Sheaves of Modules

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May 13, 2018

Some notation remarks: In this lecture, we will write $k[X]$ instead of $A(X)$, so it will be easier to distinguish from the much often used notation $\mathcal{A}(X)$, where $\mathcal{A}$ is a sheaf of modules. Also, for a sheaf $\mathcal{F}$, I will interchangeably use the notations $\Gamma(U, \mathcal{F})$ and $\mathcal{F}(U)$, which are the same. Finally, for an algebraic function f, $D(f)$ is the domain where f is nonzero.

1 Basic Definitions

Reminder 1. Let X be a topological space. A *sheaf of rings* $\mathcal{A}$ over X is a sheaf of X such that for any open $U \subseteq X$, $\mathcal{A}(U)$ is a ring, and the restrictions are ring homomorphisms.

Definition 1. Let $\mathcal{A}$ be a sheaf of rings over a topological space X. A *sheaf of $\mathcal{A}$-modules* $\mathcal{M}$ is an abelian sheaf such that for any open $U \subseteq X$ the set $\mathcal{M}(U)$ is an $\mathcal{A}(U)$-module, and restrictions satisfy multiplication - $(a \cdot m)|_V = a|_V \cdot m|_V$.

Example 1. $A^{\oplus I}$ for some set of indexes I, with multiplication and addition being the obvious ones and we additionally require $(\beta_i)_{i \in I}$ to locally only have finitely many non-zero entries. These are *free $\mathcal{A}$-modules*, and if $|I| < \infty$ then $|I|$ is called the *rank* of $\mathcal{A}^{\oplus I}$.

Definition 2. An $\mathcal{A}$-module $\mathcal{M}$ is *locally free* if there exists an open cover $X = \bigcup X_i$ such that $\mathcal{M}|_{X_i}$ is a free $\mathcal{A}|_{X_i}$-module. If all $\mathcal{M}|_{X_i}$'s have the same finite rank n, then $\mathcal{M}$ is said to have *rank n*. If $n = 1$ then $\mathcal{M}$ is said to be *invertible.*

Definition 3. An $\mathcal{A}$-module is said to be *quasi-coherent* if it is locally given by generators and relations - i.e., there is an open cover $X = \bigcup X_i$ such that for any i there is an exact relation of $\mathcal{A}|_{X_i}$-modules:

$$\mathcal{A}_{X_i}^{\oplus J} \to \mathcal{A}_{X_i}^{\oplus I} \to \mathcal{M}|_{X_i} \to 0$$

Definition 4. Now, let $\mathcal{A}$ be a sheaf of X, and let M be an $\mathcal{A}(X)$-module. We may define an $\mathcal{A}$-module $M \otimes_{\mathcal{A}(X)} \mathcal{A}$ by taking the sheaf associated to the presheaf given by $M \otimes_{\mathcal{A}(X)} \mathcal{A}(U)$ for any open $U \subseteq X$.

An exact sequence $M_1 \to M_2 \to M_3 \to 0$ gives us an exact sequence:

$$M_1 \otimes_{\mathcal{A}(X)} \mathcal{A} \to M_2 \otimes_{\mathcal{A}(X)} \mathcal{A} \to M_3 \otimes_{\mathcal{A}(X)} \mathcal{A} \to 0$$

Then, since on noetherian spaces we have $\mathcal{A}^{\bigoplus I}(U) = \bigoplus_{i \in I} A(U)$, we get that on a noetherian space we can write a quasicoherent sheaf of $\mathcal{A}$-modules as locally having the form $M_i \otimes_{\mathcal{A}(X_i)} (\mathcal{A}|_{X_i})$ for some $\mathcal{A}(X_i)$-module M_i.

2 Quasi-coherent Sheaves on Affine Varieties

Let M be a $k[X]$-module. Denote $M \otimes_{A(X)} \mathcal{O}_X$ as $\tilde{M}$

Proposition 1. *1. $M \cong \Gamma(X, \tilde{M})$*

2. For any $f \in k[X]$, $M_{(f)} \cong \Gamma(D(f), \tilde{M})$.

3. For any $x \in X$, $M_{n_x} \cong \left(\tilde{M}\right)_x$, where n_x is the ideal of vanishing function at x in $k[X]$.

Proof. 3. $\tilde{M}_x = \varinjlim_{x \in U} (M \otimes_{k[X]} \mathcal{O}_X(U)) = M \otimes_{k[X]} \mathcal{O}_{X,x} = M \otimes_{k[X]} k[X]_{n_x} = M_{n_x}$. (Recall $k[X]_{n_x} \cong \mathcal{O}_{X,x}$).

2. Use the homomorphism $\psi : M_{(f)} \to \Gamma(D(f), \tilde{M})$ given by $\psi(\frac{m}{f^i}) = m \otimes \frac{1}{f^i}$. Let us show that it is an isomorphism:

- Injectivity: Assume $\frac{m}{f^i}$ is in the kernel. Then $\frac{m}{f^i} \otimes 1 = 0$. So $f^l m = 0$ for some $l \in \mathbb{N}$. Therefore $m = 0$.
- Surjectivity: Let $m \otimes s \in \Gamma(D(f), \tilde{M})$. By definition, we can cover $D(f)$ with open sets $D(g_i)$, where $s = \frac{a_i}{g_i}$ in $D(g_i)$.
  Now, observe that $D(f)$ is covered by finitely many $D(g_i)$'s - $D(f) \subseteq \bigcup D(g_i)$ if and only if $V((f)) \supseteq \bigcap V((g_i)) = V\left(\sum (g_i)\right)$. By Hilbert's nullestensatz, this is equivalent to saying that $f \in \sqrt{\sum (g_i)}$, or equivalently $f^n \in \sum (g_i)$. So for some $n \in \mathbb{N}$, f^n can be expressed as a finite sum $f^n = \sum a_i g_i$, and thus only a finite number of g_i's will do.
  Now, for some i, j, there are two elements of $D(g_i) \cap D(g_j) = D(g_i g_j)$, namely, a_i/g_i and a_j/g_j, which s is equal to. Then, for some n:
  $$(g_i g_j)^n (g_j a_i - g_i a_j) = 0$$
  Since there are finitely many such i, j's, we may pick some n that is large enough to work for all. Then:
  $$g_j^{n+1}(g_i^n a_i) = g_i^{n+1}(g_j^n a_j)$$
  Then we may replace g_i by g_i^{n+1} and a_i by $g_i^n a_i$ and still get $s = a_i/g_i$ on $D(g_i)$, only now $g_j a_i = g_i a_j$ for all i, j. Now, write $f^n = \sum b_i g_i$. Denote $a = \sum b_i a_i$, so for each j we have:
  $$g_j a = \sum b_i a_i g_j = \sum b_i g_i a_j = f^n a_j$$
  So $s = a_j/g_j = a/f^n$ and therefore $m \otimes s = \psi\left(\frac{am}{f^n}\right)$, so we have surjectivity and injectivity, and therefore an isomorphism.

1. This comes directly from 2 by putting $f = 1$.

□

Corollary 1. *$\tilde{M}$ is quasi-coherent.*

Next, we wish to prove the converse - any quasi-coherent $\mathcal{O}_X$-module is of the form $\tilde{M}$ for some $k[X]$-module M. First we will need a few definitions:

Definition 5. Let $\mathcal{F}$ be a sheaf over a topological space X, and let U be an open subset of X. Then we define a new sheaf ${}_U\mathcal{F}$ by ${}_U\mathcal{F}(V) = \mathcal{F}(U \cap V)$.

Definition 6. Let X be a variety, and let f be a regular function on X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then we have a directed system:

$$\mathcal{F} = \frac{1}{1}\mathcal{F} \to \frac{1}{f}\mathcal{F} \to \dots \frac{1}{f^i}\mathcal{F} \to \dots$$

sending $\frac{1}{f^i}(\alpha)$ to $\frac{1}{f^{i+1}}(f\alpha)$. We denote $\mathcal{F}_{(f)} = \varinjlim \frac{1}{f^i}\mathcal{F}$.

Proposition 2. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on an affine variety. Then the following conditions are equivalent:*

1. *$\mathcal{F}$ is quasi-coherent.*
2. *For all $f \in k[X]$, there is an isomorphism $h : \mathcal{F}_{(f)} \cong {}_{D(f)}\mathcal{F}$.*
3. *$\mathcal{F} \cong \tilde{M}$ for some $k[X]$-module M.*

Proof. $3 \Rightarrow 1$ is what we proved earlier. We will prove, then, $1 \Rightarrow 2$ and $2 \Leftrightarrow 3$:

Assume 1. Then X has an open cover $\bigcup D(g_i)$, where $\mathcal{F}|_{D(g_i)} = M|_i \otimes_{k[D(g_i)]} \mathcal{O}_X(D(g_i))$, so $\mathcal{F}$ locally has the form of $\tilde{M}_i$ for some $k[D(g_i)]$ module M_i. Then if we prove $3 \Rightarrow 2$, we will have proved $1 \Rightarrow 2$. Assume 3, then.

Lemma 3. *If $U \subseteq X$ is open, then $\mathcal{F}_{(f)}(U) = (\mathcal{F}(U))_{(f)}$.*

Proof. For directed systems of abelian sheaves on a noetherian space we know that pre$-\varinjlim \mathcal{F}_i \cong \varinjlim \mathcal{F}_i$, and in our case $\mathcal{F}_{(f)}(U) = \varinjlim \left(\frac{1}{f^i}\mathcal{F}\right)(U) = \varinjlim \mathcal{F}_i(U)$ and $(\mathcal{F}(U))_{(f)} = \varinjlim \left(\frac{1}{f^i}\mathcal{F}(U)\right) = \text{pre} - \varinjlim \mathcal{F}_i(U)$. ☐

Now, let $D(g)$ be an open subset of X. Then:

$$\begin{aligned}\Gamma(D(g), \tilde{M}_{(f)}) = \Gamma(D(g), \tilde{M})_{(f)} = \left(\tilde{M}_{(g)}\right)_{(f)} = \tilde{M}_{(fg)} \\ = \Gamma(D(fg), \tilde{M}) = \Gamma(D(g) \cap D(f), \tilde{M}) = \Gamma(D(g), {}_{D(f)}\tilde{M})\end{aligned}$$

So we get an isomorphism between the two sheaves.
Now, let us prove $2 \Rightarrow 3$. Assume 2, and define $M = \Gamma(X, \mathcal{F})$. Then for any algebraic g over X, we have $\Gamma(D(g), \tilde{M}) = M_{(g)}$, but also:

$$\Gamma(D(g), \mathcal{F}) \underset{\substack{\uparrow \\ \text{By definition}}}{=} \Gamma(X, {}_{D(g)}\mathcal{F}) = \Gamma(X, \mathcal{F}_{(g)}) = \Gamma(X, \mathcal{F})_{(g)} = M_{(g)} = \Gamma(D(g), \tilde{M})$$

So we have an isomorphism between the two sheaves, proving $2 \Rightarrow 3$. ☐

Propositions 1 and 2 suggest that the functor $\sim$ from the category of $k[X]$-modules to the category of quasi-coherent sheaves on X is inverse to the functor $\Gamma(X, -)$ - so we have an equivalence of categories. We may also show exactness of this equivalence:

Proposition 4. 1. *The functor $\sim$ is exact.*

2. *The functor $\Gamma(X, -)$ is exact.*

Proof. 1. Assume an exact sequence of $k[X]$-modules $0 \to M_1 \to M_2 \to M_3 \to 0$. Then (by exactness of localization) all stalks at each point $x \in X$ give an exact sequence - $0 \to M_{1,n_x} \to M_{2,n_x} \to M_{3,n_x} \to 0$ is exact. Thus we get an exact sequence of abelian sheaves.

2. Assume $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of quasi-coherent $\mathcal{O}_X$-modules. To check right-exactness, otherwise we generally have an exact sequence of $k[X]$-modules:

$$\Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3) \to M \to 0$$

For some module M. But applying $\sim$ we get that $\tilde{M} = 0$, so $M = \Gamma(X, M) = 0$. Thus we get right-exactness, and left exactness is achieved in the same way.

□

3 Coherent Sheaves

Definition 7. Let $\mathcal{A}$ be a sheaf of rings on a topological space X. An $\mathcal{A}$-module $\mathcal{M}$ is called *coherent* if it locally has a presentation $\mathcal{A}|_U^{\oplus I} \to \mathcal{A}_U^{\oplus J} \to \mathcal{M}|_U \to 0$ with finite I, J.

If X is a variety, an $\mathcal{O}_X$-module $\mathcal{M}$ is coherent if it locally has the form $\tilde{M}$ for a finitely generated $k[U]$-module M, for an open subvariety U of X - this comes from what we proved earlier about quasi-coherent modules over affine varieties (and the fact that a variety has an open cover of affine varieties).

Lemma 5. *Let M be a $k[X]$-module where X is an affine variety. Then $\tilde{M}$ is coherent if and only if M is a finitely generated $k[X]$-module.*

Proof. If M is finitely generated, then clearly $\tilde{M}$ is coherent. Now, if $\tilde{M}$ is coherent, then we have a finite open cover $X = \bigcup D(f_i)$ for some regular f_i's, where $\tilde{M}|_{D(f_i)} = \tilde{N}_i$ for some finitely generated $k[X]_{(f_i)}$-modules N_i. Now, $N_i = \Gamma(D(f_i), \tilde{M}) = M_{(f_i)}$, so for each i we have a finite number of elements $M_{i,j} = \frac{m_{i,j}}{f_i^{p_{i,j}}} \in M_{(f_i)}$ that span N_i. Then define M_1 to be the submodule of M generated by the finitely many $m_{i,j}$. By construction, the inclusion $\alpha : \tilde{M_1} \to \tilde{M}$ is locally surjective, so we get that α is an isomorphism and hence $\Gamma(X, \alpha) : M_1 \to M$ is an isomorphism. □

Definition 8. Let X be a variety, and let $x \in X$. Define $\mathcal{F}|_x = \mathcal{F}/m_x\mathcal{F}_x$ (where m_x is the maximal ideal corresponding to x).
Let σ be a section of $\mathcal{F}$ over some neighborhood of x. Then the image of σ_x in $\mathcal{F}|_x$ is denoted by $\sigma(x)$.

Lemma 6. *If $\mathcal{F}$ is a coherent sheaf on a variety X, and $x \in X$, then $\mathcal{F}|_x = 0$ if and only if $\mathcal{F}|_U = 0$ for some neighborhood U of x.*

Proof. If $F|_U = 0$ for some neighborhood of x then the statement is clear. Otherwise (since this is a local statement) assume X is affine and then $\mathcal{F} = \tilde{M}$ for some finitely generated $k[X]$-module M. We claim $\mathcal{F}|_x = M/m_xM$ -

this is because M/m_xM is a $k[X]$ module on which $k[X]\backslash m_x$ acts invertibly, so $(M/m_xM)_{m_x} = M/m_xM$, but also $(M/m_xM)_{m_x} = M_{m_x}/m_xm_xM_{m_x} = \mathcal{F}_x/m_x\mathcal{F}_x$. Thus $M = m_xM$.
Recall Nakayama's lemma - if M is a finitely generated module over a ring A, and I is an ideal of A such that $IM = 0$, then there some $a \in A$ such that $(1+a)M = 0$. In our case, this means that there is a regular f that does not vanish on x such that $fM = 0$. Then taking $D(f)$, we get a neighborhood of x on which $\mathcal{F}|_{D(f)} = \tilde{M}_{(f)} = 0$. □

Corollary 2. *In the conditions of the lemma:*

1. *Let $\sigma_1, \ldots, \sigma_n$ be sections of $\mathcal{F}$. Then the homomorphism $\psi : \mathcal{O}_X^{\oplus n} \to \mathcal{F}$ is surjective given by sending e_i to σ_i is surjective if and only if $\sigma_1(x), \ldots, \sigma_n(x)$ span $\mathcal{F}|_x$.*

2. *The function $x \mapsto \dim_k(\mathcal{F}|_x)$ is upper semi-continuous - the subsets $\{x \in X \mid \dim_k(\mathcal{F}|_x) \geq m\}$ are closed.*

3. *This function has constant value m if and only if $\mathcal{F}$ is locally free of rank m.*

Proof.
1. Apply the lemma to the cokernel of ψ - $\operatorname{Cok}\psi|_x = (\mathcal{F}/\operatorname{Im}\psi)|_x = \mathcal{F}|_x/\operatorname{Im}\psi = \mathcal{F}|_x/Span(\sigma_i(x))$.

2. Let $n = \dim_k \mathcal{F}|_x$. We wish to show that the set $\{y \in X \mid \dim_k \mathcal{F}|_y \leq n\}$ contains a neighborhood of x. Choose, now, a basis $\sigma_1(x), \ldots, \sigma_n(x)$ which spans $\mathcal{F}|_x$ (where $\sigma_1, \ldots, \sigma_n$ are sections over a neighborhood of x). By 1, there is some neighborhood of x such that for each y in that neighborhood, $\sigma_1(y), \ldots, \sigma_n(y)$ span $\mathcal{F}|_y$, so $\dim \mathcal{F}|_y \leq n$. Thus we get 2.

3. If $\mathcal{F}$ is locally free of rank m, we clearly get what we wanted. In the other direction, let $\sigma_1, \ldots, \sigma_n$ be local sections near a point x such that $\sigma_1(x), \ldots, \sigma_n(x)$ is a basis for $\mathcal{F}|_x$. Then by 1 we have a surjection $\psi : \mathcal{O}_U^{\oplus m} \to \mathcal{F}|_x$ for some open neighborhood U of x. By our dimension assumption, $\psi|_y$ is an isomorphism for all $y \in U$. Now, let $(f_1, \ldots, f_m)$ be a section of $\ker\psi$. Then by definition $f_i(y) = 0$ for all i and for all $y \in U$, so we must have $f_i = 0$, so $\ker\psi = 0$, so ψ is an isomorphism and we get our claim.

□

4 Quasi-coherent Sheaves on Projective Varieties

Definition 9. Let m, n be integers, and U be open in $\mathbb{P}^n$. Then we define $\mathcal{O}_{\mathbb{P}^n}(m)(U)$ to be the set of regular functions on $\pi^{-1}U$ which are homogeneous of degree m ($\pi : \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ being the projection). Note that $\mathcal{O}_{\mathbb{P}^n}(0) = \mathcal{O}_{\mathbb{P}^n}$. We have homomorphisms $\mathcal{O}_{\mathbb{P}^n}(m_1) \otimes \mathcal{O}_{\mathbb{P}^n}(m_2) \to \mathcal{O}_{\mathbb{P}^n}(m_1 + m_2)$ given by $f \otimes g \mapsto fg$, which are isomorphisms. Notice that since $\mathcal{O}_{\mathbb{P}^n}(m)_{\{x_i \neq 0\}} = x_i^m \cdot \mathcal{O}_{\mathbb{P}^n}$, so $\mathcal{O}_{\mathbb{P}^n}(m)$ is invertible.

For closed subvariety of $\mathbb{P}^n$ X, we define $\mathcal{O}_X(m) = \mathcal{O}_{\mathbb{P}^n}(m)|_X$. generally, if $\mathcal{F}$ an $\mathcal{O}_X$-module, we define $\mathcal{F}(m) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$.

Now let X be a projective variety in $\mathbb{P}^n$, and let $C(X)$ be the cone over X. Then $S(X)$ is a graded ring, and let M be a graded module over $S(X)$ - we wish to define a sheaf $\tilde{M}$ over X.

Recall we have the old sheaf $\tilde{M}_{\mathbb{A}^n}$ over $C(X)$. Now, define for any open U in X $\tilde{M}(U) = (\tilde{M}(\pi^{-1}U))_0$ where π^{-1} is the lifting of U into $C(X)$. Then, by construction, we get that $\tilde{M}$ is quasi-coherent, and coherent if M is finitely generated.

Theorem 7. *Any quasi-coherent sheaf $\mathcal{F}$ on a projective variety X is of the form $\tilde{M}$. If $\mathcal{F}$ is coherent then it is of the form $\tilde{M}$ where M is a finitely generated $S(X)$-module.*

Proof. Let us have a quasi-coherent sheaf $\mathcal{F}$ on a projective variety X. We wish to construct a graded module over $S(X)$ that will satisfy $\tilde{M} = \mathcal{F}$. Consider:

$$M = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$$

We claim that $\tilde{M} \cong \mathcal{F}$. By the equivalence we saw on affine varieties we know that $S(X) \cong \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$ - so M is a graded module over $S(X)$.
Now, we will prove the statement locally. Let f be a homogeneous regular function of $C(X)$ of degree d. Since $(M_{(f)})_0$ is built by elements $\frac{\alpha}{f^i}$ for $\alpha \in \Gamma(X, \mathcal{F}(id))$, we get:

$$\begin{aligned}
\Gamma(D(f), \tilde{M}) = (M_{(f)})_0 &= \varinjlim \left(\Gamma(X, \mathcal{F}) \xrightarrow{f} \frac{1}{f}\Gamma(X, \mathcal{F}(d)) \xrightarrow{f} \dots \frac{1}{f^i}\Gamma(X, \mathcal{F}(id)) \xrightarrow{f} \dots \right) \\
&= \Gamma\left(X, \varinjlim \left(\mathcal{F} \to \frac{1}{f}\mathcal{F}(d) \to \dots \right) \right) \\
&= \Gamma\left(X, {}_{D(f)}\mathcal{F} \right) = \Gamma(D(f), \mathcal{F})
\end{aligned}$$

So indeed $\mathcal{F} \cong \tilde{M}$.
Now, if $\mathcal{F}$ is coherent then $\mathcal{F} = \bigcup \tilde{M}_i$ where the M_i's are finitely generated graded $S(X)$-submodules of M. Since $\tilde{M}_i + \tilde{M}_j = (M_i \tilde{+} M_j)$ (and the fact that $\mathcal{F}$ is coherent, and thus is locally $\tilde{M}$ with M finitely generated module on a coordinate ring of an open affine variety), we get by the ascending chain condition (as X is compact) that $\mathcal{F}$ must equal some $\tilde{M}_i$, finishing our proof. □

Corollary 3. *If $\mathcal{F}$ is a coherent sheaf on a projective variety X then there exists some n_0 such that we have a surjection $\mathcal{O}_X^{\oplus m} \to \mathcal{F}(n)$ for any finite m and any $n \geq n_0$.*

Proof. Just take n_0 to be the maximal number of generators of M, and the statement is obvious. □