Algebraic geometry seminar

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Curves

Throughout this lecture C is a smooth curve.

Invertible sheaves on curves

Definitions

1. Let $P \in C$. Then $\mathcal{O}_{C,P}$ is a DVR, denote its valuation by $ord_P : k(C) \to \mathbb{Z}$.

2. Let $D = \sum_P n_P P$ be a divisor on C. Define $\mathcal{O}_C(D)$ to be the sheaf of $\mathcal{O}_C$-modules given by
$$\mathcal{O}_C(D)(U) = \{f \in k(X) \mid (\forall P \in U)(ord_P(f) + n_P \geq 0)\}$$

3. Ler $f \in k(C)$. Define $div(f) = \sum ord_P(f) \cdot P$. This is a divisor since it is a divisor if C is affine.

4. We will say 2 divisors D_1, D_2 are *linearly equivalent* ($D_1 \equiv D_2$) if exist $f \in k(C)$ s.t. $D_1 + div(f) = D_2$. This is an equivalence relation.

5. Let $\mathcal{L}$ be an invertible sheaf on C, $P \in C$ and $f \in \mathcal{L}(U)$ (not necessary $P \in U$). Define
$$ord_P(s) = ord_P(\psi(s))$$
where $\psi : \mathcal{L}|_V \to \mathcal{O}_C|_V$ for some $V \subseteq U$. This definition is independent of the choice of ψ, V. Indeed, let $\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{O}_C|_U$ for $i = 1, 2$. Then $\psi_1^{-1} \circ \psi_2 : \mathcal{O}_C|_{U_1 \cap U_2} \to \mathcal{O}_C|_{U_1 \cap U_2}$ is an isomorphism of $\mathcal{O}_C|_{U_1 \cap U_2}$ modules so $\psi_1(s)$ and $\psi_2(s)$ differ by multiplication by a unit, so $ord_P(\psi_1(s)) = ord_P(\psi_2(s))$. Simillarly to the structure sheaf, define
$$div(s) = \sum ord_P(s) \cdot P$$

Lemma Let $V \subseteq U \subseteq C$ open.

1. The retriction maps $res_{U,V} : \mathcal{O}_C(U) \to \mathcal{O}_C(V)$ are injective.
2. If $f \in \mathcal{O}_C(V)$ with $ord_P(f) \geq 0$ for all $P \in U \backslash V$ then f is in the image of $res_{U,V}$.
3. If $\mathcal{L}$ is an invertible sheaf of $\mathcal{O}_C$-modules and $f \in \mathcal{L}(V)$ with $ord_P(f) \geq 0$ for all $P \in U \backslash V$ then f is in the image of $res_{U,V}$.

Proof In a general ringed space $(X, \mathcal{O})$, for $f \in \mathcal{O}(U)$ the set $\{P \in X | f_P \in \mathcal{O}_P^*\}$ is open since if g_p is the inverse of f_P then exist a nieghborhood U of P s.t. $g_Q \cdot f_Q = 1$ for all $Q \in U$. Thus,

1. is true for any irreducible variety. Indeed, if $s \in \mathcal{O}_C(U)$ then $\{P \in U | s(P) = 0\} = \left\{P \in U | s_P \neq \mathcal{O}_{C,P}^*\right\}$ is closed so if it contains V it contains U (because V is dense in U).
2. This follows from
$$ord_P(f) = \min_{t^r f \in \mathcal{O}_{C,P}} r$$
3. Locally we can assume $\mathcal{L} \cong \mathcal{O}_C$ and then this is just 2. The local sections we would get will agree on V and thus can be glued together.

Theorem For a sheaf $\mathcal{F}$ on C TFAE:

1. $\mathcal{F} \cong \mathcal{O}(D)$ for a divisor D.
2. $\mathcal{F}$ is a subsheaf of $\mathcal{K}(C)$.
3. $\mathcal{F}$ is invertible.

And $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$ iff $D_1 \equiv D_2$.

Proof

$\underline{1 \Rightarrow 2}$: From definition of $\mathcal{O}(D)$.

$\underline{2 \Rightarrow 3}$: If $\mathcal{F}$ is fractional ideal and $P \in C$ then $\mathcal{F}_P \subseteq k(C)$ is a fractional ideal so $\mathcal{F}_P = t^d \mathcal{O}_{C,P}$. Taking $U \subseteq C$ s.t. $ord_Q(t) = 0$ for all $Q \in U \backslash \{P\}$ we get that $\mathcal{F}|_U = t^d \mathcal{O}_C|_U \cong \mathcal{O}_C|_U$ so $\mathcal{F}$ is invertible.

$\underline{3 \Rightarrow 1}$: Pick some non zero section $s \in \mathcal{F}(U)$ (U is arbitrary open subset of C). Let $D = div(s) = \sum n_P P$. We want to define an isomorphism $\varphi : \mathcal{O}(-D) \xrightarrow{\sim} \mathcal{F}$. Let $f \in \mathcal{O}(-D)(U')$ then $fs \in \mathcal{F}(U \cap U')$ has

$$ord_P(fs) = ord_P(f) + ord_P(s) = ord_P(f) + n_P \geq 0$$

for every $P \in U'$. So, by the lemma, there exist a unique $s' \in \mathcal{F}(U')$ with $s'|_{U \cap U'} = fs$, define $\varphi(f) = s'$. The easy verification that φ is a morphism of $\mathcal{O}_C$-modules is left as an exercise (one need to check that φ commutes with restriction maps and that $\varphi(U)$ is $\mathcal{O}_C(U)$-linear). We can check that it is an isomorphism on the level of stalks where φ_P is a non-constant morphism of rank 1 free $\mathcal{O}_{C,P}$-modules.

If $D_1 \equiv D_2$ then $D_1 + div(f) = D_2$ for some $f \in k(C)$ and then

$$\mathcal{O}_C(D_1) \to \mathcal{O}_C(D_2)$$

$$g \mapsto fg$$

is an isomorphism. Conversely, let $\varphi : \mathcal{O}_C(D_1) \xrightarrow{\sim} \mathcal{O}_C(D_2)$ and $f \in \mathcal{O}_C(D)(U)$ arbitrary non zero section. First, note that φ induces isomorphism on stalks and

$$\mathcal{O}_C(D)_P = t_P^{-ord_P(D)}\mathcal{O}_{C,P}$$

so

$$ord_P(f) + ord_P(D_1) = ord_P(\varphi(f)) + ord_P(D_2)$$

We can construct an isomorphism

$$\varphi' : \mathcal{O}_C \to \mathcal{O}_C(D_1 - D_2)$$

$$g \mapsto g\frac{f}{\varphi(f)}$$

where $g\frac{f}{\varphi(f)} \in \mathcal{O}_C(D_1 - D_2)(V)$ since

$$\begin{aligned} ord_P\left(g\frac{f}{\varphi(f)}\right) + ord_P(D_1 - D_2) &= \\ ord_P(g) + ord_P(f) - ord_P(\varphi(f)) + ord_P(D_1) - ord_P(D_2) &= \\ ord_P(g) + (ord_P(f) + ord_P(D_1)) - (ord_P(\varphi(f)) + ord_P(D_2)) &= \\ ord_P(g) &\geq 0 \end{aligned}$$

and this is an isomorphism since it induces isomorphism on stalks. Now, it is easy enough to see that $\mathcal{O}_C(D_1 - D_2) = \varphi'(1) \cdot \mathcal{O}_C$ so $D_1 - D_2 = div(\varphi'(1))$.

Principal parts and Cousin problem

Definitions

1. Let $\mathcal{F}$ be a coherent sheaf on C. Define $Rat(\mathcal{F}) = \varinjlim_{U \neq \emptyset} \mathcal{F}(U)$ and $\underline{Rat}(\mathcal{F})$ be the constant sheaf associated with $Rat(\mathcal{F})$.

2. Let $\mathcal{F}$ be an invertible (locally free coherent) sheaf on C and let $P \in C$. There is a natural embedding $\mathcal{F}_P \to Rat(\mathcal{F})$, define $Prin_P(\mathcal{F}) := {}^{Rat(\mathcal{F})}/_{\mathcal{F}_P}$ *the group of principal parts of* $\mathcal{F}$.

3. Define the sheaf $Prin(\mathcal{F})$ on C by

$$U \mapsto \bigoplus_{P \in U} Prin_P(\mathcal{F})$$

with obvious restriction maps.

Lemma There is an exact sequence of sheaves

$$0 \to \mathcal{F} \to \underline{Rat}(\mathcal{F}) \xrightarrow{\alpha} Prin(\mathcal{F}) \to 0$$

and $Rat(\mathcal{F})$, $Prin(\mathcal{F})$ are flabby.

Proof First, let us define α. For $f \in Rat(\mathcal{F})$ define $\alpha(U)(f) = \sum_{P \in U}(f \mod \mathcal{F}_P)$. This is well defined because $f \in \mathcal{F}_P$ on V, where $f \in \mathcal{F}(V)$ and $U \backslash V$ is finite. This is exact because it is exact on stalks. $\underline{Rat}(\mathcal{F}), Prin(\mathcal{F})$ are obviously flabby.

Definition Taking global sections of the above exact sequence we get the exact

$$0 \to \Gamma(\mathcal{F}) \to Rat(\mathcal{F}) \xrightarrow{\alpha(C)} \bigoplus_{P \in C} Prin(\mathcal{F})$$

a sheaf is called *ordinary* if $\alpha(C)$ is surjective.

Example

1. Every sheaf on an affine curve is ordinary.
2. The structure sheaf of a projective curve C is ordinary iff $C \cong \mathbb{P}^1$.

Proof

1. $\underline{Rat}(\mathcal{F})$ and $Prin(\mathcal{F})$ are quasi-coherent, so this is follows from the general fact about quasi-coherent sheaves on affine variety.
2. Let $C = \mathbb{P}^1$. It is sufficient to show that $Prin_P(\mathcal{O}_{\mathbb{P}^1}) \in im\left(\alpha\left(\mathbb{P}^1\right)\right)$ for every $P \in \mathbb{P}^1$ and by projective change of ccoordinates we can assume $P = [0:1]$. Let $s \in Prin_P(\mathcal{O}_{\mathbb{P}^1})$ and let $t = \frac{x_0}{x_1}$, then

$$s = \sum_{1 \le i \le n} a_i t^i \left(\mod \mathcal{O}_{\mathbb{P}^1, P}\right)$$

so $s = \alpha\left(\sum_{1 \le i \le n} a_i t^i\right)$.
Conversely, let C be a curve s.t. $\mathcal{O}_C$ is ordinary.

Introduction to cohomology of curves

Definition (cohomology - which definition?)

Example

Proposition (short exact sequence of sheves yields a long exact sequence in cohomology)

Theorem Let

$$0 \to \mathcal{F} \to \mathcal{F}_1 \to \dots$$

be a flabby resolution of $\mathcal{F}$. Then $H^i(C, \mathcal{F})$ is isomorphic to the i-th cohomology of the sequence of global sections

$$0 \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}_1) \to \dots$$

Proposition $H^1(C, \mathcal{F})$ and $H^0(C, \mathcal{F})$ are finite dimensional vector spaces over k.

Lemma

$$\begin{aligned} 0 \to \Gamma(\mathcal{O}_C(D)) \to \Gamma(\mathcal{O}_C(D+P)) \to k \to \\ \to H^1(\mathcal{O}_C(D)) \to H^1(\mathcal{O}_C(D+P)) \to 0 \end{aligned}$$

Fact (Serre duality)

Theorem (Riemann-Roch theorem)