Algebraic geometry seminar

Uriel Sinichkin

March 11, 2018

1 Affine Varieties

1.1 Algebraic sets

Notation k is an algebraically closed field. $\boldsymbol{A}_k^n$ denotes the *affine n-space* which is the set of all n-tuples of elements of k. An element of $\boldsymbol{A}^n$ is called a *point*. $R = k\left[x_1, ..., x_n\right]$.

Definition Let $T \subseteq R$ be some subset. Then

$$Z(T) = V(T) = \{P \in \boldsymbol{A}^n | f(P) = 0 \text{ for all } f \in T\}$$

A subset of $\boldsymbol{A}^n$ that is equal to $V(T)$ for some $T \subseteq R$ is called algebraic or affine algebraic set.

Proposition

1. $V(T) = V(\langle T\rangle)$ where $\langle T\rangle$ is the ideal generated by T in R (From now on we are interested only in ideals).
2. If $I_1 \subseteq I_2 \subseteq R$ are ideals then $V(I_2) \subseteq V(I_1)$.
3. If $I_1, I_2 \subseteq R$ are ideals then $V(I_1 I_2) = V(I_1) \cup V(I_2)$.
4. If $I_\alpha \subseteq R$ is a collection of ideals then $V\left(\sum I_\alpha\right) = \bigcap V(I_\alpha)$.
5. The empty set and the whole set are algebraic.

Proof

1. If $f(P) = 0$ and $g(P) = 0$ then $(f+g)(P) = 0$ and $(hf)(P) = 0$ for all $h \in R$.
2. If $f(P) = 0$ for all $f \in I_2$ then in particular $f(P) = 0$ for all $f \in I_1$ so $P \in V(I_1)$.

3. Since $I_1 I_2 \subseteq I_1$ $V(I_1) \subseteq V(I_1 I_2)$ and similarly for I_2 so $V(I_1) \cup V(I_2) \subseteq V(I_1 I_2)$. In the other direction, if $P \in V(I_1 I_2)$ and $P \notin V(I_1)$ then exist $f \in I_1$ s.t. $f(P) \neq 0$ but then since $(fg)(P) = 0$ for all $g \in I_2$, $g(P) = 0$ for all such g so $P \in I_2$.

4. Since $I_\beta \subseteq \sum I_\alpha$ for all β, $V(\sum I_\alpha) \subseteq \bigcap V(I_\alpha)$. Conversely if $P \in \bigcap V(I_\alpha)$ and $f \in \sum I_\alpha$ then $f = \sum f_\alpha$ with $f_\alpha \in I_\alpha$ then $f(P) = \sum f_\alpha(P) = 0$ so $P \in V(\sum I_\alpha)$.

5. $\emptyset = V(1)$, $\boldsymbol{A}^n = V(0)$.

Corollary The algebraic sets form the closed sets of a topology which called the *Zariski topology on* $\boldsymbol{A}^n$.

Example The Zariski topology on $\boldsymbol{A}^1$ is the co-finite one. Indeed, since $k[x]$ is a PID, any algebraic set is cut out by one polynomial which have finite number of roots.

1.2 Ideal of a set and coordinate ring

Definition Let $Y \subseteq \boldsymbol{A}^n$ then

$$I(Y) = \{f \in R | f(P) = 0 \text{ for all } P \in Y\}$$

it is called the *ideal* of Y.

Proposition

1. $I(Y)$ is an ideal.

2. If $Y_1 \subseteq Y_2$ then $I(Y_2) \subseteq I(Y_1)$.

3. Let $Y_1, Y_2 \subseteq \boldsymbol{A}^n$ then $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

4. For any $Y \subseteq \boldsymbol{A}^n$ $V(I(Y)) = \overline{Y}$, the closure of Y with respect to the Zariski topology.

5. For an ideal $J \subseteq R$, $I(V(J)) = \sqrt{J}$ the radical of J.

Proof

1. Direct calculation.

2. The same as the analogous proposition for V.

3. $I(Y_1 \cup Y_2) \subseteq I(Y_1) \cap I(Y_2)$ follows from 2. Conversely, if $f \in I(Y_1) \cap I(Y_2)$ and $P \in Y_1 \cup Y_2$ then WLOG $P \in Y_1$ and thus $f(P) = 0$ since $f \in I(Y_1)$.

4. Obviously, $Y \subseteq V(I(Y))$ and since $V(I(Y))$ is closed, $\overline{Y} \subseteq V(I(Y))$. Let W be a set containing Y. Then $W = V(J)$ for some ideal J. Applying I we get
$$J \subseteq I(V(J)) \subseteq I(Y)$$
and using the fact V is inclusion reversing, $V(I(Y)) \subseteq V(J) = W$ so $V(I(Y)) = \overline{Y}$.

5. This is one of the formulations of Hilbert's Nullstellensatz.

Remarks

- We get that I, V form one-to-one inclusion reversing correspondence between closed subsets of $\boldsymbol{A}^n$ and radical ideals of R.
- If k is not algebraically closed, the last part of the proposition does not hold. For example, if $k = \mathbb{R}$, $V\left(x^2+1\right) = \emptyset$, $I\left(V\left(x^2+1\right)\right) = k[x]$ but $\sqrt{x^2+1} = \left(x^2+1\right)$.
- We get that $I(Y_1 \cap Y_2) = \sqrt{I(Y_1) + I(Y_2)}$. Indeed,
$$\sqrt{I(Y_1) + I(Y_2)} = IV(I(Y_1) + I(Y_2)) = I(VI(Y_1) + VI(Y_2)) = I(Y_1 \cap Y_2)$$

Example Let $n = 2$ $Y_1 = V\left(x_2 - x_1^2\right)$ and $Y_2 = V(x_2)$. Then the intersection $Y_1 \cap Y_2$ is the origin $(x_1, x_2) = (0,0)$ and $I(Y_1 \cap Y_2) = (x_1, x_2)$. On the other hand, $I(Y_1) + I(Y_2) = \left(x_2 - x_1^2\right) + (x_2) = \left(x_2, x_1^2\right)$.

Definition If Y is affine algebraic set then $A(Y) = \Gamma(Y) = {}^{R}/_{I(Y)}$ is called the coordinate ring of Y.

Remarks

- Essentially by the Nullstellensatz, $A(Y)$ is the ring of functions from Y to k which are restrictions of polynomial functions from $\boldsymbol{A}^n$ to k.
- $A(Y)$ is finitely generated, reduced k-algebra. Conversely, if B is finitely generate, reduced, k-algebra, $B \cong k[x_1, ..., x_n]/J$ for radical J and then $B \cong A(V(J))$.

1.3 Irreducibility

Definition A topological space X is called *irreducible* if it cannot be expressed as the union $X = Y_1 \cup Y_2$ of two closed proper subsets. A subset of a topological space is called irreducible if it is irreducible in the subspace topology.

Example $\boldsymbol{A}^1$ is irreducible since any proper subsets are finite but $\boldsymbol{A}^1$ is infinite.

Example $V(xy) \subseteq \boldsymbol{A}^2$ is not irreducible since it is the union $V(x) \cup V(y)$.

Proposition

1. X is irreducible iff every two non empty open subsets intersect.
2. X is irreducible iff every non empty open set is dense.
3. If $U \subseteq X$ is open subset of irreducible space X, U is irreducible.
4. If $Y \subseteq X$ is irreducible then $\overline{Y}$ is irreducible.

Proof

1. This is just taking complements on the definition.
2. This is a restatement of 1.
3. If $V \subseteq U$ is non empty and open, V is open in X so it is dense there so it is dense in U and thus U is irreducible.
4. If $U \subseteq \overline{Y}$ is open and non empty then $U \cap Y$ is open in Y so it is dense there and thus U is dense in $\overline{Y}$ meaning that $\overline{Y}$ is irreducible.

Definition An irreducible closed subset of $\boldsymbol{A}^n$ is called *affine variety*. An open subset of an affine variety is called *quasi-affine variety*.

Remark In some literature any affine algebraic set is called variety.

Proposition In the correspondence between closed subsets of $\boldsymbol{A}^n$ and radical ideals of R, prime ideals correspond to irreducible closed sets and maximal ideals correspond to points.

Proof Let Y be irreducible, and let $fg \in I(Y)$. Then $(Y \cap V(f)) \cup (Y \cap V(g)) = Y$ since Y is irreducible we can assume WLOG that $Y \cap V(f) = Y$ meaning that $Y \subseteq V(f)$ and so $f \in I(Y)$. Conversely, let $J \subseteq R$ be prime ideal and let $V(J) = Y_1 \cup Y_2$ where Y_1, Y_2 closed. Then

$$J = \sqrt{J} = I(V(J)) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$$

so either $J = I(Y_1)$ or $J = I(Y_2)$ and thus $V(J) = Y_1$ or $V(J) = Y_2$ meaning that $V(J)$ is irreducible.
If $P \in \boldsymbol{A}^n$ is a point then $I(P)$ is the kernel of the surjective morphism $R \to k$, $f \mapsto f(P)$ and so maximal. Conversely, if $J \subseteq R$ maximal, $V(J)$ is non empty by the Nullstellensatz, let $P \in V(J)$. Then $J \subseteq I(P)$ and so $J = I(P)$.

Corollary Y is irreducible iff $A(Y)$ is integral domain. Y is a point iff $A(Y)$ is a field.

Example $\boldsymbol{A}^n$ is irreducible since R is an integral domain.

Example If f is an irreducible polynomial in R. Then f generates a prime ideal (since R is an UFD) so $V(f)$ is an affine variety. It is called a curve if $n = 2$, a surface if $n = 3$ and hyper-surface if $n > 3$.

1.4 Noether condition

Definition A topological space X is called *Noetherian* if every descending chain of closed subsets stabilizes.

Example

- A subset of a noetherian space is noetherian. Indeed if $Y \supseteq Y_0 \supseteq Y_1 \supseteq ...$ is a descending chain of closed subsets of $Y \subseteq X$ then $\overline{Y_0} \supseteq \overline{Y_1} \supseteq ...$ is a descending chain of closed subsets of X so it stabilizes. Let n be s.t. $\overline{Y_m} = \overline{Y_n}$ for all $m > n$ then $Y_m = \overline{Y_m} \cap Y = \overline{Y_n} \cap Y = Y_n$ and so the original chain stabilizes.
- $\boldsymbol{A}^n$ is noetherian since R is noetherian ring. Indeed, let $Y_0 \supseteq Y_1 \supseteq ...$ be a descending chain of closed subsets of $\boldsymbol{A}^n$ then $I(Y_0) \subseteq I(Y_1) \subseteq ...$ is an ascending chain of ideals in R so it stabilizes and since $Y_i = V(I(Y_i))$ so is the original chain.
- Thus, every affine or pseudo-affine variety is noetherian.

Proposition In a noetherian topological space every closed subset can be expressed uniquely as a finite (possibly empty) union of irreducible closed subsets s.t. no one contained in the other.

Proof We will first show existence. Let $\mathcal{C}$ be the set of all closed subsets of X that cannot be represented in such a way. Since X is noetherian if $\mathcal{C}$ is not empty it has a minimal element Y. If Y is not empty since then it can be represented as the empty union. Y is not irreducible so exist closed $Y_1, Y_2 \subsetneq Y$ s.t. $Y = Y_1 \cup Y_2$. From the minimality of Y, $Y_1, Y_2 \notin \mathcal{C}$ so they can be written as finite unions of irreducible sets and thus so is Y which is a contradiction, thus $\mathcal{C}$ is empty. Now let $Y = Y_1 \cup ... \cup Y_r$ and $Y = Y_1' \cup ... \cup Y_{r'}'$ be 2 such representations. Then for each $1 \leq i \leq r'$ $Y_i' = (Y_i' \cap Y_1) \cup ... \cup (Y_i' \cap Y_r)$ so $Y_i' \subseteq Y_j$ for some j. Similarly, $Y_j \subseteq Y_l'$ for some l. But since for each $l \neq i$ $Y_i{}' \not\subseteq Y_l'$, $i = l$ and so $Y_i' = Y_j$.

Corollary Every affine algebraic set can be expressed uniquely as a union of varieties no one containing the other. They are called irreducible components.

Proposition A noetherian topological space is quasi-compact (every open cover has a finite sub-cover). In particular, any quasi-affine variety is quasi-compact.

Proof Let $\mathcal{U}$ be open cover of X. Since X is noetherian, $\{\bigcup\mathcal{U}_0 | \mathcal{U}_0 \subseteq \mathcal{U} \text{ finite}\}$ has a maximal element V. If $P \in X$ then exist $U_0 \in \mathcal{U}$ s.t. $P \in U_0$ and then $V \cup U_0$ is a finite union of elements in $\mathcal{U}$ so $U_0 \subseteq V$ and thus $P \in V$.

1.5 Dimension

Definition

- Let X be a topological space. The supremum of all integers n s.t. there exist a chain of strictly increasing irreducible closed subsets $Y_0 \subsetneq Y_1 \subsetneq ... \subsetneq Y_n$ is called the *dimension* of X. The dimension of an affine or quasi-affine variety is its dimension as topological space.
- If $I \subseteq R$ is a prime ideal in a ring R its *height* is defined as the supremum of n s.t. exist a chain of prime ideals $I_0 \subsetneq I_1 \subsetneq ... \subsetneq I_n = I$. The dimension (Krull dimension) of R is the supremum of heights of its prime ideals.

Proposition The dimension of an affine algebraic set Y is the dimension of $A(Y)$.

Proof The closed irreducible subsets of Y correspond to the prime ideals of R containing $I(Y)$ which correspond to the prime ideals of $A(Y)$.

Fact Let k be a field and let B be an integral domain which is finitely generated k-algebra. Then:

1. The dimension of B is equal to the transcendence degree of the field of fractions $K(B)$ over k.
2. For any prime ideal $J \subseteq B$

$$\textit{height}\, J + \dim {}^{B}\!/\!_{J} = \dim B$$

Proof

1. [Mil13, Theorem 18.17].
2. [Gat14, Lemma 11.6] together with [Gat14, Proposition 11.9].

Proposition $\dim \boldsymbol{A}^n = n$.

Proof

$$\dim \boldsymbol{A}^n = \dim k[x_1, ..., x_n] = tr.\,deg.\, {}^{k(x_1,...,x_n)}\!/\!_{k} = n$$

Proposition If Y is quasi-affine variety, $\dim Y = \dim \overline{Y}$.

Proof If $Y_0 \subsetneq Y_1 \subsetneq ... \subsetneq Y_n$ is a chain of irreducible closed subsets of Y then $\overline{Y_0} \subsetneq \overline{Y_1} \subsetneq ... \subsetneq \overline{Y_n}$ is such a chain in $\overline{Y}$ so $\dim Y \leq \dim \overline{Y}$. Thus $\dim Y$ is finite so we have a maximal chain of irreducible closed subsets $Y_0 \subsetneq Y_1 \subsetneq ... \subsetneq Y_n$ with $n = \dim Y$. Since the chain is maximal, Y_0 is a point P. There is a one-to-one correspondence between irreducible closet subsets of Y containing P and irreducible closed subsets of $\overline{Y}$ containing P given by

$$Z \mapsto \overline{Z}$$

$$Z' \cap Y \leftarrow\!\shortmid Z'$$

and thus the co-dimension of P in $\overline{Y}$ is n which means that $height\, I\,(P) = n$ in $A\left(\overline{Y}\right)$. Thus, since ${}^{A\left(\overline{Y}\right)}/_{I(P)} \cong k$,

$$\dim \overline{Y} = \dim A\left(\overline{Y}\right) = height\, I\,(P) + height\, {}^{A\left(\overline{Y}\right)}/_{I(P)} = n + 0 = n$$

Facts

1. Let A be a noetherian ring, and let $f \in A$ be an element which is neither zero divisor nor a unit. Then every minimal prime ideal containing f has height 1.

2. A noetherian integral domain A is a UFD iff every prime ideal of height 1 is principal.

Proof

1. [Mil13, Proposition 21.3].

2. One direction is [Mil13, Corollary 21.4]. The other direction (which is the one we will use) is the easier one and the proof will presented here. Let A be a noetherian UFD and let $I \subseteq A$ be a prime ideal of height 1. If $I = (0)$ it is of height 0 (and also principal) let $0 \neq a \in I$. Since A is an UFD, $a = p_1^{n_1} \cdot ... \cdot p_m^{n_m}$ for prime $p_1, ..., p_m$ and natural exponents $n_1, ..., n_m$. Since I is prime, one of the prime factors of a contained in it, assume WLOG that $p_1 \in I$. Then $(p_1) \subseteq I$ is prime ideal, so since I is of height 1, $I = (p)$ and so principal.

Proposition A variety Y in $\boldsymbol{A}^n$ has dimension $n-1$ iff it is the zero set $V\,(f)$ of a non constant irreducible polynomial in R.

Proof If f is an irreducible polynomial in R then (f) is prime so $V\,(f)$ is a variety and (f) has height 1 so $V\,(f)$ has dimension $n-1$. Conversely, if $Y \subseteq \boldsymbol{A}^n$ is a variety of dimension $n-1$ then $I\,(Y)$ has height 1 and since R is an UFD, $I\,(Y)$ is principal. Obviously the generator of $I\,(Y)$ has to be irreducible since $I\,(Y)$ is prime.

2 References

[Mil13] Milne, J. S. (2013). A Primer of Commutative Algebra. Freely available at http://jmilne.org/math/xnotes/CA.pdf.
[Gat14] Gathmann, A. (2014). Commutative Algebra, class notes. Freely available at http://www.mathematik.uni-kl.de/˜gathmann/class/commalg-2013/commalg-2013.pdf.