18k + 8)/(24k + 12) < 3/4$. Let ABC be a triangle in G, and without loss of generality, we assume that $A \in V_1$. Since there is no edge between V_1 and V_2, then we know that $B, C \notin V_2$. Using this argument, we know that there have to be two vertices in the same V_i. Hence, every triangle has at least one edge from $G[V_1] \cup G[V_2] \cup G[V_3] \cup G[V_4]$, so we should have $|E(G[V_1] \cup G[V_2] \cup G[V_3] \cup G[V_4])| \geq |E(G)|/3$, which in fact does not hold. Hence, this graph G does not have a triangle decomposition.*

In this essay, we shall present two different ways to get some approximate decompositions, both of which will lead to a result in the K_3-decomposition problem. The first way utilizes the approximate threshold that is defined in section 1.1, while the second way is to require G to be typical. We shall present the first approach in details from section 2 to 6, and then we shall briefly discuss the second approach in section 7. Now we can state our main theorem of this essay which is the main result in [1] by Barber-Glock-Kühn-Lo-Montgomery-Osthus.

Theorem 1.13 (Main theorem). *Let $\delta := \max\{3/4, \delta^{0+}\}$. For all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every K_3-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta+\epsilon)n$ admits a K_3-decomposition.*

Corollary 1.13.1. $\delta_{K_3} \leq \max\{3/4, \delta^{0+}\}$.

Proof. It follows directly from the main theorem and definition 1.3. □

However, one may notice that the definition of δ^{0+} might seem too strong to be useful. One may wonder whether δ^{0+} is too close to 1, if it is, then this theorem would be not very interesting. In fact, the current best known bound on δ^{0+} shows that it is smaller than or equal to 0.9 (see [5]), which already makes the main theorem a nontrivial result.

Indeed, in [5], Dross shows that every graph G with $\delta(G)/|V(G)| \geq 0.9$ assumes a fractional triangle decomposition. In other words, [5] shows $\delta^* \leq 0.9$. We shall see in [7] how to use δ_F^* for exact decomposition problems. Similar to an approximate decomposition, we can also transfer a fractional decomposition into an exact one thanks to the theorem by Haxell and Rödl [9]. The major result in [9] shows that one can always transfer a fractional decomposition into an approximate decomposition whose leftover edges is of size $o(n^2)$. In [3], it is shown that this leftover can be further transferred into a leftover with small minimum vertex degree and thus we get the result $\delta^* \leq \delta^{0+}$. Hence, one corollary of this result in [5] is $\delta^{0+} \leq 0.9$. Furthermore, from corollary 1.13.1 we see that any improvement on δ^{0+} would lead to an improvement on δ_{K_3}.

It is also worth mentioning that determining whether a graph has a K_3-decomposition is NP-hard ([4]), thus making the above results more important.

1.3 Iterative absorption method

The method used successfully in [1] and [3] is called the iterative absorption method. Although this method is still quite new, the absorption method has already been used for several decades for some other graph-related problems. In [14], the absorption method was used to show the existence of a triangle factor in a random graph. And in [6], the absorption method was used to show vertex coverings by monochromatic cycles, which can be seen as a graph decomposition problem into some different graphs. In [19], the absorption method serves an important role when proving the existence of Hamilton cycle in 3-uniform hypergraphs.

As the name suggests, the iterative absorption method utilizes the absorption method many times, iteratively. The first appearance of this new method was in [15], in which the method was used to show the decomposition of large regular tournaments into Hamilton cycles. It has been used successfully in some graph decomposition problems as in [1], [3], and [11], which solves the decomposition problem of K_n into a sequence of bounded degree trees. In [13], a variation of the iterative absorption method was used to show the existence of a large collection of edge-disjoint Hamilton cycles in a random graph with certain properties.

Indeed, the proof of Theorem 1.13 is already contained in [3], but in this essay, we shall introduce the approach as in [1] because it is easier than [3]. Let us note however that [3] uses the same method to prove something more general (Theorem 1.5).

2 Scheme of the proof

In this section, we shall briefly introduce the idea behind the proof of Theorem 1.13, which is the major focus of this essay. The proof uses a technique called the iterative absorption method as a crucial tool. But before we introduce the iterative absorption method, we need the definition of absorbers, which serves an important purpose in the whole context.

The problem we want to tackle is about K_3-decomposition of K_3-divisible graphs. One possible approach might be trying to remove triangles incident to a vertex greedily. However, because of the possible poor choice of triangles removed, we might end up with no K_3-decomposition. Consider figure 1, which clearly has a K_3-decomposition. But if we choose the triangles 1,2,3 to be in the decomposition, then all of the leftover edges which form a C_{12} cannot be decomposed any further. This example tells us that although a K_3-divisible graph G has a K_3-decomposition, we still need to use some "clever" way to find it. So the takeaway of the above example is that we will constantly encounter some leftover subgraphs of the original graph that cannot be decomposed. And the concept "absorber" which we shall introduce next gives a possible instruction on how to deal with those leftover edges.

Definition 2.1 (Absorber for L)**.** *For a given graph L, an absorber for L is a graph A such that $V(L) \subset V(A)$ is independent in A and both A and $A \cup L$ have a K_3-decomposition.*

The letter L above indicates that L is the leftover subgraph. Let us note that the concept of absorber now gives us a new idea of showing some graph G is K_3-decomposable. The first step is to find a large subgraph $G' \subset G$ which has a K_3-decomposition. If $G' = G$, then we are already done. But in general, there are some "leftover" subgraphs $L = G - G'$. If we can find an absorber A for L such that both $A \cup L$ and $G' - A$ have a K_3-decomposition,

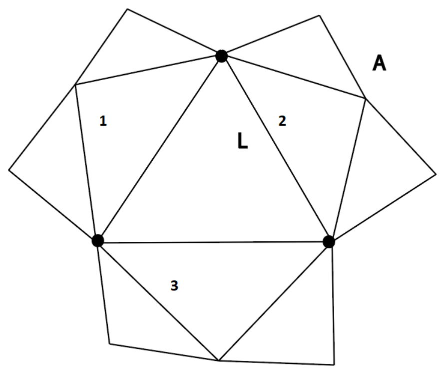

Figure 1: Example

then $G = (A \cup L) \cup (G' - A)$ has a K_3-decomposition. Then in such a case, the problem is solved.

Let us keep in mind that the above procedure may be difficult, as we do not know how to show $G' - A$ has a K_3-decomposition. So a more promising approach should be in the "reversed" direction.
At first, we somehow force our possible leftover subgraph L to be small, and then we find an absorber A for L. Then let $G' = G - A$ for we want to show that $G' - L$ has a K_3-decomposition. We shall see that in section 6 (the complete proof), roughly speaking, we let L to be the set of all possible leftovers which is in a small subset of $V(G)$, and A is the absorber for all such possible leftovers. And in order to show $G' - L$ has a K_3-decomposition, we need to use the iterative absorption method.

The idea behind iterative absorption method is not very difficult, as we are just applying the absorption method iteratively. The main theorem is talking about graph G with a very large number of vertices. As discussed regarding the possible leftovers, we want to control the size of them. In other words, we want to restrict the leftover edges to a small subset, considering that constant size would be ideal.

This is why we should introduce a structure called vortex. A vortex is a nested subset of $V(G) : U_0 = V(G) \supset U_1 \supset ... \supset U_\ell$, where $|U_\ell|$ is a constant. These nested subsets also satisfy some additional properties. Once we have this structure, we absorb iteratively as follows: In step 1, the leftover is the whole graph (in U_0), then we use some edges in $G[U_1] - G[U_2]$ to "absorb" all leftover edges in $G[U_0] - G[U_1]$. In step i, we use some edges in $G[U_i] - G[U_{i+1}]$ to "absorb" all leftover edges in $G[U_{i-1}] - G[U_i]$. So after step i, the leftover edges would be in $G[U_i]$. We do this iteratively until all possible leftovers are in

$G[U_\ell]$. Then we shall have rather good control of the leftover so that the size would be constant.

And now the only difficult part is how to "absorb" in the above procedure. This is different from "absorber" of a leftover subgraph which requires an important lemma: the cover down lemma. If $U \subset V(G)$ is a relatively small subset, then the cover down lemma tells us the following: If certain conditions are satisfied, then there is a subgraph H of G with $H \supset G - G[U]$ that is K_3-decomposable, that is, we can find an approximate K_3-decomposition and the only possible leftovers are in $G[U]$, which is smaller than G. In other words, we use some edges in $G[U]$ to absorb everything in $G - G[U]$, and that is exactly what we want for the preceding iterative absorption.

Interestingly, later when we prove the cover down lemma in section 5, we may notice that the idea behind the proof of the cover down lemma and the main theorem has something in common. In both cases, we wish to get an approximate decomposition first, then absorb the leftover edges. We already stated that this approach may not work in the main theorem case, as we want to get a precise K_3-decomposition of the whole graph. However, in the cover down lemma case, we do allow certain leftover edges to be left inside $G[U]$ so that this natural approach works more easily.

The above discussion is just a rough structure of the proof. As one can see, there are three parts that we need to work with before giving the final proof: absorber, vortex and cover down lemma. And we shall devote the following three sections to these aspects. And then proceed with the proof of Theorem 1.13 in section 6.

3 Absorber

In this section, we study in details what an absorber is and some properties regarding it. Then at the end of the section, we shall prove a lemma which serves an important role in the final proof of the main theorem (section 6). To begin with, let us recall the definition of an absorber.

Definition 3.1 (Absorber for L)**.** *For a given graph L, an absorber for L is a graph A such that $V(L) \subset V(A)$ is independent in A and both A and $A \cup L$ have a K_3-decomposition.*

Remark 3.2. *Note that since $V(L) \subset V(A)$ is independent in A, then we know that A and L are edge-disjoint. Furthermore, both A and $A \cup L$ have a K_3-decomposition implies that both are K_3-divisible. As a result, we know that L is K_3-divisible.*

Example 3.3. *Let L be a triangle, we give an example of an absorber for L as illustrated in the picture.*

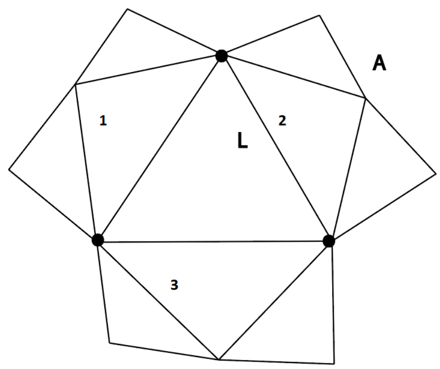

Figure 2: Example

The example above is not a coincidence. In fact, it is a special case of something general which we will introduce next. In brief, one can generalize the above construction to try to find an absorber for any given K_3-divisible graph. From remark 3.2, we know that if L admits an absorber, then it has to be K_3-divisible. In fact, the absorber lemma will tell us that the converse is also true, that is, for every K_3-divisible graph L, there is an absorber. But in order to prove this lemma, we need some more preparations.

Definition 3.4 (Transformer)**.** *Given vertex-disjoint graphs L, L', an (L, L')-transformer is a graph T such that $V(L \cup L') \subset V(T)$ is independent in T and both $T \cup L$ and $T \cup L'$ have a K_3-decomposition.*

Corollary 3.4.1. *If T is an (L, L')-transformer, and L' is K_3-decomposable, then $A := T \cup L'$ is an absorber for L.*

Proof. By definition, we know that $V(L) \subset V(A)$ is independent. The fact that A has a K_3-decomposition follows from T is a transformer. Note that $A \cup L = (T \cup L') \cup L = (T \cup L) \cup L'$. $(T \cup L)$ is K_3-decomposable as T is a transformer. L' is K_3-decomposable by assumption. Also, we know that L' and $T \cup L$ are edge-disjoint, then their union is also K_3-decomposable. $\square$

From the name of transformers and also the context of corollary 3.4.1, we know that the usage of a transformer would be to transform one leftover L to another L'. Since we can define L' ourselves, this will give us more flexibility to construct the absorber. As a result, transformers would be used as a tool to show the existence of an absorber.

Definition 3.5. *Given graphs G, G', a function $\phi : V(G) \to V(G')$ is said to be an edge-bijective homomorphism from G to G' if $\phi(x)\phi(y) \in E(G')$ whenever $xy \in E(G)$, and $|E(G)| = |E(G')| = |\{\phi(x)\phi(y) : xy \in E(G)\}|$. If such a function exists, then we write $G \rightsquigarrow G'$.*

Definition 3.6. *For $m \in \mathbb{N}$, define L_m to be the (canonical) graph with*

$$V(L_m) = \{v^*, v_1, ..., v_{3m}\}, E(L_m) = \bigcup_{i\in[m]} \{v^*v_{3i-2}, v_{3i-2}v_{3i-1}, v_{3i-1}v_{3i}, v_{3i}v^*\}.$$

Definition 3.7. *For any graph L, define $\tilde{\nabla} L$ to be the graph with*

$$V(\tilde{\nabla} L) = V(L) \cup \{v_e : e \in E(L)\}, E(\tilde{\nabla} L) = E(L) \cup \{xv_e, yv_e : e = xy \in E(L)\}.$$

Define $\nabla L := \tilde{\nabla} L - L$, and $\nabla\nabla L = \nabla(\nabla L)$.

Remark 3.8. *It is very obvious that $\tilde{\nabla} L = \nabla L \cup L = \{(e = xy, xv_e, yv_e) : e \in E(L)\}$ has a K_3-decomposition. And now we see that in example 3.3, our choice of absorber is nothing but $\tilde{\nabla}(\nabla L)$.*

Remark 3.9. *For every graph L with m edges, we have $\nabla\nabla L \rightsquigarrow L_m$.*

Proof. Define $\phi : \nabla\nabla L \to L$ by $\phi(v) = v^*$ for all $v \in V(L)$. And there are $3m$ vertices left in each graph, map them bijectively in the natural way. And this ϕ is an edge-bijective homomorphism. □

Now we have enough recipe to prove the lemma, but note that from our above definition of the canonical graph L_m and ∇L, one can imagine that our proof will be constructive, that is, we will give an explicit construction of an absorber of a K_3-divisible graph L using some transformer in the middle step. Note that through this construction, some additional structure comes almost for free, namely the degeneracy of the graph. And this property will turn out to be useful in the final proof. So next, we will give the definition of degeneracy first, and then proceed to present and prove the lemma.

Definition 3.10 (Degeneracy)**.** *For a graph G and a subset $U \subset V(G)$, the degeneracy of G rooted at U is the smallest $d \in \mathbb{N}$ such that there exists an ordering $v_1, ...v_{|V(G)|-|U|}$ of the vertices of $V(G) - U$ such that for all $i \in [|V(G)| - |U|]$,*

$$d_G(v_i, U \cup \{v_j : 1 \le j < i\}) \le d.$$

We need one more lemma in order to prove the final absorber lemma.

Lemma 3.11 (Absorber lemma)**.** *For any K_3-divisible graph L, there exists an absorber A for L such that the degeneracy of A rooted at $V(L)$ is at most 4.*

We need one more lemma in order to prove the final absorber lemma.

Lemma 3.12. *Let L, L' be vertex-disjoint graphs such that $L \rightsquigarrow L'$ and $2|d_L(x)$ for all $x \in V(L)$. There exists an (L, L')-transformer T such that the degeneracy of T rooted at $V(L \cup L')$ is at most 4.*

Proof. Let $\phi : L \to L'$ be an edge-bijective homomorphism.
Since $2|d_L(x)$ for all $x \in V(L)$, then we can decompose L into cycles $\mathcal{C}$, then $\{\phi(C)\}_{C\in\mathcal{C}}$ is a decomposition of L'. Then I claim that as long as we prove the case where L is a cycle, we are done. That is, assume that there is a $(C, \phi(C))$-transformer T_C for every $C \in \mathcal{C}$ such that the degeneracy of T_C rooted at $V(C \cup \phi(C))$ is at most 4. Clearly, we may assume that for all C, $V(T_C) \cap V(L \cup L') = V(C \cup \phi(C))$. And we can always choose distinct vertices for distinct transformers, that is, we may also assume that $V(T_C) \cap V(T_{C'}) \subset V(L \cup L')$ for every $C \neq C' \in \mathcal{C}$. We order the vertices of $T := \bigcup_{C\in\mathcal{C}} T_C$ so that for every C, if we only look at the sequence of vertices of T_C in T, it has degeneracy at most 4. For any $v \in V(T) - V(L \cup L')$, there is $C \in \mathcal{C}$ such that $v \in V(T_C)$. For any C' that has some vertex with a smaller index, we have $V(T_C) \cap V(T'_C) \cap V(L \cup L') \subset V(C \cup \phi(C))$. Hence, the degeneracy of T rooted at $V(L \cup L')$ is also at most 4.
What remains to show is the case when L is a cycle $x_1x_2...x_s$. In this case, we construct a transformer T explicitly as follows.

$$V(T) = V(L \cup L') \cup \{u_i, v_i, w_i : i \in [s]\},$$

where $\{u_i, v_i, w_i : i \in [s]\}$ is a set of $3s$ vertices that are disjoint from $V(L \cup L')$. Define the set of edges as follows:

$$\begin{aligned}
E &:= \{x_iu_i, x_iv_i, x_iw_i, x_iu_{i+1} : i \in [s]\}\\
E' &:= \{\phi(x_i)u_i, \phi(x_i)v_i, \phi(x_i)w_i, \phi(x_i)u_{i+1} : i \in [s]\}\\
\tilde{E} &:= \{u_iv_i, w_iu_{i+1} : i \in [s]\}\\
E^* &:= \{v_iw_i : i \in [s]\},
\end{aligned}$$

where the indices are modulo s. Let $E(T) := E \cup E' \cup \tilde{E} \cup E^*$. Then T is the desired transformer.
By construction, we know that the vertices $V(L \cup L')$ are independent in T. Moreover, $E' \cup \tilde{E} = \{\phi(x_i)u_iv_i : i \in [s]\} \cup \{\phi(x_i)w_iu_{i+1} : i \in [s]\}$, $E(L) \cup E \cup E^* = \{x_iu_iu_{i+1} : i \in [s]\} \cup \{x_iv_iw_i : i \in [s]\}$ are K_3-decomposable, so $T \cup L$ has a K_3-decomposition. Similarly,

$E \cup \tilde{E} = \{x_i u_i v_i\} \cup \{x_i w_i u_{i+1}\}$, and $E(L') \cup E' \cup E^* = \{\phi(x_i) u_i u_{i+1}\} \cup \{\phi(x_i) v_i w_i\}$ are K_3-decomposable, so $T \cup L'$ admits a K_3-decomposition as well. We order the vertices so that $V(L \cup L')$ come before $\{u_i\}$ before $\{v_i, w_i\}$. Then the number of neighbors of u_i with a smaller index is at most 4, namely $x_i, x_{i-1}, \phi(x_i), \phi(x_{i-1})$. The number of neighbors of v_i with a smaller index is at most 4, namely $x_i, \phi(x_i), u_i, w_i$ and that of w_i is $x_i, \phi(x_i), u_{i+1}, v_i$. Hence, the degeneracy of T rooted at $V(L \cup L')$ is at most 4. And this finishes the proof of lemma 3.12. □

Proof of the absorber lemma. Let $m := |E(L)|$. Then $3|m$ as L is K_3-divisible. Let L' be the vertex-disjoint union of $m/3$ triangles. Let L_m be the canonical graph defined in definition 3.6. Let $\nabla L, \nabla\nabla L, \nabla L', \nabla\nabla L'$ be defined as in definition 3.7 and assume that $\nabla\nabla L, \nabla\nabla L', L_m$ are vertex-disjoint.
By remark 3.9, we know that $\nabla\nabla L \rightsquigarrow L_m$ and $\nabla\nabla L' \rightsquigarrow L_m$, by lemma 3.12, we know there exists an $(\nabla\nabla L, L_m)$-transformer T such that the degeneracy of T rooted at $V(\nabla\nabla L \cup L_m)$ is at most 4. Similarly, there exists an $(\nabla\nabla L', L_m)$-transformer T' such that the degeneracy of T' rooted at $V(\nabla\nabla L' \cup L_m)$ is at most 4. Furthermore, we may assume that we use new vertices for T and T' whenever it is possible to do so, that is, we assume that $V(T) \cap V(T)' = V(L_m), V(T) \cap V(\nabla\nabla L') = \emptyset, V(T') \cap V(\nabla\nabla L) = \emptyset$.
Define

$$A := \nabla L \cup \nabla\nabla L \cup T \cup L_m \cup T' \cup \nabla\nabla L' \cup \nabla L' \cup L'.$$

Then this graph A is the absorber for L. It is obvious that $V(L)$ is independent in A. $A = (\nabla L \cup \nabla\nabla L) \cup (T \cup L_m) \cup (T' \cup \nabla\nabla L') \cup (\nabla L' \cup L')$. By remark 3.8 and the definition of transformer, using the fact that each graph in the parentheses are edge-disjoint, we know that A is K_3-decomposable. Moreover, $A \cup L = (L \cup \nabla L) \cup (\nabla\nabla L \cup T) \cup (L_m \cup T') \cup (\nabla\nabla L' \cup \nabla L') \cup L'$ is also K_3-decomposable. So we've shown that A is an absorber for L. We order the vertices of A such that the degeneracy of T rooted at $V(\nabla\nabla L' \cup L_m)$ is at most 4 and the degeneracy of T' rooted at $V(\nabla\nabla L \cup L_m)$ is at most 4. This can be done because we assume that $V(T) \cap V(\nabla\nabla L') = \emptyset, V(T') \cap V(\nabla\nabla L) = \emptyset$. Then we can see that the degeneracy of A rooted at $V(L)$ is at most 4. □

Remark 3.13. *Notice that by examining the construction, we also know that* $|V(A(L))| = O(|V(L) + E(L)|)$, *because each* $\nabla L, \nabla\nabla L, L', T, L_m, T', \nabla L', \nabla\nabla L'$ *has linear size.*

We end this section by trying to give an explanation of why the degeneracy condition will be useful. As stated in section 2, in our final proof of the main theorem, our L would be a small leftover subgraph inside some small subset U_ℓ, and we wish to find an absorber for it at the very beginning. The absorber lemma ensures that there is an absorber A for L, but the proof given above is very constructive, which means that it may not be easy for us to find this absorber A in a general graph G. Hence, instead of finding A in G directly, we can try to embed it into the graph G, which is an easier task thanks to the degeneracy condition. Assume $v_1, ..., v_s$ is an enumeration of the vertices in $V(A) - V(L)$ with respect to which the degeneracy is at most 4. We will embed the vertices of A into G according to this order. Assume $v_1, ..., v_i$ are embedded and now we want to embed v_{i+1}. Since the degeneracy is at most 4, then as long as any four vertices in $\{v_1, ..., v_i\} \cup V(L)$ still have an available common neighbor u, we can define u to be the image of v_{i+1} under this embedding. And the existence of common neighbors of four vertices is not so difficult to show as our graph G is a very dense graph. For the more detailed explanation, see section 6 where we prove the main theorem.

4 Vortex

We mentioned the idea of a vortex in section 2 very briefly. In this section, we will state the definition of a vortex and then prove a lemma about the existence of vortex, which will be used for the iterative step in the final proof.

Definition 4.1 (Vortex). *Let G be a graph on n vertices. A (δ, ϵ, m)-vortex in G is a sequence $U_0 \supset U_1 \supset ... \supset U_\ell$ with the following properties:*
(V1) $U_0 = V(G)$;
(V2) $|U_i| = \lfloor \epsilon |U_{i-1}| \rfloor$ for all $i \in [\ell]$;
(V3) $|U_\ell| = m$;
(V4) $d_G(x, U_i) \geq \delta |U_i|$ for all $i \in [\ell]$ and $x \in U_{i-1}$.

Let us try to understand why we define vortex as above. As discussed in the early section, the vortex would be the main structure for us to iteratively absorb the "leftover" edges. And in order to control the size of the final possible leftovers, we need $|U_\ell| = m$ to have a constant size. And since we do want to absorb everything of G, so we also need $U_0 = V(G)$. Now if we look back at the discussion on how to absorb in one particular step, (that is, the cover down lemma), we wish to cover the leftover edges in $G[U_i] - G[U_{i+1}]$ using some edges from $G[U_{i+1}] - G[U_{i+2}]$, so that the remaining leftovers have smaller size (in $G[U_{i+1}]$ instead of $G[U_i]$). Hence, we wish to exponentially decrease the size of possible leftovers, which means that U_i's are supposed to be decreasing exponentially as well. Thus, we require $|U_i| = \lfloor \epsilon |U_{i-1}| \rfloor$ for all $i \in [\ell]$.

Notice that if we wish to achieve the above goal of absorbing, the first three conditions are not sufficient. In other words, there has to be some additional structure on the U_i's such that we could absorb the leftover edges. The main theorem we wish to prove (Theorem 1.13) requires that the minimum degree of the graph to be large. So naturally, we wish each of our sets U_i in the vortex to inherit similar properties. On the other hand, if we wish to use edges from $G[U_{i+1}]-G[U_{i+2}]$ to absorb almost everything from $G[U_i]-G[U_{i+1}]$, analogous to the statement of the main theorem, we wish the vertices in U_i to have large degree in U_{i+1} and that is why we need $V4$ in the definition.

Remark 4.2. *If we wish to prove some other theorems other than theorem 1.13, we may change condition (V4) to other conditions which fits with the statement of the theorem. For example, in the second part of this essay where we wish to prove theorem, where G is a $(\xi, 4, p)$-typical graph, instead of the large degree condition we will need the vortex to possess a similar property as the $(\xi, 4, p)$-typical property instead of the degree property.*

Let us go back and stick with this definition of a vortex for a while and show the vortex existing lemma now:

Lemma 4.3 (Vortex existing lemma)**.** *Let $\delta \in [0,1]$ and $1/m' \ll \epsilon < 1$. Suppose that G is a graph on $n > m'$ vertices with $\delta(G) \geq \delta n$. Then G has a $(\delta-\epsilon, \epsilon, m)$-vortex for some m with $\lfloor \epsilon m' \rfloor \leq m \leq m'$.*

Remark 4.4. *Observe that if we choose a random subset U_{i+1} of U_i of size $\lfloor \epsilon |U_i| \rfloor$, then the expected value of $d_G(v, U_{i+1})$ is $\delta' |U_{i+1}|/|U_i|$, where $v \in U_i$ and $\delta' = d_G(v, U_i)$. This inspires us to use the probabilistic method and merely choose a random subset $U_{i+1} \subset U_i$. Then we can use Chernoff inequality to show that such a random subset with certain properties exists with positive possibility. And this is also why we have $\delta - \epsilon$ instead of δ in the vortex degree condition, because each time we use Chernoff inequality, we need to have a little variation off the expected value, which in this case is ϵ.*

Proof. We will prove this by induction. In fact, we will induct on some statement which is a bit stronger than the lemma, but turns out to be easier for the induction to carry on.
Some setups: define $n_0 := n, n_i := \lfloor \epsilon n_{i-1} \rfloor$. Then we have $\epsilon^i n \geq n_i \geq \epsilon^i n - 1/(1-\epsilon)$. Let $l := 1 + \max\{i \geq 0 : n_i \geq m'\}$ and let $m := n_l$. Then we have $\lfloor \epsilon m' \rfloor \leq m \leq m'$. For $i \in [\ell]$, let

$$\epsilon_i := n^{-1/3} \sum_{j=1}^{i} \epsilon^{-(j-1)/3} = \epsilon n^{-1/3} \frac{\epsilon^{-i/3}-1}{\epsilon(\epsilon^{-1/3}-1)}.$$

For any i, $\frac{\epsilon^{-i/3}-1}{(\epsilon^{-1/3}-1)\epsilon}$ is an expression only depending on ϵ, since $1/n \ll \epsilon$, we know that we can choose n so that $n^{-1/3}\frac{\epsilon^{-i/3}-1}{\epsilon(\epsilon^{-1/3}-1)} \leq 1$, hence $\epsilon_i \leq \epsilon$.

Now suppose that for some $i \in [\ell]$, we have already found a $(\delta - \epsilon_{i-1}, \epsilon, n_{i-1})$-vortex $U_0, ..., U_{i-1}$ in G. Now we wish to find U_i and carry on the induction. Let $U_i \subset U_{i-1}$ be a random subset of size n_i. For any $x \in U_{i-1}$, let X be the random variable for the number of neighbors of x in U_i. Then $X := \sum_{u \in N_{U_{i-1}}(x)} X_u$, where X_u is the indicator random variable of the event $u \in U_i$. By the inductive hypothesis, we have $|N_{U_{i-1}}(x)| \geq (\delta - \epsilon_{i-1})n_{i-1}$, and $\mathbb{P}[X_u = 1] = \frac{n_i}{n_{i-1}}$, thus, $\mathbb{E}X \geq (\delta - \epsilon_{i-1})n_i$. By the Chernoff bound, we have, for some constant c,

$$\mathbb{P}[X \leq (\delta - \epsilon_i)n_i] \leq \mathbb{P}[\mathbb{E}X - X \geq n_i \cdot \epsilon^{-(i-1)/3}] \leq ce^{-(\epsilon^{-2/3(i-1)}n_i)/(\delta - \epsilon_{i-1})}.$$

Thus, by union bound and the fact that $1/n \ll \epsilon$, we have $\mathbb{P}[d_G(x, U_i) \geq (\delta - \epsilon_i)n_i] \leq n_{i-1}ce^{-(\epsilon^{-2/3(i-1)}n_i)/(\delta - \epsilon_{i-1})} \leq 1$ for all $x \in U_{i-1}$, thus, with positive probability, our random choice of U_i works. So we obtain a $(\delta - \epsilon_i, \epsilon, n_i)$-vortex in G. By induction, we finally get a $(\delta - \epsilon_l, \epsilon, m)$-vortex $U_0, ..., U_\ell$ in G. Since $\epsilon_i \leq \epsilon$, the lemma follows. □

5 Cover down lemma

Now we introduce the cover down lemma, which is the final missing piece for our proof.

Lemma 5.1 (Cover down lemma)**.** *Suppose* $1/n \ll \epsilon$ *and let* $\delta := \max\{2/3, \delta^{0+}\}$*. Let* G *be a graph on* n *vertices and* $U \subset V(G)$ *with* $|U| = \lfloor \epsilon n \rfloor$*. Suppose that* $\delta(G) \geq (\delta + 3\epsilon)n$ *and* $d_G(x, U) \geq (\delta + 2\epsilon)|U|$ *for all* $x \in V(G)$*, and assume that* $d_G(x)$ *is even for all* $x \in V(G) - U$*. Then there exists a* K_3*-decomposable subgraph* H *of* G *such that* $G - G[U] \subset H$ *and* $\Delta(H[U]) \leq \epsilon^{10}n$*.*

The proof of the cover down lemma is a bit complicated. But before proving it, let us discuss the statement of this lemma. Although there is some imprecision, we want to apply this lemma to $(G[U_i], U_{i+1})$ in the role of (G, U), where U_i's are in the vortex of G. And one can see that the subsets in the vortex actually satisfy a similar condition about the degree as the cover down lemma.

The existence of H is as expected when we discussed the usage of this lemma back in section 2. Just as discussed, H serves the role as "absorber" of $G - G[U]$. But if we only need to absorb once, we do not need the property of H where $\Delta(H[U]) \leq \epsilon^{10}n$. We require our "absorber" H to have this property, that is, we wish H to use not too many edges inside $G[U]$ because we need to iteratively apply the cover down lemma to the subsets of the vortex. And this condition on H makes it possible that if the leftover edges in $G[U_{i+1}]$ still have the large minimum degree property, then we can continue to apply this cover down lemma until all the possible leftover edges are in U_ℓ.

Next, we present the rough idea about the proof of the cover down lemma. As discussed before, the general procedure to such problems is always to try to find an approximate K_3-decomposition first (of $G - G[U]$), and we are left with some leftover subgraph L. Then we try to absorb L using only a few edges from $G[U]$.

This time, the approximate K_3-decomposition will be guaranteed by the definition of δ^{0+}. So let us look at L more closely. Notice that $L \subset G - G[U]$ so $L = L[W] \cup L[W, U]$, where $W = V(G) - U$. Hence we can try to absorb $L[W]$ and $L[W, U]$ separately. For an edge $e \in L[W], e = xy$, where $x, y \in W$. The degree condition will make sure that there exists some $u \in U$ such that $u \in N_G(x) \cap N_G(y)$. Hence $\{ux, uy\}$ will absorb the edge e. Now as long as we can find exclusive u_e for all $e \in L[W]$, we get an absorber for $L[W]$. For edges of the form $u_1x, u_2x, ..., u_kx$, where $x \in W, u_i \in U$, since $d_G(x)$ is even and we remove even number of edges when removing triangles incident to x, then we know k is even. Thus, it would be ideal if there was a perfect matching M_x between $\{u_1, ..., u_k\}$. This way, we can say for sure that M_x is an absorber for $\{u_1x, ..., u_kx\}$ and we know that M_x does not use too many edges from $G[U]$. Hence, in order to absorb $L[W, U]$, we need to find edge-disjoint perfect matchings M_x for all possible x.

But notice that the above ideal conditions are not very easy to achieve, that is why in the real proof, we need to set aside a sparse graph R and some sets $U_1, ..., U_N$ before doing the approximate decomposition, just to give some additional structure so that the discussion above is easier to be satisfied. We will see the power of R later in the real proof. But before the proof, we still need to give a lemma regarding the property of the perfect matching M_x that would be useful in the proof.

Lemma 5.2. *Let $1/n \ll \rho$ and $N \in \mathbb{N}$. Let H be a graph on n vertices and suppose there are sets $U_1, ..., U_N \subset V(H)$ with the following properties:*
(a), $2||U_i|$ and $\delta(H[U_i]) \geq (1/2 + 4\rho^{1/6})|U_i|$ for all $i \in [N]$;
(b), $|U_i| \geq \rho^{4/3}n$ for all $i \in [N]$;
(c), $|U_i \cap U_j| \leq \rho^2 n$ for all $i \neq j$;
(d), every vertex $u \in V(H)$ is contained in at most ρn of the sets U_i.
Then for every $i \in [N]$, there exists a perfect matching M_i of $H[U_i]$, such that all the matchings $\{M_i\}_{i\in[N]}$ are pairwise edge-disjoint.

Remark 5.3. *Dirac's theorem on Hamilton cycles says that if H is a graph with n vertices and $\delta(H) \geq n/2$, then there is a Hamilton cycle in H. Therefore, by (a), we know that there exist perfect matchings M_i in U_i for all $i \in [N]$. But as discussed above, what we want is edge disjoint union of perfect matchings, and that is why we need three more conditions.*

Remark 5.4. *We will use randomized algorithm in the proof to show the existence of such matchings.*

Proof. Set $t := \lceil 2\rho^{3/2}n \rceil$ and define $H_i := H[U_i]$ for all $i \in [N]$.
We use a randomized algorithm inductively. Suppose $M_1, ...M_{i-1}$ are found such that they are perfect matchings and they are edge-disjoint, now we find M_i. First, define $L_{i-1} := \bigcup_{j=1}^{i-1} M_j$. Let $H_i' := (H - L_{i-1})[U_i]$.
If we have

$$\Delta(L_{i-1}[U_i]) \leq \rho^{3/2}n, \tag{2}$$

then $\delta(H_i') \geq |U_i|/2 + t$, so there are t edge-disjoint prefect matchings $A_1, ..., A_t$ of G_i' by applying remark 5.3 successively. If condition (1) does not hold, then we set $A_1, ..., A_t$ to be the empty graph. In either case, we choose $s \in [t]$ uniformly at random and set $M_i = A_s$. Then the lemma holds if (1) holds for all $i \in [N]$.
For $u \in U_i$, define $Y_j^{i,u}$ be the indicator variable of the event that there exists some $u' \in U_i$ with $uu' \in E(M_j)$. Since M_j is a perfect matching, then for fixed j, there could be at most one $u' \in U_i$ with $uu' \in E(M_j)$. Hence, we get

$$d_{L_{i-1}[U_i]}(u) = \sum_{j \in [i-1]} Y_j^{i,u}.$$

But by property (d), there can be at most ρn indices $j \in [i-1]$ such that $Y_j^{i,u}$ could be 1 as there are at most such number of j with $u \in U_j$. For an index $j \in [i-1]$ with $u \in U_j$, by the inductive procedure above, we know that we have t edge-disjoint choices of perfect matchings, but there are at most $\rho^2 n$ number of u' for us to choose, thus

$$\mathbb{P}[Y_j^{i,u} | u \in U_j] \leq \frac{\rho^2 n}{t} \leq \frac{\rho^{1/2}}{2}.$$

Hence, we know that $\mathbb{E}(\sum_{j \in [i-1]} Y_j^{i,u}) \leq \rho n \cdot \frac{\rho^{1/2}}{2} = \rho^{3/2}n/2$. By a variation of Chernoff inequality, we get that

$$\mathbb{P}[\sum_{j \in [i-1]} Y_j^{i,u} > \rho^{3/2}n] \leq 2e^{-\rho^2 n/2}.$$

Furthermore, using (d) again, we know that there are at most ρn such i with $u \in U_i$, and at most n such u, thus, by a union bound, $\mathbb{P}[(1) \text{ does not hold}] \leq 2e^{-\rho^2 n/2}\rho n^2 \ll 1$. Hence, there is positive probability such that (1) holds, so the induction procedure works and thus lemma holds. □

Remark 5.5. *In the proof of the cover down lemma, we will apply lemma 5.2 where $G[U]$ serves the role as H in lemma 5.2.*

Now we will prove the cover down lemma. But in order to make the proof more accessible, we will make a few more preparations.
From now on, the settings will be the same as the cover down lemma, and furthermore, set $W := V(G) - U$ and $N := |W|$, order the vertices in W by $w_1, ..., w_N$.

Lemma 5.6. *There exist sets $U_1, ..., U_N$ with the following properties:*
(a) $U_i \subset N_G(w_i) \cap U$ for all $i \in [N]$;
(b) $(1-\rho)\rho|N_G(w_i) \cap U| \leq |U_i| \leq (1+\rho)\rho|N_G(w_i) \cap U|$ for all $i \in [N]$;
(c) $\rho^2|U|/4 \leq |U_i \cap U_j| \leq 2\rho^2|U|$ for all $1 \leq i < j \leq N$;
(d) $|N_G(u) \cap U_i| \geq (1-\rho)\rho(1/2 + 3\epsilon)|N_G(w_i) \cap U|$ for all $U \in U, i \in [N]$;
(e) each $u \in U$ is contained in at most $2\rho n$ of the U_i's.

Proof. We use a probabilistic argument. In fact, for any $u \in N_G(w_i) \cap U$, include u in U_i with probability ρ independently of all other choices. Then (a) is clearly satisfied. Next we will show that with positive probability, the other four conditions can also be satisfied.
For (b), let $X = |U_i|$ be a random variable. Then $X = \sum_{u \in N_G(w_i) \cap U} X_u$, where X_u is the indicator random variable which takes the value 1 if $u \in U_i$ and 0 otherwise. Then $\mathbb{E}X = \rho|N_G(w_i) \cap U|$. Since X is the sum of Bernoulli random variables, by the Chernoff bound, we have

$$\mathbb{P}[|X - \mathbb{E}X| \geq \rho^2|N_G(w_i) \cap U|] \leq 2e^{-2\rho^4|N_G(w_i)\cap U|}.$$

By union bound, we know that the probability that (b) does not hold is at most

$$2Ne^{-2\rho^4|N_G(w_i)\cap U|} \leq 2ne^{-2\rho^4(2/3\epsilon n)} \ll 1,$$

since $1/n \ll \rho \ll \epsilon$.
For (c), similar to (b), let $X := |U_i \cap U_j| = \sum_{u \in N_G(w_i) \cap N_G(w_j) \cap U} X_u$, where $X_u = 1$ if $u \in U_i \cap U_j$. Hence, $\mathbb{E}X = \rho^2|N_G(w_i) \cap N_G(w_j) \cap U|$. Since $d_G(x, U) \geq 2/3|U|$ for all $x \in V(G)$, then we have $|U| \geq |N_G(w_i) \cap N_G(w_j) \cap U| \geq 2/3|U| + 2/3|U| - |U| \geq 1/3|U|$. Therefore, $\rho^2|U|/3 \leq \mathbb{E}X \leq \rho^2|U|$. Hence, by the Chernoff bound again, the probability that (c) does not hold is at most

$$cn^2e^{-\epsilon\rho^4 n} \ll 1,$$

where c is a constant and the inequality holds again by $1/n \ll \rho \ll \epsilon$.
For (d). For any $x, y \in V(G)$, let $X := N_G(x) \cap U$ and $Y := N_G(y) \cap U$. Then $|X| - |Y| \leq |U| - |Y|$. Since $|Y| \geq (2/3 + 2\epsilon)|U|$, then we know that

$$|X \cap Y| = |X| - |X - Y| \geq (2/3 + 2\epsilon)|U| - |U| + |Y| \geq (1/2 + 3\epsilon)|Y|,$$

where the last inequality holds because ϵ is small, in particular, we can choose $\epsilon \leq 1/6$. Take $x = u, y = w_i$, and by the Chernoff bound, one can see that the probability that (d) holds is less than 1 (using the fact that $\mathbb{E}[N_G(U) \cap U_i] \geq \rho(1/2 + 3\epsilon)|N_G(w_i) \cap U|$).
For (e), let $X = \sum_{i=1}^N X_i$ be the number of U_i that $u \in U$ is contained in, where $X_i = 1$ if $u \in U_i$. Since $|U| \geq |N_G(w_i) \cap U| \geq 2|U|/3$, then we know that $2/3\rho n \leq \mathbb{E}X \leq \rho$. Hence, by Chernoff bound, the probability that (e) does not hold is at most $c\epsilon ne^{-\rho^2 n} \ll 1$, where c is a constant and the inequality holds by $1/n \ll \rho \ll \epsilon$. $\square$

The usage of these sets is still not very clear at this stage. Although one can already see that the conditions that these U_i's satisfy look really similar to those conditions in lemma 5.2. This is not a coincidence, in fact, we will make slight changes to make U_i into U_i' later in the proof and those U_i' will serve the role as in lemma 5.2.
We point out one more observation before the proof so that it is not so tedious: for any $u \in U, i \in [N]$,

$$d_G(u, U_i) = |N_G(u) \cap U_i| \geq (1-\rho)\rho(1/2+3\epsilon)|N_G(w_i) \cap U| \geq \frac{1-\rho}{1+\rho}(1/2+3\epsilon)|U| \geq (1/2+2\epsilon)|U|, \tag{3}$$

by (d), (b), and $\rho \ll \epsilon$.

Proof of the cover down lemma. As mentioned before, we set aside a sparse graph R at first. Let $R = \{uw_i : u \in U_i, w_i \in W\}$ be a subgraph of $G[U, W]$. Then by lemma 5.6 (e), we know that $\Delta(R) \leq 2\rho n$.
Set $G' := G - G[U] - R$, then

$$\delta(G') \geq \delta(G) - |U| - \Delta(R) \geq (\delta + 3\epsilon - \epsilon - 2\rho)n \geq (\delta^{0+} + \epsilon)n.$$

Hence, by the definition of δ^{0+}, there exists a subgraph L of G' such that $\Delta(L) \leq \gamma n$ and $G' - L$ is K_3-decomposable.
As the name suggests, L is the leftover subgraph and we need to absorb it. Notice that $L = L[W] \cup L[U, W]$ so we absorb each part separately using edges from R.

First, we absorb $L[W]$. For every edge $e = w_i w_j \in L[W]$, we would like to choose $u_e \in U_i \cap U_j$, and we would like to have $u_e \neq u_{e'}$ when $e \cap e' \neq \emptyset$. But this can be done greedily. Since by lemma 5.6 (c), there are at least $\rho^2|U|/4$ choices of u_e, but there are at most $2\Delta(L) \leq 2\gamma n$ such e' with $e' \cap e \neq \emptyset$. Since we can choose γ so that $\gamma \ll \rho$, then $2\Delta(L) \leq \rho^2|U|/4$, so the greedy algorithm works. Let $A = \{u_{w_i w_j} w_i, u_{w_i w_j} w_j\}$ contains all such edges for $w_i w_j \in L[W]$, then it is a subgraph of R such that $A \cup L[W]$ has a K_3-decomposition.

Now, we absorb $L[U, W]$ using the rest of edges from R and possibly some edges (ideally not too many) from $G[U]$ if necessary. Let $R' = L[U, W] \cup (R - A)$, then the goal is to absorb R' using a few edges from $G[U]$. We will use lemma 5.2 now.
Define

$$U_i' := N_{R'}(w_i) = (U_i - N_A(w_i) \cup N_{L[U,W]}(w_i).$$

Now we wish to show that U_i' satisfy the conditions in lemma 5.2.

Define $H := G[U]$, $n' := |U|$, and $\rho' := 4\rho/\epsilon$. Since $d_G(w_i)$ is even and we only remove edge-disjoint triangles to get $G[U] \cup R'$, then we know that $|U_i'|$ is even. By inequality (2)

that appears in the remark before the proof of the cover down lemma, we know $d_G(u,U_i) \geq (1/2+2\epsilon)|U_i|$. Since $d_A(w_i) \leq \Delta(A) \leq \gamma n$, then $\delta(H[U_i']) = \delta(G[U_i']) \geq d_G(u,U_i) - \Delta(A) \geq (1/2+\epsilon')|U_i'|$. And this proves condition (a) in lemma 5.2. $|U_i'| \geq |U_i| - |A| \geq (1-\rho)\rho(2/3+2\epsilon)|U| - \gamma|U| \geq \rho|U|/2 \geq \rho' n'$, then condition (b) is shown. Since $d_{L[U,W]}(w_i) \leq \gamma n$, then $|U_i' \cap U_j'| \leq |U_i \cap U_j| + |N_{L[U,W]}(w_i)| + |N_{L[U,W]}(w_j)| \leq 2\rho^2 n' + 2\gamma n \leq \rho'^2 n'$, which leads to condition (c). Similarly, each $u \in U$ is contained in at most $2\rho n$ of the U_i's and $d_{L[U,W]}(w_i) \leq \gamma n$, then each u is contained in at most $3\rho n$ of the U_i's, which is smaller than or equal to $\rho' n'$, thus (d) is shown.
Hence, we can apply lemma 5.2 and get a perfect matching M_i for each $i \in [N]$ such that they are edge-disjoint. Then $\bigcup_{i\in[N]} M_i \cup R'$ is K_3-decomposable. Thus, $H := (G - G[U]) \cup \bigcup_{i\in[N]} M_i$ is $K-3$-decomposable and $\Delta(H[U]) = \Delta(\bigcup_{i\in[N]} M_i) \leq \epsilon^{10} n$. □

As we can see from the last inequality of the proof, $\Delta(H[U]) \leq \epsilon^2 n$ might be a better bound. However, in the final proof (section 6), we actually prove the theorem 1.13 with 8ϵ instead of ϵ, and dealing with $\epsilon^{10} n$ would make the final proof easier because of that reason. In fact, when proving lemmas of similar nature, the constants are usually not so important.

6 Proof of the main theorem

Finally, we have enough preparation to prove the main theorem (theorem 1.13).
Recall section 2 on the outline of the proof. We will apply the vortex existing lemma to G to get a platform to apply the cover down lemma iteratively. For convenience, we prove the theorem with 8ϵ instead of ϵ, that is, we assume G to be a graph on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + 8\epsilon)n$. Then by lemma 4.3, there exists a $(\delta + 7\epsilon, \epsilon, m)$-vortex $U_0, ..., U_\ell$ in G for some $\lfloor \epsilon m' \rfloor \leq m \leq m'$.

The idea of the proof is to use edges of $G[U_{i+1}]$ to absorb the leftover edges from $G[U_i] - G[U_{i+1}]$, so that we are eventually left with only possible leftover edges inside U_ℓ. So it is better for us to deal with the "final" leftovers first.

Let $\mathcal{L}$ be the collection of all spanning K_3-divisible subgraphs of $G[U_\ell]$. Then since $|U_\ell| = m$, we know $|\mathcal{L}| \leq 2^{\binom{m}{2}}$. The goal is to find absorbers A_L for each $L \in \mathcal{L}$ such that they are edge-disjoint. Apply the absorber lemma (lemma 3.11), we know that there exists an absorber B_L for L as $|L| \leq m'$ such that $|B_L| \leq M$ and the degeneracy of B_L rooted at $V(L)$ is at most 4. Since B_L is not subgraph of G in general, we need to use them to find those A_L which are subgraphs of G. We show that this is possible by induction. Suppose some absorbers ($A_{L'}$'s) are already found as subgraphs of $G - G[U_1]$, we would like to find A_L.

Consider the graph K obtained from $G - G[U_1]$ by deleting those edges of the previously chosen absorbers. Then $\delta(K) \geq \delta(G) - |U_1| - M2^{\binom{m}{2}} \geq (3/4+\epsilon)n$. Thus, any four vertices in K would have at least $4 \cdot (3/4+\epsilon)n - (4-1)n = 4\epsilon n \geq M$ common neighbors. Since on the other hand there is an ordering of the vertices in $V(B_L) - V(L)$ such that every vertex is connected to at most 4 preceding vertices in B_L. And $|V(B_L) - V(L)|$ thus we can embed the vertices of $V(B_L) - V(L)$ according to this order into K to obtain A_L. Let $A := \bigcup_{L \in \mathcal{L}} A_L$, then A has the property:

$$\text{For any } K_3\text{-divisible subgraph } L^* \text{ of } G[U_\ell],\ A \cup L^* \text{ has a } K_3\text{-decomposition.} \tag{4}$$

Now it is obvious what to do next. Let $G' := G - A$. Since $\Delta(A) \leq M|\mathcal{L}| \leq \epsilon^2 n$, and $A[U_1]$ is empty, then for any $x \in U_0$, $d_{G'}(x, U_1) \geq d_G(x, U_1) - \epsilon^2 n \geq (\delta + 6\epsilon)|U_1|$. Hence, we know that $U_0, ..., U_\ell$ is now a $(\delta + 6\epsilon, \epsilon, m)$-vortex of G'. Since we also have $\Delta(A) \leq M|\mathcal{L}| \leq \epsilon n$, then $\delta(G') \geq (\delta + 7\epsilon)n$. Since A is K_3-decomposable, then it is K_3-divisible, thus G' is also K_3-divisible. Now we iteratively apply the cover down lemma to vortex in G'.
Precisely speaking, for every $i \in [\ell] \cup \{0\}$, we wish to show there exists a subgraph $G_i \subset G'[U_i]$ such that $G' - G_i$ has a K_3-decomposition, and the following holds (where $U_{l+1} := \emptyset$):

$$\delta(G_i) \geq (\delta + 4\epsilon)|U_i|; \tag{5}$$

$$d_{G_i}(x, U_{i+1}) \geq (\delta + 5\epsilon)|U_{i+1}| \text{ for all } x \in U_i; \tag{6}$$

$$G_i[U_{i+1}] = G'[U_{i+1}]. \tag{7}$$

Clearly, this holds for $i = 0$ with $G_0 := G'$ so the induction can start.

The induction step: Assume for some $i \in [l-1]] \cup \{0\}$, we found G_i with the above properties. Then G_i is K_3-divisible. Define $G_i' := G_i - G_i[U_{i+2}]$. Then for all $x \in U_i - U_{i+1}$, $x \notin U_{i+2}$, so $d_{G_i'}(x)$ is even since G_i' is K_3-divisible. By (4) and (5), we know that we can apply the cover down lemma with $(G_i', U_{i+1}, \epsilon)$ in the role of (G, U, ϵ), and get a K_3-decomposable subgraph $H \subset G_i'$ with $G_i' - G_i'[U_{i+1}] \subset H$ and $\Delta(H[U_{i+1}]) \leq \epsilon^{10}|U_i| \leq \epsilon|U_{i+1}|$. Define $G_{i+1} := (G_i - H)[U_{i+1}]$, then it is a subgraph of $G'[U_i]$ and clearly $G' - G_{i+1}$ has a K_3-decomposition and $G_{i+1}[U_{i+2}] = G'[U_{i+2}]$. So we only need to show that (4) and (5) still hold and that will finish the induction.
By (5),

$$\delta(G_{i+1}) \geq (\delta + 5\epsilon)|U_{i+1}| - \Delta(H[U_{i+1}]) \geq (\delta + 4\epsilon)|U_{i+1}|,$$

so (4) holds. For every $x \in U_{i+2}$, we have

$$d_{G_{i+1}}(x, U_{i+2}) \geq (\delta + 6\epsilon)|U_{i+2}| - \Delta(H[U_{i+1}]) \geq (\delta + 5\epsilon)|U_{i+2}|,$$

so (5) holds, where the first inequality above uses (6) and condition (V4) of definition of vortex (definition 4.1). Therefore, the induction works and there exists a subgraph $G_l \subset G'[U_\ell]$ such that $G' - G_l$ has a K_3-decomposition.

Since G_l is K_3-divisible, then $A \cup G_l$ has a K_3-decomposition. Thus, $G = (G' - G_l) \cup (G_l \cup A)$ is K_3-decomposable and the theorem is proved.

7 The second approach to K_3-decomposition problem

Recall section 1.2 of this essay, we would like to tackle the K_3-decomposition problem of a graph. A promising idea is to get some approximate decompositions first, then transfer it to a complete decomposition. In order to get an approximate decomposition, our first approach was to introduce a new constant called the approximate decomposition threshold and require our graph G to have large enough minimum degree. The problem with the above approach is that the result really depends on the value of δ^{0+}, which we do not know yet. Therefore, in this section, we shall give a second way to get approximate decompositions, which will eventually lead to another theorem on the triangle decomposition problem.

The major concept we will be dealing with in section 7 is called typicality, which is a natural way to describe the quasi-randomness of a graph.

Definition 7.1. *Given $p, \xi > 0, h \in \mathbb{N}$, a graph G with n vertices is said to be (ξ, h, p)-typical if for every set $A \subset V(G)$ with $|A| \leq h$, $(1-\xi)p^{|A|}n \leq |N_G(A)| \leq (1+\xi)p^{|A|}n$.*

The analogous result of theorem 1.13 is the following theorem:

Theorem 7.2. *For all $p > 0$, there exist $n_0 \in \mathbb{N}, \xi > 0$ such that every $(\xi, 4, p)$-typical K_3-divisible graph on $n \geq n_0$ vertices is K_3-decomposable.*

This theorem is proved as a corollary in the paper by Keevash in 2014 ([8]), but the proof used in [8] was different, and here we present an outline of a new proof using the iterative absorption method. In fact, the proof is very similar to that of the main theorem (theorem 1.13), that is, we will also use the iterative absorption method on the vortex structure. Therefore, we need an analogous version of the vortex existing lemma as well as an analogous cover down lemma.

Just as we mentioned in remark 4.2, the definition of vortex should be changed here to reflect the property of G. The new definition of a vortex is defined below:

Definition 7.3 (Vortex'). *Let G be a graph on n vertices. For any fixed $p \in (0,1)$, a (ξ, ϵ, m)-vortex in G is a sequence $U_0 \supset U_1 \supset \ldots \supset U_\ell$ with the following properties:*
(V1) $U_0 = V(G)$;
(V2) $|U_i| = \lfloor \epsilon |U_{i-1}| \rfloor$ for all $i \in [\ell]$;
(V3) $|U_\ell| = m$;
(V4) for every set $A \subset U_{i-1}$ with $|A| \leq 4$, we have $(1-\xi)p^{|A|}|U_i| \leq |N_{U_i}(A)| \leq (1+\xi)p^{|A|}|U_i|$.

Regarding this new definition, we can state and prove the analogous version of the vortex existing lemma:

Lemma 7.4 (Vortex existing lemma'). *Let $\xi, p \in [0,1]$ and $1/m' \ll \epsilon < 1$. Suppose that G is a $(\xi, 4, p)$-typical graph on $n > m'$ vertices. Then G has a $(\xi + \epsilon, \epsilon, m)$-vortex for some m with $\lfloor \epsilon m' \rfloor \leq m \leq m'$.*

Proof. The proof will be very similar to the proof of the previous vortex existing lemma. We will use the same notation for n_i, l, m, ϵ_i. We also use mathematical induction here. Suppose for some $i \in [l]$, we already get a $(\xi + \epsilon_{i-1}, \epsilon, n_{i-1})$-vortex. Let $U_i \subset U_{i-1}$ be a random subset of size n_i. Now we would like to show that

$$(1 - \xi - \epsilon_i)p^{|A|}|U_i| \leq |N_{U_i}(A)| \leq (1 + \xi + \epsilon_i)p^{|A|}|U_i|. \tag{8}$$

Let $X := \sum_{u \in U_i} X_u$, where X_u is the indicator random variable that takes the value 1 if u is in the common neighborhood of A. But by the inductive hypothesis, we know that $\mathbb{E}X = |U_i|\frac{|N_{U_{i-1}(A)}|}{|U_{i-1}|}$, which leads to

$$(1 - \xi - \epsilon_{i-1})p^{|A|}|U_i| \leq \mathbb{E}X \leq (1 + \xi + \epsilon_{i-1})p^{|A|}|U_i|.$$

Hence, in order for equation (7) to hold, we need to show

$$\mathbb{P}[|X - \mathbb{E}X| \geq t] \leq 1,$$

where $t = n^{-1/3}\epsilon^{-(i-1)/3}p^{|A|}|U_i|$. By the Chernoff inequality,

$$\mathbb{P}[|X - \mathbb{E}X| \geq t] \leq 2e^{-2t^2/|U_i|} \leq 1.$$

In fact, the basic step ($i = 1$) for the induction uses the same inequality, but instead of inductive hypothesis, we use the condition that G is $(\xi, 4, p)$-typical. Hence, the induction shows that we can finally get a $(\xi + \epsilon_l, \epsilon, m)$-vortex, and thus a $(\xi + \epsilon, \epsilon, m)$-vortex. $\square$

We also need to use the analogous cover down lemma:

Lemma 7.5 (Cover down lemma'). *Let $\xi, p \in [0,1]$ be given, and suppose $1/n \ll \epsilon$. Let G be a $(\xi, 4, p)$-typical graph on n vertices and $U \subset V(G)$ with $|U| = \lfloor \epsilon n \rfloor$. Suppose that for all $x \in V(G)$, $G[N_G(x) \cap U]$ is $(\sqrt{\xi}, 3, p)$-typical and assume that $d_G(x)$ is even for all $x \in V(G) - U$. Then there exists a K_3-decomposable subgraph H of G such that $G - G[U] \subset H$ and $\Delta(H[U]) \leq \epsilon^{10} n$.*

Note that in the proof of the cover down lemma (lemma 5.1), we use the definition of δ^{0+} to get an approximate decomposition first. However, in this setting, we do not have δ^{0+}. So we need to get some approximate decompositions using the typicality condition. A so-called nibble method turns out to be handy in this situation, the method was introduced by Rödl in [18]. The next lemma shows how to use the typicality condition to get an approximate decomposition:

Lemma 7.6. *Let $1/n \ll \xi, p$ and $\xi \leq p^7/20$. Then any $(\xi, 4, p)$-typical graph on n vertices contains a collection of triangles $\mathcal{T}$ such that: for every edge $e \in E(G)$, the number of triangles in $\mathcal{T}$ that contains e is between $(1 - n^{-1/3})p^2 n/4$ and $(1 + n^{-1/3})p^2 n/4$.*

After stating the vortex existing lemma and the cover down lemma, the proof of theorem 7.2 is the same as that of 1.13. In the first step, we obtain a vortex $U_l \subset U_{l-1} \subset ... \subset U_0$ whose existence is guaranteed by lemma 7.4. Then let $\mathcal{L}$ be the collection of all possible leftovers that are in $G[U_l]$ and let A be an absorber of it. Let $G' = G - A$, then the previous vortex is also a vortex of G'. We then apply the cover down lemma iteratively on the subsets U_i, at each step, the size of the leftover subgraph decreases with a factor of ϵ until finally we end up with only possible leftover edges in $G[U_l]$. By construction, these leftover edges can be absorbed by A and we finish the proof. Of course, the actual proof requires much more details, but the above outline is sufficient to illustrate the idea of iterative absorption again. For more details of the proof, please see [1].

8 Conclusion

In this essay, we briefly discuss the graph decomposition problem, in particular, the K_3-decomposition problem. Of all the methodologies that could be used for this problem, we introduce the iterative absorption method, which is especially handy when it comes to the decomposition problem of dense graphs. There are also some interesting applications of this method regarding combinatorial designs, for example in [10]. Back to the K_3-decomposition problem, we might also work with some other properties of the graph than minimal degree and typicality. For example, we can assume that every edge is in roughly the same number of triangles. Such "regularity" condition may also fall into the same framework as we are dealing with in this essay, that is, we may also use the iterative absorption method to tackle it. One can also generalize the problem further to hypergraphs, in [8], the iterative absorption method is used for the hypergraph case.

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