The Picard Group

And some more about sheaves

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Outline

Coherence
- The story
- Reminders
- Locally Free
- Quasi-Coherence
- Coherence

The Picard Group
- Invertible sheaves
- The Picard group
- Divisors
- Examples

Bird's view

We want to define invariants for algebraic varieties, to be able to test whether varieties are isomorphic. To do that we will study the intrinsic geometry of a variety; In this lecture we will study the line bundles of the variety, that give rise to an invariant called the **Picard group**. To define it for a genral variety we define coherent modules. In the next lecture we shall define a **differential form** on an algebraic variety, and use it to give an intrinsic definition of the tangent and cotangent bundles on a variety. Then constructions from differential geometry will lead to additional numeric invariants. The Picard group will also later be seen to be a particular case of a cohomology group.

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The story

X an affine variety, $A = k[X]$.
For every A-module M, attach a module $\tilde{M}$ over the sheaf of rings $\mathscr{O}_x$. This defines an equivalence of the category of A-modules with the category of **quasi-coherent** $\mathscr{O}_x$-modules.

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Reminders and philosophical addenda

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Now, we are allowed to call a section of an arbitrary sheaf $\mathscr{F}$ by this name because every sheaf $\mathscr{F}$ has a space $Y = \cup_{x \in X} \mathscr{F}_x$, its **étalé space**, such that there is a local homeomorphism $\pi : Y \to X$.

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Indeed, we define the topology on Y by the following basis. Note that an element of Y is a germ. For every U open in X and for every $s \in \mathscr{F}(U)$, take the set of all germs of s in U to be in the basis.

Reminders II

Let X be a topological space with a sheaf of rings $\mathscr{O}_X$.
An $\boldsymbol{\mathscr{O}_x}$**-module** is a sheaf $\mathscr{F}$ on X with two morphisms of sheaves defining addition and scalar multiplication,

- $\mathscr{F} \times \mathscr{F} \to \mathscr{F}$
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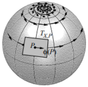
$\mathscr{O}_X(U)$-module morphism $=$ (pre-)sheaf morphism $+$ module morphism on every open $U \subseteq X$.
Examples:

- If X is a single point then this is just a usual module.
- If $\mathscr{F}$ is the constant sheaf $\mathbb{Z}$ we get a sheaf of abelian groups.

Why Bother?

For a reason to study $\mathscr{O}_X$-modules, consider $\mathbb{P}^1$ over $\mathbb{C}$, the



Riemann sphere.

Next week's tangent sheaf has sections given by tangent vectors at each point (varying "nicely"). Then the right description of this sheaf is not just as a ring, since we can multiply by the corresponding regular functions; we want to capture its structure as an $\mathscr{O}_X$-module. To help motivate further development, notice also that this sheaf is locally $\cong \mathscr{O}_X$ but not globally, since it does not even have constant global sections by the hairy ball theorem (Every continuous tangent field on $\mathbb{S}^2$ has a zero). Thus this is an example of an **invertible sheaf**.

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The specific details of the construction that we saw, such as the sheaf of discontinuous sections and $\flat, \sharp$, may be forgotten; the sheafification is used via its universal property implied by it being a functor from presheaves to sheaves. Namely, for every morphism $\mathscr{F} \to \mathscr{G}$, there is a unique $\tilde{\mathscr{F}} \to \tilde{\mathscr{G}}$ making the square diagram commutative.

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However, the cokernel must be checked globally.

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- Every morphism $u : X \to Y$ has a kernel with $i : Ker(u) \to X$ with $u \circ i = 0$ "largest": For every object Z there is a bijection $g \mapsto i \circ g$ between $\mathrm{Hom}(Z, Ker(u)) \to \{f \in \mathrm{Hom}(Z, X) : u \circ f = 0\}$

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- Analogously, $\exists p : Y \to Coker(u)$, $p \circ u = 0$ [**preabelian category**]
- Letting $Im(u) := Ker(Coker(u)), Coim(u) := Coker(ker(u))$, The path $X \to Coim(u) \to Im(u) \to Y$ is the same as $u : X \to Y$. [**abelian category**]

Locally-Free Module

An $\mathscr{O}_X$-module is **locally free** if for all $x \in X$ there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is isomorphic to a direct sum $\oplus_I O_X|_U$, where I might depend on x.

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The cardinality of I, called the **rank**, depends only on $\mathscr{F}$ and x because $\mathscr{F}_x \cong \oplus_I \mathscr{O}_{X,x}$ as $O_{X,x}$-modules. (And every commutative ring with identity satisfies IBN, the invariant basis number.)

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Useful Preliminaries

We will use these later.

- Given a continuous function $f : X \to Y$ between topological spaces, we have the **pushforward** of a sheaf $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$ which is also a sheaf.
- Given a sheaf $\mathcal{G}$ on Y, we define the pullback by the sheafification of $f^*\mathcal{G}(U) = \lim \to \mathcal{G}(V)$ over all $V \supset f(U)$. Note this is necessary since f might not be an open map.
- in particular, stalks are given by $\mathcal{F}_x = i_x^*\mathcal{F}$ for the embedding $i_x : \{x\} \to X$.
- Let $p \in \mathbb{P}^1$, then we define the **ideal sheaf** $\mathscr{O}(-p)$ of this point to be the subsheaf of $\mathscr{O}$ given by:

$$\mathscr{O}(-p)(U) = \{f \in \mathscr{O}(U) | f(p) = 0\}.$$

- More generally, the ideal sheaf of a closed subvariety of an algebraic variety has the sections of the structure sheaf which vanish on the closed subset.
- The twisting sheaf $\mathscr{O}(d)$ is defined on $\mathbb{P}^n$ by $\mathscr{O}(d)(U) = k[\tilde{U}]_d = \{\frac{p}{q} | deg(p) - deg(q) = d, q|_{\tilde{U}} \neq 0\}$

Quasi-Coherent modules

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Affine Case II

The key property in showing the other direction of the equivalence in the affine case is the fact that for $f \in A$ any section u of $\mathscr{F}$ over the principal open subset $D(f)$ can be extended essentially uniquely to X after multiplying u with some f^i.

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We give a sketch of the proof of the other direction. For details see Roi's notes or Kempf 5.2.2.
Proposition. Let f^{-n} be a formal notation denoting copies of $\mathscr{F}$, combined in a directed system by the maps $\mathscr{F} \to f^{-1}\mathscr{F} \to \dots$ given by f. Let $j : D(f) \to X$ be the embedding. Then $j_* j^* \mathscr{F} = \lim_{\to} f^{-n}\mathscr{F}$, where $j_*\mathscr{F}(V) := \mathscr{F}(U \cap V)$. (Kempf uses ad-hoc notation for this.)

Projective Case

In the projective case, too, we have a global description of quasi-projective sheaves. For a graded module M, define a quasi-coherent sheaf on $\mathbb{P}^n$, denoted $\tilde{M}_{\mathbb{P}^n}$, as follows. Its section on U is $(\tilde{M}_{\mathbb{A}^{n+1}}(\tilde{U}))_0$, where $\tilde{U}$ is the lift of U to the cone $\mathbb{A}^{n+1} \setminus \{0\}$.

Theorem

- *$M \mapsto \tilde{M}_{\mathbb{P}^n}$ is an exact functor*
- *Every quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is of the form $\tilde{M}_{\mathbb{P}^n}$, and for coherent sheaves we may take M finitely generated. Moreover, concretely $M = \oplus n \geq 0 \Gamma(\mathcal{F}(n))$.*

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Let $\mathcal{F} \in QCoh(\mathbb{P}^n)$, and $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ are j, π respectively. One may compute $\pi_* \pi^* \mathbb{F} = \oplus_{nin\mathbb{Z}} \mathbb{F}(n)$ and $j_* \pi^* \mathbb{F} = \oplus \mathbb{F}(n)$ is a quasi-coherent sheaf on $\mathbb{A}^{n+1}$ with the same global sections as $\pi^* \mathbb{F}$, which is the same as for $\pi_* \pi^* \mathbb{F}$, and these global sections are $\oplus_{n \in \mathbb{Z}} \Gamma(\mathcal{F}(n)) =: M'$. Moreover let $M = \oplus_{n \geq 0} \Gamma(\mathcal{F}(n))$, then M'/M has only negative degrees and its associated sheaf is $(\frac{\tilde{M'}}{M})_{\mathbb{P}^n} = 0$ (as there are no zero degree elements). Thus $\tilde{M'}_{\mathbb{P}^n} = \tilde{M}_{\mathbb{P}^n}$. Finally $\tilde{M'}_{\mathbb{P}^n} = \mathcal{F}$ because $\tilde{M'}_{\mathbb{A}^n} = j_* \pi^* \mathcal{F}$ so $\tilde{M'}_{\mathbb{P}^n}(U) = j_8 \pi^* (\mathcal{F}(\tilde{U}))_0 = \pi^*(\mathcal{F})(\tilde{U})_0 = \mathcal{F}(U)$. The proofs for the other claims are omitted. □

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Corollary. if $\mathcal{F}$ is coherent, then $\exists d, k$ such that $\mathcal{O}(-d)^{\oplus k} \to \mathcal{F}$ is a surjection, or equivalently $\mathcal{O}^{\oplus k} \to \mathcal{F}(d)$. In other words, every coherent sheaf is a quotient of a vector bundle.

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Proof. $\mathcal{F} = \tilde{M}$ for M finitely generated; pick $d >$ the degrees of all generators, then $M_{\geq d}$ is generated by M_d, so $\tilde{M}_{\geq d} = \tilde{M}$. By def'n of finitely generated, $A^{\oplus k[-d]} \to M$ is surjective. □

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Coherence

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The general definition, where the situation is as in quasi-coherence, is that there is a local **finite** presentation.

Theorem

Let X be an affine variety. Then a module M is finitely generated iff $\tilde{M}$ is coherent.

Proof.

If M is finitely generated then it is in particular finitely presented, so $\tilde{M}$ is coherent. Conversely, suppose $\tilde{M}$ is coherent. Cover X by principal open sets $D(f_i)$ such that on each $D(f_i)$ the restriction $\tilde{M}_{f_i}$ is a finitely generated $k[X]_{f_i}$-module. Since there are only finitely many f_i, after clearing denominators we get a finite generating set for M. $\square$

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(Coherence is useful outside algebraic geometry: the deep *Oka coherence theorem* in complex geometry that the sheaf of holomorphic functions on $\mathbb{C}^n$ is coherent.)

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Intro

Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module, then the dual sheaf $\mathscr{F}^\vee = \mathrm{Hom}(\mathscr{F}, \mathscr{O}_X)$, which is an O_X-module by itself, is also locally free of the same rank. (Since direct sum commutes with Hom)

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Proof.

the map

$$\mathscr{L}^\vee \otimes_{\mathscr{O}_X} \mathscr{L} \to \mathscr{O}_X$$

given by $(\phi, m) \mapsto \phi(m)$ is an isomorphism. Indeed, isomorphism is a local property so it is enough to check locally, where we know that $\mathscr{L} \cong \mathscr{O}_X \cong \mathscr{L}^\vee$ and so we're done. □

Ignore this I

We will now see in two steps that the dual of $\mathscr{L}$ is its inverse with respect to $\otimes$. **Lemma 1.** There is an isomorphism

$$\mathscr{L}^{\vee} \otimes_{\mathscr{O}_X} \mathscr{L} \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L})$$

.

Theorem (Giving lemma 1)

Let $\mathscr{F}, \mathscr{G}, \mathscr{H}$ be $\mathscr{O}_X$-modules where $\mathscr{F}$ is locally-free, then there is an isomorphism

$$\mathrm{Hom}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{G}) \otimes_{\mathscr{O}_{\mathscr{X}}} \mathscr{H} \to \mathrm{Hom}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{G} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathscr{H})$$

Ignore this II

Lemma 2. The canonical homomorphism

$$\mathscr{O}_X \to \mathrm{Hom}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{L}, \mathscr{L})$$

that sends a section $s \in \mathscr{O}_X(U)$ to scalar multiplication by $\mathscr{F}$ is an isomorphism.

Proof. This is a local question, so we may assume $\mathscr{L} = \mathscr{O}_X$

Ignore this III

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Together we obtain the desired:

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Together we obtain the desired:

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We define an **invertible sheaf** to be a locally-free sheaf of rank 1. This is justified since the converse is also true: if $M \otimes_{\mathscr{O}_X} N \cong \mathscr{O}_X$, then M is locally-free of rank 1. (Proof omitted. See ▸ Link For quasi-coherent sheaves, and 19.11 in ▸ Link for the general case.)

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The Story I

We define the **Picard Group** of a general variety as the isomorphism classes of ”line bundles”.

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We define the **Picard Group** of a general variety as the isomorphism classes of "line bundles".
The Picard group is an invriant of isomorphism, and so is useful to show that two varieties are not isomorphic. For example, it can be proven that the Picard group of a smooth cubic curve in $\mathbb{P}^2$ of a smooth cubic curve in $\mathbb{P}^2$ is nontrivial, and so it is never isomorphic to $\mathbb{P}^1$.

The Story II

We follow an idea from number theory to represent the Picard group as a quotient of **invertible fractional ideals** by its subgroup of principal ideals, using a short exact sequence.

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The idea is the **(ideal) class group**, that measures the extent to which unique factorization fails in the ring of integers of an algebraic number field, and more generally in Dedekind domains, by the quotient of the fractional ideals by the principal ideals.

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We mention both a geometric heuristical reasoning and an algebraic justification: the local rings of smooth varieties are UFD, since regular local rings are UFD. This is the **Auslander–Buchsbaum** theorem.

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We mention both a geometric heuristical reasoning and an algebraic justification: the local rings of smooth varieties are UFD, since regular local rings are UFD. This is the **Auslander–Buchsbaum** theorem.

We use this characterization to compute the Picard group for $\mathbb{A}^n$ and for $\mathbb{P}^n$.

Invertible sheaf

Let $\mathscr{L}$ be an invertible sheaf. Set $\mathscr{L}^{\otimes -n} := (\mathscr{L}^{\vee})^n$. Then

$$\mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m} \cong \mathscr{L}^{\otimes n+m},$$

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so the isomorphism classes of invertible $\mathcal{O}_X$ modules form an abelian group under this operation, called the **Picard group**. The neutral element is the class of $\mathscr{O}_X$.

Let X be an irreducible variety. Let $\mathcal{K}_X$ be the constant sheaf equal to the field $k(X)$ of rational functions on X,

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(The definition of the sheaf of rational functions is less trivial for schemes, where there might be zero divisors.)

A **sheaf of fractional ideals** $\mathcal{I}$ is a coherent subsheaf of $\mathcal{K}_X$.

The **invertible** sheaves of fractional ideals form a group $\mathrm{IFI}(X)$ under multiplication.

There is a subgroup $P(X)$ of IFI consisting of the **principal ideals** $f \cdot \mathscr{O}_X$. It is important because of the following theorem.

Exact sequence of Picard group

Theorem

There is a short exact sequence

$$0 \to P(X) \to \mathrm{IFI}(X) \to^{\psi} \mathrm{Pic}(X) \to 0,$$

where ψ sends an invertible fractional ideal to its isomorphism class.

Proof.

$\ker(\psi) = P(X)$: By definition $\psi((I)) = 0$ iff $\mathcal{I} \cong \mathscr{O}_X$. Let f be the image of 1 under this isomorphism, then $\mathcal{I} = f \cdot \mathscr{O}_X$, so $\mathcal{I} \in P(X)$. Conversely $f \cdot \mathscr{O}_X \cong \mathscr{O}_X$, so indeed $\ker(\psi) = P(X)$.
All left is showing ψ is surjective. Let $\mathscr{L}$ be an invertible sheaf, then we construct an invertible fractional ideal $\mathcal{I}$ such that $\mathcal{I} \cong \mathscr{L}$. Let $\sigma \in \mathscr{L}(V)$ for V open and dense. (Note we cannot simply take $V = X$ as there may be no global sections.) Then $\mathcal{I}(U) = \{f \in k(X) | f \cdot \sigma = \tau \in \mathscr{L}(U)\} \cong \mathscr{L}$ via multiplication by σ.

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Motivation

It is a classical question to determine the sets of zeros and poles of rational ("meromorphic") functions on a given variety.

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It is a classical question to determine the sets of zeros and poles of rational ("meromorphic") functions on a given variety.
We want to determine the local/global relationship: given zero/pole configurations on an open covering, are these configurations induced from a global rational function? (E.g. in complex analysis: Mittag-Leffler, Weierstrass product)

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- For a general, non-smooth variety, the Weil divisor does not work and we need the **Cartier divisor**. Here we abstractly define a "configuration" as an equivalence class of rational functions where $f \sim g$ on some open subset U iff $f = ug$ for some $u \in \Gamma(U, \mathscr{O}_X)^\times$ — u is a unit, so f and g should be thought of having the same zero/pole configuration.

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  On smooth varieties, the two notions are equivalent.

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On singular varieties we need to distinguish between the two:

- codimension-1 subvarieties $=$ Weil divisors
- varieties which can locally be defined by one equation $=$ Cartier divisors

We will also see a correspondence between divisors and invertible sheaves, which are by definition **line bundles**;

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Thus much of algebraic geometry studies an arbitrary variety by analyzing its codimension-1 subvarieties and the corresponding line bundles.

Weil Divisors

Definition

Let X be an irreducible variety. An **irreducible Weil divisor** on X is an irreducible subvariety D with $\dim D = \dim X - 1$. The group $\mathrm{Div}_W(X)$ is the free abelian group generated by the irreducible divisors. Thus a Weil divisor is of the form $D = \sum n_i D_i$ for $n_i \in \mathbb{Z}$. A divisor D is **effective** if $n_i \geq 0 \forall i$.

Cartier Divisors

Let X be an irreducible variety. The **Cartier divisor group**, $\mathrm{Div}_C(X)$, consists of subvarieties locally given by a nonzero rational function defined up to multiplication by a nonvanishing function.

Definition

An element of $\mathrm{Div}_C(X)$ is given by an open cover U_i together with rational functions $f_i \neq 0$ on U_i, such that on the intersection $U_i \cap U_j$ we have $f_i = phi_{ij} f_j$ for some $\phi_{ij} \in \mathscr{O}^*(U_i \cap U_j)$. A more abstract way of viewing this is as $\mathrm{Div_C(X)} = \Gamma(\mathrm{Rat}_\mathrm{X}^* / \mathscr{O}_\mathrm{X}^*)$, where $*$ signifies being nowhere zero.

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Claim. The Cartier divisors are equivalent to the category of invertible sheaves. Indeed, match a divisor D to the sheaf defined locally by $f_i \mathscr{O}_X$ which is invertible, and conversely an invertible sheaf has local data defining a divisor.

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Claim. There is an exact sequence $\Gamma(\mathcal{K}) \to Div_C(X) = IFI(X) \to Pic(X)$. Indeed, the latter morphism is just $\mathscr{L} \to [\mathscr{L}$, and it is surjective: Choose a local trivialization of a given line bundle $\mathscr{L}$, then we have an isomorphism $\mathscr{L} \otimes_{\mathscr{O}_x} \mathcal{K} \cong \mathcal{K}$. Now for the kernel: it consists of sections of $\mathcal{K}^*/\mathscr{O}^*$ isomorphic to $\mathscr{O}_X$, which is the set of nowhere zero rational functions.

Reduction in smooth case

We saw in a previous lecture that for smooth varieties the local rings $\mathscr{O}_{X,x}$ are regular local rings.

Theorem (Auslander, Buchsbaum, 1959)

Any local regular ring (R, m) is UFD.

(Sketch to give a sense of the difficulty).

- Let $k = R/m$, then the completion of R is isomorphic to $k[[x_1, .., x_d]]$ where $d = \dim R$.
- For Noetherian local rings, if the completion is UFD then the ring itself is UFD.
- $k[[x_1, .., x_d]]$ is UFD.

$\square$

Theorem

Let X be a locally factorial variety, meaning that all $\mathscr{O}_{X,x}$ are regular local rings. Then $Div_C(X) = Div_W(X)$.

Proof of Weil = Cartier

Consider the map $Div_W(X) \to Div_C(X)$ given by $D \mapsto \mathscr{O}(-D) \subseteq \mathscr{O}_X \subseteq \mathcal{K}$, where $\mathscr{O}(-D)$ denotes the sheaf of functions vanishing on D.

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- $\mathscr{O}(-D)$ is locally principal. In a UFD, every prime ideal of height one is principal: $\mathscr{O}(-D)$ is locally induced by a prime ideal of height 1 by definition, so when we pass to the stalk it is induced by (f_x) for some $f_x \in \mathcal{K}$. Now (f_x) and $\mathscr{O}(-D)$ only differ on components that do not pass x (as they agree on the stalk), which can only happen on finitely many other components, so after shrinking our local neighborhood we can have (f_x) agreeing with $\mathscr{O}(-D)$ on some neighborhood.

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- The map $D \mapsto \mathscr{O}(-D)$ is injective: enough to see that $nD \mapsto 0$ does not hold for $n > 0$, but the image is $\mathscr{O}(-D)^n \neq 0$.

Proof Cont.

- The map is surjective: Assume first $\mathscr{L} \subset \mathscr{O}$, we want to find a Weil divisor $D \mapsto \mathscr{L}$. We can asssume that we know this for all bigger $\mathscr{L}'$, $\mathscr{L} \subset \mathscr{L}' \subset \mathscr{O}_X$. Working locally, there is some $\mathscr{L} = (f)$. Let D be an irreducible component of $D(f)$, $\mathscr{O}(-D) \supset \mathscr{L}$. Let $(\phi) = \mathscr{L}$, then $\phi^{-1}\mathscr{L} \supset \mathscr{L}$ comes from some D', and we have $D + D' \mapsto \mathscr{L}$.

Proof Cont.

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  Finally, in the general case where we don't assume $\mathscr{L} \subset \mathscr{O}$, we still have locally $\mathscr{L} = (\frac{g}{h})$ for some $g, h \in \mathscr{O}(U)$. By the first case we know that there are $D \mapsto \alpha, D' \mapsto \beta$, so all in all $D - D' \mapsto \mathscr{L}$. $\square$

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$\mathrm{Pic}(\mathbb{A}^n)$

Example

$$\mathrm{Pic}(\mathbb{A}^n) = 0$$

.

Indeed: **Proposition.** Let A be a UFD. Then every polynomial ring $A[x_1, ..., x_n]$ is also a UFD, and for every multiplicative set the localization $S^{-1}A$ is a UFD.
Thus $\mathbb{A}^n$ is locally factorial, and we conclude by noting that every variety of codimension 1 is given by a single polynomial, so is principal. $\square$
$\mathbb{C}^n$ is a contractible manifold, and hence has no nontrivial topological vector bundles, which is analogous to this result.

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$\mathbb{C}^n$ is a contractible manifold, and hence has no nontrivial topological vector bundles, which is analogous to this result. So is it true that $\mathbb{A}^n$ has no nontrivial vector bundles?
This is the **Quillen-Suslin Theorem**, formerly known as **Serre's Conjecture**, part of Quillen's work leading to his 1978 Fields Medal.

$\mathrm{Pic}(\mathbb{P}^n)$

Example

$$\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$$

.

Concretely, this means that every codimension-1 subvariety of $\mathbb{P}^n$ is defined by the vanishing of a single homogeneous polynomial.
Indeed, $\mathbb{Z} \subseteq Pic(\mathbb{P}^n)$ because we have
$\mathscr{O}(d_1) \otimes \mathscr{O}(d_2) \cong \mathscr{O}(d + 1 + d_2)$. Moreover these are pairwise nonisomorphic: enough to show specific two are nonisomrphic, but $\mathscr{O}$ is not isomorphic to $\mathscr{O}(d)$ for negative d since there the global sections of $\mathscr{O}(d)$ vanish, as we saw.
Now for the other inclusion, let D be of codimension 1, then there is a homogeneous polynomial p of degree d that generates the homogeneous ideal of D, so $\mathscr{O}(-D) \cong \mathscr{O}(-d)$ by multiplication by p. $\square$