Nonsingular Curves

Roi Blumberg

April 28, 2018

1 Introduction

When one wants to classify algebraic varieties (up to isomorphisms), one way of tackling this problem is by classifying the varieties via a weaker type of equivalence, namely, birational equivalence. Now, a nonsingular variety is considered more "well-behaved" than a general variety, and has some stronger conditions that one can work with. Therefore, one can reach two sub-problems, which may help him classify varieties up to birational equivalence:

- Within any birational equivalence class, find a nonsingular projective variety.
- Classify all nonsingular projective varieties within one birational equivalence class.

Generally, both questions are very difficult. However, in one dimension, this is a problem we can tackle. The two problems will get answered as we will show that in each birational equivalence class, there is a single nonsingular curve. We will therefore be able to talk, for each field K of trancendence degree 1 over the base field k, about its nonsingular curve C_K. Furthermore, we will show an equivalence of categories between the category of such field, with k-homomorphisms, to the category of nonsingular projective curves, with dominant morphisms, to the category of quasi-projective curves, with dominant rational maps.

In order to show this, we will also define the notion of an "abstract algebraic curve". While this will only be used as a tool here (and we will show that we have added nothing new to our category of curves), it will be interesting to see this notion as another, much more purely algebraic, way, of defining varieties.

2 Valuations and Valuation Rings

Before we continue with nonsingular curves, we need to note a few facts from commutative algebra - mainly, valuations, DVR's, and Dedekind domains.

Definition 1. Let us have a field K, and a totally ordered abelian group G. A *valuation* v of K with values in G is a function $v : K\backslash\{0\} \to G$ that satisfies for all $x, y \in K$ such that $x, y \neq 0$:

$$v(xy) = v(x) + v(y) \tag{1}$$
$$v(x+y) \geq \min(v(x), v(y)) \tag{2}$$

Definition 2. Given a valuation $v : K\backslash\{0\} \to G$, we may define the *valuation ring of* v as the subring of K $R = \{x \in K \mid v(x) \geq 0\} \cup \{0\}$. It has an ideal $\mathbf{m} = \{x \in K \mid v(x) > 0\} \cup \{0\}$.

Remark. Many courses and textbooks use the equivalent definition that R is a valuation ring if in its field of fraction K, for any $x \in K$ we have either $x \in R$ or $x^{-1} \in R$.

Definition 3. An integral domain R is called a *valuation ring* if it is a valuation ring of some valuation of its quotient field. If we have a subfield $k \subseteq K$ for which $v(x) = 0$ for all $x \in k\backslash\{0\}$, we say that v is a *valuation of* K/k, and R is a *valuation ring of* K/k.

Definition 4. We say a local ring B *dominates* a local ring A if $A \subseteq B$ and $\mathbf{m}_B \cap A = \mathbf{m}_A$ ($\mathbf{m}_A, \mathbf{m}_B$ being the corresponding rings' unique maximal ideals).

Fact 1. *Let K be a field. A local ring $R \subseteq K$ is a valuation ring of K if and only if it is maximal in the set of local rings contained in K, with respect to domination. Furthermore, every local ring contained in K is dominated by some valuation ring of K.*

Proof. See Atiyah-Macdonald [Ch. 5, page 65] □

Definition 5. A valuation v is *discrete* (in short *DVR*) if it takes values in $\mathbb{Z}$. The corresponding valuation ring is called a *discrete valuation ring.*

Fact 2. *Let A be a local noetherian ring of dimension 1, with maximal ideal* $\mathbf{m}$. *Then the following are equivalent:*

- *A is a discrete valuation ring.*
- *A is integrally closed.*
- *A is a regular local ring*
- $\mathbf{m}$ *is a principal ideal.*

Proof. See [Atiyah-Macdonald Proposition 9.2, page 94]. □

Definition 6. A *Dedekind domain* is an integrally closed noetherian domain of dimension 1.

Fact 3. *An integral domain R is integrally closed if and only its localization at any nonzero prime ideal R_p is integrally closed.*

Proof. See [Atiyah-Macdonald, Prop. 5.13, page 63]. □

This implies that a localization of a Dedekind domain at a nonzero prime ideal is a DVR.

Fact 4. *If R is a Dedekind domain and k is its fraction field, then the integral closure of R in a finite extension of k is also a Dedekind domain.*

Proof. See [Zariski-Samuel, vol.1, Theorem 19, page 281]. □

Fact 5. *Let A be an integral domain which is a finitely generated algebra over a field k. Let K be the quotient field of A, and L be a finite algebraic extension of K. Then the integral closure of A in L is a finitely generated A-module, and also a finitely generated k-algebra.*

Proof. See [Zariski-Samuel, vol. 1, Ch. 5, Theorem 9, page 267]. □

3 Preliminaries on Algebraic Curves

Now that we have covered the necessary commutative algebraic background, we can go back to algebraic curves. Let us have a function field K of dimension 1 over k, with k algebraically closed.

Take a point P on a nonsingular curve Y over k. By last week's lecture $\mathcal{O}_P$ is a regular local ring of dimension 1, and thus is a DVR. Its quotient field is K, and since $k \subseteq \mathcal{O}_P$, it is a valuation ring of K/k.

Now, define C_K to be the set of all DVR's of K/k. We see, therefore, that the set of local rings of Y is a subset of C_K. This will motivate later our definition of an abstract nonsingular curve, but only after a few more preliminaries.

Lemma 1. *Let Y be a quasi-projective variety. Let $P, Q \in Y$, and suppose that $\mathcal{O}_Q \subseteq \mathcal{O}_P$. Then $P = Q$.*

Proof. Let us embed Y in $\mathbb{P}^n$ for some n. By replacing Y with its closure, we may assume Y is a projective variety. By linearly changing coordinates, we may assume that P, Q are not in the hyperplane H_0 defined by $x_0 = 0$. Therefore $P, Q \in Y \cap (\mathbb{P}^n \backslash H_0)$, which is an affine variety. So we may assume Y to be affine.
Let $A = A(Y)$. Then there are maximal ideals $\mathbf{m}, \mathbf{n} \subseteq A$ for which $\mathcal{O}_P = A_{\mathbf{m}}, \mathcal{O}_Q = A_{\mathbf{n}}$ (this fact was proven in the lecture about morphisms that Omer gave - there is a 1-1 correspondence between points of Y and maximal ideals of A). Now, since $\mathcal{O}_Q \subseteq \mathcal{O}_P$, we have $\mathbf{n} \subseteq \mathbf{m}$, so by maximality of $\mathbf{n}$ we have $\mathbf{m} = \mathbf{n}$, and therefore $P = Q$.
□

Lemma 2. *Let us have some $x \in K$. Then the set $\{R \in C_K \mid x \notin R\}$ is finite.*

Proof. For any valuation ring R, $x \notin R$ if and only if $x^{-1} \in \mathbf{m}_R$, where $\mathbf{m}_R$ is the maximal ideal of R (this comes directly from the well-known fact that all non-units in a ring are contained in a maximal ideal).
Therefore, let us denote $y = x^{-1}$, so we have to show that $\{R \in C_K \mid y \in \mathbf{m}_R\}$ is finite. If $y \in k$ then y is invertible in R so there are no such R. Let us assume, then, that $y \notin k$.
Now consider the subring $k[y]$ in K. k is algebraically closed, and therefore y is trancendental over k. Then $k[y]$ is a free polynomial ring so K is a finite extension over $k(y)$.
Now let B be the integral closure of $k[y]$ in K. It is an integral closure of a Dedekind domain ($k[y]$ is noetherian, integrally closed, and of dimension 1), and therefore a Dedekind domain itself, and also a finitely generated k-algebra by a theorem shown in the lecture about morphisms.

Now, if $y \in R$ for any DVR of K/k, then $k[y] \subseteq R$, and since R is equivalently integrally closed in K, we have $B \subseteq R$. Now, let $\mathbf{n} = \mathbf{m}_R \cap B$. Then is a maximal ideal of B, and B is dominated by R. But $B_{\mathbf{n}}$ is also a DVR of K/k by our note from earlier, so $B_{\mathbf{n}} = R$ by maximality of valuation rings with respect to domination.
If, now, $y \in \mathbf{m}_R$, then $y \in \mathbf{n}$. Aside from that, B, as a finitely generated k-algebra which is an integral domain, is the affine coordinate ring of some affine variety Y (we mentioned this in the first lecture). B is Dedekind, and therefore $\dim Y = 1$ and it is nonsingular (all its localizations are regular local). Now, since $y \in \mathbf{n}$, then y, as a regular function on Y, vanishes at the point of Y corresponding to $\mathbf{n}$. But $y \neq 0$, so it vanishes only at a finite set of points. This is in 1-1 correspondence with the maximal ideals of B, and therefore there can be only a finite number of maximal ideals that y is in. Since $R = B_n$ is dependent only on $\mathbf{n}$, we may conclude that $y \in \mathbf{m}_R$ for only finitely many $R \in C_K$.

□

Corollary. *Any DVR of K/k is isomorphic to the local ring of a point on some nonsingular affine curve.*

Proof. Given a DVR R, let $y \in R \backslash k$. Then we may construct such a curve in the same way as in the proof above (as the corresponding curve to the integral closure of $k[y]$). □

4 Abstract Nonsingular Curves

Now we have enough preliminaries so we can define an abstract nonsingular curve. This construction is a particular case of the more general concept of abstract varieties that we had talked about briefly in the lecture about morphisms - a ringed space that is locally isomorphic to an affine variety.

Let K, k and C_K be as before. Note that following the previous corollary, we may call elements of C_K *points*, and write $P \in C_K$ where P corresponds to the DVR R_P. Note, that C_K is infinite - it contains all local rings of any nonsingular curve with function field K. Those are all distinct by a previous lemma and there is an infinite number of them [Hartshorne Ex. 4.8].
Now, define the finite complement topology on C_K. Now, if $U \subseteq C_K$ is open, define the *ring of regular functions* on U to be $\mathcal{O}(U) = \bigcap_{P \in U} R_P$. Given an element $f \in \mathcal{O}(U)$, it defines a function from U to k by taking f modulo the maximal ideal of R_P. If two elements $f, g \in \mathcal{O}(U)$ define the same function, then $f - g \in \mathbf{m}_P$ (the maximal ideal of R_P) for infinitely many $P \in C_K$, so $f - g = 0$ by the previous theorem. Therefore, we may identify the elements of $\mathcal{O}(U)$ with functions $f : U \to k$. By the theorem we just proved, any $f \in K$ is regular on some open subset $U \subset C_K$.

Definition 7. An *abstract nonsingular curve* is an open subset $U \subseteq C_K$ (with K, k, and C_K as before), with the induced topology, and induced notion of regular functions on its open subsets.

Definition 8. A *morphism* $\varphi : X \to Y$ between abstract nonsingular curves is a continuous map such that for every open $V \subseteq Y$ and every regular function $f : V \to k$, $f \circ \varphi$ is a regular function on $\varphi^{-1}(V)$.

Our main mission for the rest of the lecture will be to show that in fact, the category of nonsingular quasi-projective curves is equivalent to the category of abstract nonsingular curves.

Proposition 1. *Every nonsingular quasi-projective curve Y is isomorphic to an abstract nonsingular curve.*

Proof. Let K be the function field of Y. Then each local ring $\mathcal{O}_P$ is a regular local ring of dimension 1, and hence a DVR of K/k. Furthermore, we know that distinct points give rise to distinct subrings of K, by a previous lemma. Then let $U \subseteq C_K$ be the set of local rings of Y and let $\varphi : Y \to U$ be the bijective map:

$$\varphi(P) = \mathcal{O}_P$$

We want to show that φ is our required isomorphism. We will need to show several things then - that U is an abstract nonsingular curve, and that φ is an isomorphism.
First, let us show that U is an open subset of C_K. Since U has the finite complement topology, it is enough to show that U contains a nonempty open subset. Combining that with the fact we have seen in the lecture about rational maps that any variety has a topological base of affine varieties, we may assume that Y is affine, and denote by A its affine coordinate ring, which is a finitely generated k-algebra. Additionally, K is A's field of fractions, and U is the set of localizations of A at its maximal ideals (all this was proven in the Omer's lecture about morphisms). We see, then, that U is a set of DVR's of K/k containing A. Now, let $x_1, ..., x_n$ be generators of A over k.
$A \subseteq R_P$ for some DVR R_P if and only if $x_1, ..., x_n \in R_P$. Then $U = \bigcap U_i$, with $U_i = \{P \in C_K \,|\, x_i \in R_P\}$. But we know that $\{P \in C_K \,|\, x_i \notin R_P\}$ is a finite set - so for each i U_i is a finite complement and therefore U is open.
Now, let us show that φ is an isomorphism. By construction, φ is a bijection. Additionally, a nonempty set in Y is open if and only if its complement is finite, and that is true also in U, so φ is bi-continuous. Finally, for any open set $V \subseteq Y$, $\mathcal{O}(V) = \bigcap_{P \in V} \mathcal{O}_{P,Y}$, (and the same is true of sheaves in U), so φ is an isomorphism.

□

We have seen, then, that any quasi-projective curve is an abstract curve. We will see soon the converse to this, that will tell us that every abstract nonsingular curve is isomorphic to a projective curve.

5 Equivalence of Categories

For the converse, we will need first the following proposition:

Proposition 2. *Let X be an abstract nonsingular curve, let $P \in X$, let Y be a projective variety, and let $\varphi : X \backslash \{P\} \to Y$ be a morphism. Then φ can be uniquely extended to a morphism $\bar{\varphi} : X \to Y$.*

Proof. Embed Y as a closed set in $\mathbb{P}^n$ for some n. It is enough, then, to assume that $Y = \mathbb{P}^n$. Indeed, if we have a morphism $\varphi : X\backslash\{P\} \to \mathbb{P}^n$ that extends to $\bar{\varphi} : X \to \mathbb{P}^n$ and sends $X\backslash\{P\}$ into Y, it will send P, which is in the closure of $X\backslash\{P\}$, to the closure of $\varphi(X\backslash\{P\})$, which is contained in Y - so we lose nothing from assuming $Y = \mathbb{P}^n$. Where $x_0, ..., x_n$ are the homogeneous coordinates of $\mathbb{P}^n$, let U be the open set:

$$U = \{[x_0 : \cdots : x_n] \mid \forall 1 \leq i \leq n \,:\, x_i \neq 0\}$$

If $\varphi\left(X\backslash\{P\}\right) \cap U = \emptyset$, then $\varphi\left(X\backslash\{P\}\right)$ (by irreducibility) is contained in one of the hyperplanes defined by $\{x_i = 0\}$. However, those hyperplanes are isomorphic to $\mathbb{P}^{n-1}$ and we will be done by induction on dimension.
Assume, then, that $\varphi(X\backslash\{P\}) \cap U \neq \emptyset$. Then for every i, j the function $f_{ij} = \varphi^*\left(\frac{x_i}{x_j}\right)$ is a regular function on $X\backslash\{P\}$, and in particular, is in $K(X)$. Now, let v be the valuation associated to R_P. Define $r_i = v(f_{i0})$ for each $1 \leq i \leq n$. Then for each i, j:

$$v(f_{ij}) = v\left(\frac{f_{i0}}{f_{j0}}\right) = r_i - r_j$$

Choose k such that r_k is minimal among $r_0, ..., r_n$. Then $v(f_{ik}) \geqslant 0$ for all i, so $f_{0k}, ..., f_{nk} \in R_P$. Now extend φ by defining $\bar{\varphi}(P) = (f_{0k}(P), ..., f_{nk}(P))$. To show that this is a morphism, we just need to show that regular functions in a neighbourhood of $\bar{\varphi}(P)$ pull back to regular functions in a neighbourhood of P. Now notice that $f_{kk} = 1$, so:

$$\bar{\varphi}(P) \in U_k = \{[x_0 : ... : x_n] \,|\, x_k \neq 0\} \simeq k\left[\frac{x_0}{x_k}, ..., \frac{x_n}{x_k}\right]$$

These coordinates pull back to $f_{0k}, .., f_{nk}$, which are regular by construction, so any regular functions on U_i pull back to regular functions on X. From this it follows that the assertion is true for any smaller neighbourhood $\bar{\varphi}(P) \in V \subseteq U_K$. Hence, $\bar{\varphi}$ is a morphism.
Uniqueness follows from the fact that two morphisms that extend $X\backslash\{P\}$ must agree on a closed set.

□

We now come to the main results of this lecture:

Theorem 1. *Let K be a function field of dimension 1 over k. Then the abstract nonsingular curve C_K is isomorphic to a nonsingular projective curve.*

Proof. We saw earlier that given $R \in C_K$ there is a nonsingular affine curve X and a point $x \in X$ such that $R \simeq \mathcal{O}_{x,X}$. The curve X is isomorphic (as we saw earlier) to the abstract nonsingular curve $U \subseteq C_K$ where $U = \{\mathcal{O}_{x,X}\}$. Then the set $\{U_R\}_{R \in C_K}$ is an open cover of C_K. but since C_K has the finite complement topology, which has the property that every subset is compact, then we have a finite cover of C_K:

$$C_K = U_1 \cup ... \cup U_t$$

Where each U_i is isomorphic to a nonsingular affine curve X_i via some isomorphism $\varphi_i : U_i \to X_i$. Now, let Y_i be the closure of X_i in some projective space

$\mathbb{P}^{n_i}$. applying the previous lemma successively (remember we are in the finite complement topology), there exists a morphism

$$\varphi_i : C_K \to Y_i$$

extending the morphism on U_i. Define the product morphism:

$$\varphi : C_K \to \prod_i Y_i, \quad \varphi(R) = (\varphi_1(R), ..., \varphi_n(R))$$

and let Y be the closure of the image of φ. As the closure of an image of a morphism, it is a projective curve. Let us show that $\varphi : C_K \to Y$ is an isomorphism.

For any point $P \in C_K$, we have $P \in U_i$ for some i. Then if $\pi : Y \to Y_i$ is the projection, then $\pi \circ \varphi = \varphi_i$ on U_i. This induces inclusions of the local rings:

$$\mathcal{O}_{\varphi_i(P),Y_i} \xrightarrow{\pi^*} \mathcal{O}_{\varphi(P),Y} \xrightarrow{\varphi^*} \mathcal{O}_{P,C_K}$$

Moreover, since φ_i is an isomorphism on U_i, we get that all three local rings are isomorphic. In particular, for every $P \in C_K$ we have that $\mathcal{O}_{P,C_K}$ and $\mathcal{O}_{\varphi(P),Y}$ are isomorphic under φ^*.
Next let us show that φ is surjective. Let us have $Q \in Y$. Then $\mathcal{O}_Q$ is dominated by some DVR of K/k, R. We know that $R = R_P$ for some $P \in C_K$, and $\mathcal{O}_{\varphi(P)} \simeq R$, so $Q = \varphi(P)$. Therefore φ is surjective. Also, since distinct point of C_K correspond to distinct subrings of K, we have that φ is injective.
We get, then, that φ is a bijective morphism of C_K to Y, that satisfies for every $P \in C_K$ that φ_P^* is an isomorphism. Therefore, by [Hartshorne, Ex. 3.3b] we have that φ is an isomorphism.

□

This theorem is the main result of this lecture. It produces the following two corollaries, which we have promised to prove in the beginning of the lecture:

Corollary. *Every curve is birationally equivalent to a nonsingular projective curve.*

Proof. For any curve Y with function field K, Y is birationally equivalent to C_K which is nonsingular and projective.

□

Corollary. *The following three categories are equivalent:*

(i) Nonsingular projective curves, and dominant morphisms.

(ii) Quasi-projective curves, and dominant rational maps.

(iii) Function fields of dimension 1 over k, and k-homomorphisms.

Proof. The functors from (i) to (ii) (taking the dominant morphism and mapping it to its corresponding rational map) and (ii) to (iii) ($Y \to K(Y)$) are already known. We need, then, a functor from (iii) to (i).
Let us have some function field K. Associate with it its nonsingular projective curve C_K. A homomorphism $K_1 \to K_2$ induces (by the equivalence of (ii) to (iii)) a rational map of $C_{K_1} \to C_{K_2}$. This means, in particular, that we have a

morphism $\varphi : U \to C_{K_2}$, where $U \subseteq C_{K_1}$ is open. Therefore (finite complement topology) this extends to a morphism $\bar{\varphi} : C_{K_1} \to C_{K_2}$. By uniqueness, it is immediate to verify that this respects compositions, so $K \mapsto C_K$ is a functor from (iii) to (i). It is inverse to the functor given by (i) to (ii) to (iii), so we have an equivalence of categories.

$\square$

6 An (Important) Example

This is taken from exercise 6.2 in Hartshorne, and is intended to show that not all curves are birationally equivalent. Let us look at the curve $y^2 = x^3 - x$:

This curve is easily shown to be nonsingular. Now, notice the following facts, about rational curves not isomorphic to $\mathbb{P}^1$:

Let us have such a curve, X. Then:

- X is isomorphic to $\mathbb{A}^1$ minus a finite number of points: This is straightforward from the fact that X is isomorphic to a proper open subset of $\mathbb{P}^1$.
- $A(X)$ is a UFD: Notice $X = \mathbb{A}^1 \setminus \{a_1, \ldots, a_n\}$ for some a_i's. Then each element of $A(X)$ may be written uniquely as $a(x - b_1)^{c_1} \cdot \cdots \cdot (x - b_n)^{c_n}$ for some integers c_i and some $b_i \neq a_1, ..., a_n$.

But now, returning to our curve Y, we have in $A(Y)$ that $x|y^2$, but y and x are irreducible (I will not prove it here, but in the book Hartshorne show how to prove this - it is not very hard once you know what a norm is), so y is not a unit times x. This implies that $A(Y)$ is not a UFD, and hence not a rational curve!