Non-singular varieties.

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Definition 1:

Let $Y \subseteq \boldsymbol{A}^n$ be an affine variety and $(f_1, \dots, f_n)$ be the generator set of $I(Y)$. Then Y is called **non-singular in point** $P \in Y$ if

$$rank\, ||{}^{\partial f_i}/_{\partial x_j}(P)|| = n - r$$

where $r = \dim Y$.

Y is called **non-singular** if it is non-singular in every point.

Note: The formal derivatives of polynomials are well defined.

The matrix of partial derivatives is called **Jacobian matrix**.

Now we have defined non-singular affine variety but not general variety. For this we will need the following definition and theorem.

Definition 2:

Let A be Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = {}^A/_{\mathfrak{m}}$.

A is called **regular local ring** if $\dim_k {}^{\mathfrak{m}}/_{\mathfrak{m}^2} = \dim A$.

Theorem 1:

Let $Y \subseteq \boldsymbol{A}^n$ be an affine variety and $P \in Y$ be some point.

Y is non-singular at P iff the local ring $\mathcal{O}_{P,Y}$ is regular local ring.

Proof: Let $P = (a_1, \dots, a_n) \in \boldsymbol{A}^n$ be some point. Look at the ideal $\mathfrak{a}_P = (x_1 - a_1, \dots, x_n - a_n)$ in $A = k[x_1, \dots, x_n]$ it is a maximal ideal since it ideal of one point. Define a linear map $\theta : A \longrightarrow k^n$ s.t.

$$f \mapsto \left\langle \frac{\partial f}{\partial x_1}(P), \dots, \frac{\partial f}{\partial x_n}(P) \right\rangle$$

Notice that $\theta(x_i - a_i)$ form basis of k^n and then the restriction $\theta' : {}^{\mathfrak{a}_P}/_{\mathfrak{a}_P^2} \longrightarrow k^n$ is an isomorphism.

Now let $\mathfrak{b} = I(Y) \subseteq A$ with generators $(f_1, \dots, f_t)$. Notice that

$$rank J = rank\, ||{}^{\partial f_i}/_{\partial x_j}(P)|| = \dim(\theta(\mathfrak{b}))$$

But we have and isomorphism θ so we can say that

$$\dim\left(\theta\left(\mathfrak{b}\right)\right) = \dim\left({}^{\mathfrak{b}+\mathfrak{a}_P^2}/_{\mathfrak{a}_P^2}\right)$$

Remember how did we obtain $\mathcal{O}_{Y,P}$ we took quotient of A with $\mathfrak{b}$ and then localized at $\mathfrak{a}_P$. Then if $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{Y,P}$ it holds:

$$^{\mathfrak{m}}/_{\mathfrak{m}^2} \cong {}^{\mathfrak{a}_P}/_{\left(\mathfrak{b}+\mathfrak{a}_P^2\right)}$$

And thus we can see $\dim {}^{\mathfrak{m}}/_{\mathfrak{m}^2} + \text{rank} J = n$. For a theorem with a lot of statement (around 4). $\dim \mathcal{O}_P = \dim Y := r$. Hence $\dim_k {}^{\mathfrak{m}}/_{\mathfrak{m}^2} = r$ iff $\mathcal{O}_P$ is regular local ring. But is is essentially the same as $\text{rank} J = n - r$ which defines Y to be non-singular at P.

□

Example 1 multiplicities:

Let $Y \subseteq \boldsymbol{A}^2$ be a curve defined by equation $f(x,y) = 0$ and $P = (a,b) \in \boldsymbol{A}^2$ be some point. Change the coordinates so $P = (0,0)$. Rewrite f as sum of homogeneous polynomials in x and y, i.e. $f = \sum_{i=1}^{d} f_i$. Then define **multiplicity of** P **on** Y to be the smallest r s.t. $f_r \neq 0$ denoted by $\mu_P(Y)$. The linear factors of f_r are called **tangent directions** at P.

Now if $\mu_P(Y) = 1$ then $\text{rank} J = 1 = 2 - 1$ and then Y is not singular at P. Otherwise J is a null matrix and Y is singular at P.

1. $x^2 = x^4 + y^4$ is a tacnode as sum of homogeneous it is

   $$-x^2 + \left(x^4 + y^4\right) = f_2 + f_4 = 0$$

   and then multiplicity of $(0,0)$ is 2. Tangent direction is $x/$

2. $xy = x^6 + y^6$ is a node and as some of homogeneous polynomials:

   $$-xy + \left(x^6 + y^6\right) = f_2 + f_6 = 0$$

   the multiplicity of $(0,0)$ is 2. Tangent directions are x and y.

3. $x^3 = y^2 + x^4 + y^4$ - cusp same multiplicity. Tangent direction is y.

4. $x^2y + xy^2 = x^4 + y^4$ - triple point the multiplicity is 3. Tangent directions are x and y.

All of them have one singular point which can be shown by direct calculation.

The form are determined from tangent directions (where does curve go).

Definition 3:

Let Y be a variety and $P \in Y$ some point. Then Y is called **non-singular at point** P if $\mathcal{O}_{P.Y}$ is regular local ring.

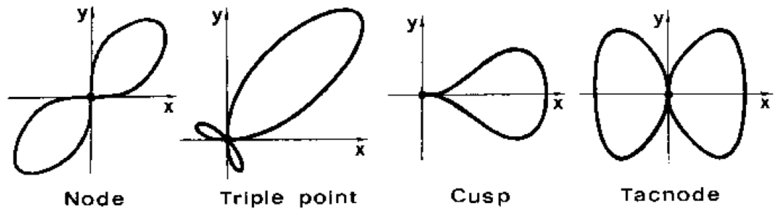

Figure 1: Singularities of plane curves

Theorem 2(algebra):

If A is Noetherian local ring with maximal ideal $\mathfrak{m}$ and quotient field k, then $\dim_k {}^{\mathfrak{m}}/_{\mathfrak{m}^2} \geq A$.

Proof: See see Atityah-Macdonald 11.15

Theorem 3:

Let Y be a variety. Denote Sing$Y$ the set of all singular points of Y. Then Sing$Y$ is proper closed subset of Y.

Proof:

Let $Y = \bigcup Y_i$ be an open covering of Y. Then we are first of all to show that SingY_i is closed for each i. By theorem[1] we may assume Y is affine variety. Then by previous theorems we have rank$J \leq n - r$ for every point $P \in Y$. Hence for any singular point on Y holds rank$J < n - r$. Thus we have an algebraic definition of Sing$Y$ as the set generated by $I(Y)$ with all determinants[2] of $(n-r) \times (n-r)$ submatricies of the matrix J. Hence Sing$Y$ is closed.

Now apply theorem [3] to Y to obtain a birational equivalence to $\boldsymbol{P}^n$. Note that birational varieties have isomorphic typologies we can reduce to the case of hypersurface in $\boldsymbol{P}^n$. It is enough to consider any open affine set of Y, so we may assume that is a hypersurface in $\boldsymbol{A}^n$, defined by a single polynomial $f(x_1, \ldots, x_n) = 0$.

Sing$Y$ is the set of points in Y s.t. ${}^{\partial f}/_{\partial x_i}(P) = 0$ for all $i \leq n$. If $Y = $ Sing$Y$ then ${}^{\partial f}/_{\partial x_i} \in I(Y)$ but $I(Y) = \langle f \rangle$ and hence $\deg({}^{\partial f}/_{\partial x_i}) \leq \deg f - 1$ but this means ${}^{\partial f}/_{\partial x_i} = 0$ for every $0 \leq i \leq n$.

[1] 4.3 On any variety Y, there is a base for the topology consisting of open affine subsets.

[2] We take all the polynomial which show existence of singularity and all the polynomial which define Y and they all together Sing$Y$ and then we have algebraic set.

[3] 4.9 Any variety X of dimension r is birational to a hypersurface Y in $\boldsymbol{P}^{r+1}$.

If char$k = 0$ then we already have a contradiction. Otherwise we conclude that f is a polynomial in $x_i^{p^k}$ for each i. But then we can take p-th root of the coefficients to get polynomial g s.t. $f = g^p$ but f is irreducible.

Hence Sing$Y < Y$.

$\square$

Blowing up singularities:

Cusp: Let Y be the cusp $x^3 = y^2 + x^4 + y^4$. Show that $\tilde{Y}$ is non singular at $O = (0, 0)$.

Let t, u be homogeneous coordinates on $\boldsymbol{P}^1$. Then X blowing-up of $\boldsymbol{A}^2$ at O is defined by the equation $xu = ty$ inside $\boldsymbol{A}^2 \times \boldsymbol{P}^1$. O is replaced by $\boldsymbol{P}^1$ and the rest looks as $\boldsymbol{A}^2$. Denote $E := \boldsymbol{P}^1$.

Consider equations $x^3 = y^2 + x^4 + y^4$ and $xu = ty$. Now E is covered by open sets $t \neq 0$ and $u \neq 0$.

If $t \neq 0$ denote $t = 1$ and then we get

$$x^3 = y^2 + x^4 + y^4$$

$$y = xu$$

In $\boldsymbol{A}^3$ with coordinates x, y, u substitute equations and get

$$-x^3 + x^2u^2 + x^4 + x^4u^4 = x^2\left(u^2 - x + x^2\left(1 + u^4\right)\right) = 0$$

Then we have two irreducible components one defined by $x = 0 = y$ with any u which is E and the other $u^2 - x + x^2\left(1 + u^4\right) = 0$ and $y = xu$ this is $\tilde{Y}$. Note $\tilde{Y}$ meets E at one point $u = 0$.

Case $u = 1$. Then $ty = x$ and the second equation of the curve

$$t^3y^3 = y^2 + y^4t^4 + y^4 = y^3\left(1 - yt^3 + y^4\left(1 + t^4\right)\right) = 0$$

And thus there is no intersection.

$\square$

Node: As we saw previously the node is a double point of a plane curve with two distinct tangent directions. Show that if P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular point on the blown-up curve $\tilde{Y}$.

First of all make a linear change of coordinates s.t. $P \mapsto O$. Then we can write $f = f_r + \cdots + f_d$ since the multiplicity of P was 2 then $r = 2$. And f_2 factors into x and y. So $f = xy + g$ with $\deg g \geq 3$. Consider the pullback of $V(f)$ in the blow-up X in two parts $t \neq 0$ and $u \neq 0$. If $t = 1$ then $f = 0$ and $y = ux$ so substitute

$$ux^2 + g(x, ux) = 0$$

But again $x^2 \mid g(x, ux)$ because $\deg g \geq 3$. The f in this form factors as

$$x^2 \left(u + {}^{g(x,ux)}/_{x^2}\right) = 0$$

Then again $x^2 = 0$ with any u gives us E. And $u + {}^{g(x,ux)}/_{x^2}$ gives us $\tilde{X}$. If $x = 0$ then $u = 0$ hence this cover meets E in one point $x = u = 0$. Multiplicity of the curve is 1 and hence there is no singularity.

The same analysis for $u \neq 0$ gives us the second non-singular point.

$\square$

Tacnode: The same as previous: $x^2 = x^4 + y^4$ the blow up of $\boldsymbol{A}^2$ is obviously the same. $f = x^2 - \left(x^4 + y^4\right)$, notice x^2 factors into x and x. The E is covered by two open sets $t \neq 0$ and $u \neq 0$ and $x^2 - \left(x^4 + y^4\right) = 0$ and $xu = ty$. If $t \neq 0$ then $xu = y$ and substitute:

$$x^2 - x^4\left(1 + u^4\right) = 0$$

As previously we can factor $x^2\left(1 - x^2\left(1 + u^4\right)\right) = 0$. Same as above $x = 0$ with any u gives us E. And $\tilde{Y}$ is

$$\begin{cases} y = ux \\ x^2\left(1 + u^4\right) = 1 \end{cases}$$

Which has no intersections with E.

Now let $u = 1$. Then $x = yt$. Substitute

$$y^2t^2 - y^4\left(1 + t^4\right) = 0 \Rightarrow$$

$$\Rightarrow y^2\left(t^2 - y^2\left(1 + t^4\right)\right) = 0$$

Then as above E is $x = y = 0$ with arbitrary t and $\tilde{Y}$ is

$$\begin{cases} x = ty \\ t^2 - y^2 = y^2t^4 \end{cases}$$

The intersection with E is $t = 0 = x = y$. Which is the singular point of $t^2 - y^2 = y^2t^4$ which is a node since the $\tilde{f}_2 = (t - y)(t + y)$.

And then one more blow-up reserves the singularity.

$\square$

$\boldsymbol{y^3 = x^5}$ Obvious that $r = 3$ and hence the point $(0, 0)$ is a triple point. The same as above $xu = yt$.

$t = 1$ then $y = xu$ and the second curve equation is $x^3\left(u^3 - x^2\right) = 0$. E is $x = 0 = y$ for any u. And $\tilde{Y} \cap E$ is $x = y = u = 0$. The resulting form is $x^2 = u^3$ which is cusp.

$u = 1$ then $x = ty$ and the second curve equation is $y^3\left(1 + y^2t^5\right) = 0$. E is $y = 0 = x$ for any t. And there is no intersection.

$\square$

Completion:

For clarification and proofs see Atityah-Macdonald Chapter 10&11):

There are to cases of completion that we probably already seen:

1. Construction of formal power series from polynomials.

2. Formation of p-adic numbers. Take a prime number p and then work in various rings $\mathbb{Z}/p^k\mathbb{Z}$. In some sense it is a limit of $\mathbb{Z}/p^k\mathbb{Z}$ as $n \to \infty$.

The two examples are close but p-adic numbers does not have a natural embedding of $\mathbb{Z}/p^k\mathbb{Z}$.

Completion as localization a method of simplifying around concrete point. We will usually perform completion after localization, e.x. local ring of a non-singular point on n dimensional variety. has always for its completion the ring of formal power series in n variables. So we use completion after localization when we need more simplification.

Also completion preserves exactness and Noetherian property.

Definitions 4 and 5:

Let G be a topological Abelian group, i.e. topological space with addiction and substraction defined and continuous. Let $a \in G$ be a fixed point, define translation $T_a : G \longrightarrow G$ with $x \mapsto x + a$ it is homeomorphism of G since it is continuous and it's inverse T_{-a} is continuous. We see that U is any neighborhood of 0 in G then $U + a$ is a neighborhood of a in G and every neighbourhood of a can be represented in this form.

Assume for simplicity that 0 has a countable fundamental system of neighborhoods. Then completion $\hat{G}$ can be defined through Cauchy sequences[4]On the set of Cauchy sequences we define equivalence classes by

$$x_n \sim y_n \iff x_n - y_n \longrightarrow 0$$

And then $\hat{G}$ is exactly the set of all the equivalence classes. It is an Abelian group. Also for $x \in G$ take $(x) \in \hat{G}$ and then we have homomorphism of Abelian groups (which is injective whenever G is Hausdorff, because $\ker \phi = \bigcap U$ for U neighbourhood of 0).

Purely algebraic definition: Let's look at less general case: $0 \in G$ has a fundamental system of neighborhoods consisting of subgroups. Then we have a sequence

$$G = G_0 \supseteq G_1 \supseteq \dots$$

And then U is a neighbourhood of 0 iff it contains some G_n, e.x. $G_n = p^n\mathbb{Z}$. Note that in this case G_n's are clopen.

Now consider $\{A_n\}$ be a sequence of groups and and family of homomorphisms $\theta_{ij} : A_j \longrightarrow A_i$ for $i \leq j$. We call this an **inverse system** if

[4] $(x_n) \subseteq G$ s.t. for any U neighbourhood of 0 there exists natural N s.t. $x_n - x_m \in U$ for all $m, n \geq N$.

- $\theta_{ii} = id$
- $\theta_{ik} = \theta_{ij} \circ \theta_{jk}$ for $i \leq j \leq k$

Then define

$$A = \lim_{\leftarrow} A_i = \left\{ \overrightarrow{a} \in \prod_{i=1}^{\infty} A_i : a_i = \theta_{ij}(a_j) \text{ for all } i \leq j \in \mathbb{N} \right\}$$

Using this definition of the **inverse limit** we can say

$$\hat{G} = \lim_{\leftarrow} {}^{G}/_{G_n}$$

Is the **completion** of G.

By the way we see that the inverse limit is just a subspace of the product equipped with induced topology from the product topology.

Maybe: Let (x_n) be a Cauchy sequence in G. Then the image of (x_k) in ${}^{G}/_{G_n}$ in finely constant equal to ε_n. If we pass from $n+1$ to n then $\varepsilon_{n+1} \mapsto \varepsilon_n$ under ${}^{G}/_{G_{n+1}} \xrightarrow{\theta_{n+1}} {}^{G}/_{G_n}$.

Thus the Cauchy sequence (x_k) defines a **coherent sequence** (ε_n) s.t. $\theta_{n+1}\varepsilon_{n+1} = \varepsilon_n$.

We see that equivalent Cauchy sequences define the same coherent sequence. Also we construct Cauchy sequence from the coherent sequence: x_n can be any element in the coset of ε_n s.t. $x_{n+1} - x_n \in G_n$. Then the **completion** $\hat{G}$ can be defined as the set of of the coherent sequences with the obvious group structure.

$\mathfrak{a}$-adic completions: Let $A = G$ be a ring and $G_n = \mathfrak{a}^n$ for an ideal $\mathfrak{a}$. Then the topology defined on A is called **$\mathfrak{a}$-adic topology** and the completion is called **$\mathfrak{a}$-adic completion**. Easy to see that with this topology A is a topological ring and Hausdorff (since the closure of 0 is 0). The completion $\hat{A}$ of A is again a topological ring and $\phi : A \longrightarrow \hat{A}$ is a continuous ring homomorphism with kernel $\bigcap \mathfrak{a}^n$.

The same goes for an A-module M, take $M = G$ and $\mathfrak{a}^n M = G_n$. This defines the $\mathfrak{a}$-adic topology on M and the completion $\hat{M}$ of M is a topological $\hat{A}$-module. If $f : M \longrightarrow N$ is any A-module then $f(\mathfrak{a}^n M) = \mathfrak{a}^n f(M) \subseteq \mathfrak{a}^n N$ and then f is continuous and so $\hat{f} : \hat{M} \longrightarrow \hat{N}$ is defined.

Theorem 4(algebra):

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let $\hat{A}$ be its completion.

1. $\hat{A}$ is a local ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$ and there is natural injective homomorphism $A \to \hat{A}$.
2. If M is finitely generated A-module its completion $\hat{M} \cong M \otimes_A \hat{A}$ with respect to its $\mathfrak{m}$-adic topology.

3. $\dim A = \dim \hat{A}$.

4. A is regular iff $\hat{A}$ is regular.

Theorem 5(algebra):

If A is a complete[5] regular local ring of dimension n containing some field, then $A \cong k\left[[x_1, \ldots, x_n]\right]$ the ring of formal power series over the residue field k of A.

Definition 7:

Two points $P \in X$ and $Q \in Y$ are **analytically isomorphic** if there is an isomorphism $\hat{\mathcal{O}}_P \cong \hat{\mathcal{O}}_Q$ as k-algebras.

Example 3:

If X and Y have analytically isomorphic points then $\dim X = \dim Y$. See 5.4.c and ex 3.12 which tells us the local ring of a point on a variety has the same dimension as the variety.

Example 4:

If $P \in X$ and $Q \in Y$ are nonsingular points on varieties of the same dimension, then P and Q are analytically isomorphic. This follows from 5.4 and 5.5. And now we finally have the same kind of locality as in differential manifolds, i.e. we know that two differential manifolds of the same dimension are locally isomorphic.

Example 5:

char$k \neq 2$

Let X be the plane nodal cubic curve given by the equation $y^2 = x^2(x+1)$. Let Y be the algebraic set in $\boldsymbol{A}^2$ defined by the equation $xy = 0$. We will show that the point $O = (0,0)$ on X is analytically isomorphic to the point O on Y. Notice $\mathcal{O}_{O,Y} = \left({}^{k[x,y]}/_{(xy)} \right)_{(x,y)}$ and then $\hat{\mathcal{O}}_{O,Y} = {}^{k[[x,y]]}/_{(xy)}$.

This example corresponds to the geometric fact that near O, X looks like two lines crossing.

Notice that $\hat{\mathcal{O}}_{O,X} \cong {}^{k[[x,y]]}/_{(y^2-x^2-x^3)}$. The leading form (i.e. the first homogeneous polynomial) is $y^2 - x^2$ it factorizes as $(y-x)(x+y)$. Now we want to find power series s.t. their multiplication is f. We want all the polynomials h_i, g_i to be homogeneous:

$$g = y + x + g_2 + g_3 + \ldots$$

$$h = y - x + h_2 + h_3 + \ldots$$

[5] if $\phi : G \longrightarrow \hat{G}$ is an isomorphism then G is called **complete**.

Construct them by induction:

$$(y-x)\,g_2 + (y+x)\,h_2 = -x^3$$

This is a legitimate construction since $y-x$ and $y+x$ generate a maximal ideal in $k\,[[x,y]]$. Now construct g_{n+1} and h_{n+1} s.t.

$$(y-x)\,g_{n+1} + (y+x)\,h_{n+1} = -g_n h_n$$

This is again possible.

Thus $\hat{\mathcal{O}}_{O,X} = {}^{k[[x,y]]}/_{(gh)}$ but g and h begin with linearly independent linear term and thus there is an automorphism of $k\,[[x,y]]$ s.t. $g,h \mapsto x,y$. Thus $\hat{\mathcal{O}}_{O,X} = {}^{k[[x,y]]}/_{(xy)}$ as requested.

□

One more blow-up:

Let $Y \subseteq \boldsymbol{P}^2$ be a non-singular plane curve of degree > 1, denote d, defined by equation $f\,(x,y,z) = 0$. Let $X \subseteq \boldsymbol{A}^3$ be the affine variety defined by f (this is the cone over Y). Let P be the point $(0,0,0)$ which is the vertex of the cone. Let $\varphi : \tilde{X} \longrightarrow X$ be the blowing up of X at P.

- Are there any singularities on X?

Remark: If $f\,(x_0,\ldots,x_n)$ is a homogeneous polynomial in $k\,[x_1,\ldots,x_n]$ then

$$\frac{\partial f}{\partial x_i}\,(x_0,\ldots,x_j=1,\ldots,x_n) = \frac{\partial}{\partial x_i}\,(f\,(x_0,\ldots,x_j=1,\ldots,x_n))$$

From the remark obvious that $X\setminus\{0\}$ is non-singular. And $\deg f > 1$ gives us O is singular.

- Does one blow-up resolve them?
  Look at $\tilde{X} \cap E$ all the possible singularities are there. Let $(x,y,z,t:u:v)$ the coordinates on $\boldsymbol{A}^3 \times \boldsymbol{P}^2$. The equations $\tilde{X}$ are $f=0$ and $xu = yt, xv = tz$ and $yv = uz$.
  Then if $t=1$ holds $f\,(x,xu,xv)=0$ then we can extract x^d i.e. $x^d f\,(1,u,v)$. Now $x^d = 0$ gives us E and $f\,(1,u,v)$ gives the strict transformation. Note that if the point $(x,u,v) = (0,u,v)$ satisfies $f\,(1,u,v)=0$ then it is not singular since f is singular only at P. The same goes for $u=1$ and $v=1$.

- The affine cover is $f\,(1,u,v)\,, f\,(t,1,v)$ and $f\,(t,u,1)$

□