Sheaves of Modules

Roi Blumberg

June 3, 2018

1 Zariski Cotangent Space

Insert Motivation

Definition 1. Let us have some variety X. Let us have some $x \in X$, with local ring $\mathcal{O}_{X,x}$ and maximal ideal m_x. Then the finite dimensional vector space m_x/m_x^2 is called the *Zariski cotangent space of X at x*, and denoted $\mathrm{Cot}_x(X)$.

Definition 2. Let f be a germ in $\mathcal{O}_{X,x}$. The *differential* $df|_x \in \mathrm{Cot}_x(X)$ of f at x is $(f - f(x)) \mod m_x^2$ (intuitively,this definition lets us study the first-order variation of f at x - which is what we want from a definition of the differential).

Claim 1. *$d - |_x : \mathcal{O}_{X,x} \to Cot_x(X)$ satisfies the following properties:*

1. $d\ constant|_x = 0$
2. $d(f+g)|_x = df|_x + dg|_x$
3. $d(f \cdot g)|_x = f(x)dg|_x + g(x)df|_x$

The proof is omitted here, but is completely straightforward.

Lemma 2.
1. $\dim_k Cot_x(X) \geq \dim_x X$
2. $Cot_{(x_1,x_2)}(X_1 \times X_2) = Cot_{x_1}(X_1) \oplus Cot_{x_2}(X_2)$
3. *If X is affine and n_x is the maximal ideal of $k[X]$ in corresponding to x then $Cot_x(X) = n_x/n_x^2$.*

Proof.
1. We already saw this in the lecture on nonsingular varieties.
3. Any element $k[X] - n_x$ acts invertibly on n_x/n_x^2. Therefore:
$$\left(n_x/n_x^2\right) = \left(n_x/n_x^2\right)_{n_x} = \left(n_{x_{n_x}}/n^2_{x_{n_x}}\right) = m_x/m_x^2$$
2. This is a local argument, so we may assume that X_1, X_2 are affine. Then $X_1 \times X_2$ is affine and:
$$\begin{aligned}\mathrm{Cot}_{(x_1,x_2)}(X_1 \times X_2) &= n_{(x_1,x_2)}/n^2_{(x_1,x_2)} \\ &= n_{x_1} \otimes k[X_2] + k[X_1] \otimes n_{x_2} / \left(n_{x_1} \otimes k[X_2] + k[X_1] \otimes n_{x_2}\right)^2 \\ &= n_{x_1}/n_{x_1}^2 \otimes k \oplus k \otimes n_{x_2}/n_{x_2}^2 \\ &= \mathrm{Cot}_{x_1}(X_1) \oplus \mathrm{Cot}_{x_2}(X_2)\end{aligned}$$

□

Recall we call a variety *smooth* (or *nonsingular*) if $\dim \mathrm{Cot}_x(X) = \dim X$.

2 The Sheaf of Differentials

Definition 3. Let A be a k-algebra and M be an A-module. Then a *derivation* $D : A \to M$ over k is a function satisfying:

1. $D(a+b) = D(a) + D(b)$.
2. $D(ab) = aD(b) + bD(a)$.
3. For all $k \in K$, $D(k) = 0$.

Definition 4. Let X be a variety. Define $\epsilon : k[X] \otimes_k k[X]$ by $\epsilon(b \otimes b') = bb'$, and let $I = \ker \epsilon$. Then we define the *module of (Kahler) differentials* of $k[X]$ over k by $\Omega[X] = I/I^2$. We have a derivation, then, $d : k[X] \to \Omega[X]$ given by:

$$b \mapsto 1 \otimes b - b \otimes 1 + I^2$$

Proposition 3. $\Omega[X]$ *is generated by* $\{df \mid f \in k[X]\}$.

Proof. It suffices to show that I is generated by elements of the form $1 \otimes f - f \otimes 1$ where $f \in k[X]$. For $f, g \in k[X]$ we have:

$$f \otimes g = fg \otimes 1 + f \cdot (1 \otimes g - g \otimes f)$$

And hence:

$$\sum f_i \otimes g_i = \left(\sum f_i g_i\right) \otimes 1 + \left(\sum f_i \left(1 \otimes g_i - g_i \otimes 1\right)\right)$$

But being in I means that $\sum f_i g_i = 0$, and therefore we get our claim. $\square$

Proposition 4. *We have the following universal property - if δ is another derivation of $k[X]$ over k into some $k[X]$-module M, then there is a unique homomorphism of modules $\ell : \Omega[X] \to M$ with $D = \ell \circ d$.*

Proof. The previous proposition gives us uniqueness - if ℓ_1, ℓ_2 both satisfy the claim, then $\ell_1(g) = \sum (\ell_1(df)) = \sum D(f) = \ell_2(g)$.
For existence, set $B = k[X] \otimes_k k[X]$ and look at the $k[X]$-module $A \oplus M$. Then look and the homomorphism $\phi : B \to A \oplus M$:

$$\phi(f \otimes g) = (fg, f \cdot D(g))$$

Notice that $I^2 \subseteq \ker \phi$ (since $\phi(f_1 f_2 \otimes g_1 g_2) = (f_1 f_2 g_1 g_2, f_1 f_2 \cdot (g_1 D(g_2) + g_2 \cdot D(g_1))) = 0$), we get an induced homomorphism of $k[X]$-algebras $\phi' : B/I^2 \to A \oplus M$ with:

$$1 \otimes y - y \otimes 1 \mapsto (0, D(y))$$

Therefore having ℓ be the restriction of ϕ' to $\Omega[X]$ via the second coordinate yields the desired result. $\square$

Definition 5. Let X be a variety. We have the diagonal morphism $\Delta : X \to X \times X$. The *sheaf of differentials* of X, Ω_X, is the (sheafification of) the pullback $\Delta^*(\mathcal{I}/\mathcal{I}^2)$, where $\mathcal{I}$ is the *ideal sheaf* of the diagonal in $X \times X$, i.e. the sheaf of functions that vanish on $\Delta(X)$. We also define the (easily seen to be) derivation $d : \mathcal{O}_X \to \Omega_X$ sending g to $(g(x_1) - g(x_2)) \mod \mathcal{I}^2$.

Proposition 5. *If X is affine, then $\Omega_X = \widetilde{\Omega[X]}$.*

Proof. On any open subset $U \times U \subseteq X \times X$, where $U \subseteq X$ is open, and assume B is its coordinate ring. We have that $U \times U$ is isomorphic to the affine variety corresponding to $B \otimes B$. Now, $\Delta(X) \cap (U \times U)$ is the closed subvariety defined by the kernel of the multiplication homomorphism $B \otimes B \to B$. Therefore $\mathcal{I}/\mathcal{I}^2 = (\widetilde{I/I^2})$ on any open set, so globally $\mathcal{I}/\mathcal{I}^2 = (\widetilde{I/I^2})$. ☐

Claim 6. $\Omega_X|_x = Cot_x(X)$.

Proof. As this is a local claim, assume that X is affine, and let n be a maximal ideal of x in $k[X]$. Then we want an isomorphism $\Omega[X] \otimes_{k[X]} k[X]/n \simeq n/n^2$. Now, recall that $d - |_x : k[X] \to n/n^2$ is a derivation, and hence we have a unique $k[X]$-linear mapping $\ell : \Omega[X] \to n/n^2$ with $df|_x = \ell(df)$. We get, then, a k-linear mapping $\bar{\ell} : \Omega[X] \otimes_{k[X]} k[X]/n \to n/n^2$ (with $\bar{\ell}(df \otimes u) = \ell(udf)$) satisfying $df|_x = \bar{\ell}(df \otimes 1)$. Now, the mapping $k[X] \to \Omega[X] \otimes_{k[X]} k[X]/n$ defined by $f \mapsto df \otimes 1$ is also a derivation. Therefore there is a unique linear mapping $m : n/n^2 \to \Omega[X] \otimes_{k[X]} k[X]/n$ with $m(df|_x) = df \otimes 1$. $\bar{\ell}$ and m are inverse to each other, so we are done. ☐

Claim 7. *Let f be a regular function on $\mathbb{A}^n$. Then $df = \frac{\partial f}{\partial X_1} dX_1 + \cdots + \frac{\partial f}{\partial X_n} dX_n$.*

Proof. Just write $f = \sum a_{i_1,\ldots i_n} x^{i_1} \cdots x^{i_n}$ and compute by induction and the formal rules of a derivation. ☐

We now wish to calculate $\Omega_{\mathbb{P}^n}$. Define $\pi : U = \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ to be the projection, then we have an induced homomorphism $\pi^* : \Omega_{\mathbb{P}^n} \to \Omega_U$. Then π^* identifies $\Omega_{\mathbb{P}^n}$ with a subsheaf of $\bigoplus_{1 \le i \le n} dX_i \cdot \mathcal{O}_{\mathbb{P}^n}(-1)$. Now, we may also define $\alpha : \bigoplus_{1 \le i \le n} dX_i \cdot \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}$ by $\alpha(dX_i \cdot \sigma) = X_i \cdot \sigma$.

Claim 8. *(Euler's Exact Sequence) We have an exact sequence:*

$$0 \to \Omega_{\mathbb{P}^n} \xrightarrow{\pi^*} \bigoplus_{1 \le i \le n} dX_i \cdot \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n} \to 0$$

Proof. The proof will not be 100 percent rigorous, so apologies in advance. Exactness at $\Omega_{\mathbb{P}^n}$ and at $\mathcal{O}_{\mathbb{P}^n}$ are immediate. For exactness at $\bigoplus_{1 \le i \le n} dX_i \cdot \mathcal{O}_{\mathbb{P}^n}(-1)$, let us compute explicitly π^*. For $f = f_1 d\left(\frac{X_1}{X_0}\right) + \cdots + f_n d\left(\frac{X_n}{X_0}\right)$, we have:

$$\pi^* f = dX_0 \left(-\frac{X_1}{X_0^2} f_1 - \cdots - \frac{X_n}{X_0^2} f_n \right) + dX_1 \frac{f_1}{X_0} + \cdots + dX_n \frac{f_n}{X_0}$$

Then one easily checks that $\alpha \pi^* f = 0$, so $\mathrm{Im}\pi^* \subseteq \ker \alpha$. On the other hand, if $\alpha\left(g_0 dX_0 + \cdots + g_n dX_n\right) = 0$, then take $f = X_0 g_1 d\left(\frac{X_1}{X_0}\right) + \cdots + X_0 g_n d\left(\frac{X_n}{X_0}\right)$, so $\ker \alpha = \mathrm{Im}\pi^*$. ☐