Algebraic geometry seminar

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Curves

Throughout this lecture C is a smooth curve.

Invertible sheaves on curves

Definitions

1. Let $P \in C$. Then $\mathcal{O}_{C,P}$ is a DVR, denote its valuation by $ord_P : k(C) \to \mathbb{Z}$.

2. Let $D = \sum_P n_P P$ be a divisor on C. Define $\mathcal{O}_C(D)$ to be the sheaf of $\mathcal{O}_C$-modules given by

$$\mathcal{O}_C(D)(U) = \{f \in k(X) \mid (\forall P \in U)(ord_P(f) + n_P \geq 0)\}$$

3. Ler $f \in k(C)$. Define $div(f) = \sum ord_P(f) \cdot P$. This is a divisor since it is a divisor if C is affine.

4. We will say 2 divisors D_1, D_2 are *linearly equivalent* ($D_1 \equiv D_2$) if exist $f \in k(C)$ s.t. $D_1 + div(f) = D_2$. This is an equivalence relation.

5. Let $\mathcal{L}$ be an invertible sheaf on C, $P \in C$ and $f \in \mathcal{L}(U)$ (not necessary $P \in U$). Define

$$ord_P(s) = ord_P(\psi(s))$$

where $\psi : \mathcal{L}|_V \to \mathcal{O}_C|_V$ for some $V \subseteq U$. This definition is independent of the choice of ψ, V. Indeed, let $\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{O}_C|_U$ for $i = 1, 2$. Then $\psi_1^{-1} \circ \psi_2 : \mathcal{O}_C|_{U_1 \cap U_2} \to \mathcal{O}_C|_{U_1 \cap U_2}$ is an isomorphism of $\mathcal{O}_C|_{U_1 \cap U_2}$ modules so $\psi_1(s)$ and $\psi_2(s)$ differ by multiplication by a unit, so $ord_P(\psi_1(s)) = ord_P(\psi_2(s))$. Simillarly to the structure sheaf, define

$$div(s) = \sum ord_P(s) \cdot P$$

Lemma Let $V \subseteq U \subseteq C$ open.

1. The retriction maps $res_{U,V} : \mathcal{O}_C(U) \to \mathcal{O}_C(V)$ are injective.
2. If $f \in \mathcal{O}_C(V)$ with $ord_P(f) \geq 0$ for all $P \in U \backslash V$ then f is in the image of $res_{U,V}$.
3. If $\mathcal{L}$ is an invertible sheaf of $\mathcal{O}_C$-modules and $f \in \mathcal{L}(V)$ with $ord_P(f) \geq 0$ for all $P \in U \backslash V$ then f is in the image of $res_{U,V}$.

Proof In a general ringed space $(X, \mathcal{O})$, for $f \in \mathcal{O}(U)$ the set $\{P \in X | f_P \in \mathcal{O}_P^*\}$ is open since if g_p is the inverse of f_P then exist a nieghborhood U of P s.t. $g_Q \cdot f_Q = 1$ for all $Q \in U$. Thus,

1. is true for any irreducible variety. Indeed, if $s \in \mathcal{O}_C(U)$ then $\{P \in U | s(P) = 0\} = \left\{P \in U | s_P \neq \mathcal{O}_{C,P}^*\right\}$ is closed so if it contains V it contains U (because V is dense in U).
2. This follows from
$$ord_P(f) = \min_{t^r f \in \mathcal{O}_{C,P}} r$$
3. Locally we can assume $\mathcal{L} \cong \mathcal{O}_C$ and then this is just 2. The local sections we would get will agree on V and thus can be glued together.

Theorem For a sheaf $\mathcal{F}$ on C TFAE:

1. $\mathcal{F} \cong \mathcal{O}(D)$ for a divisor D.
2. $\mathcal{F}$ is a subsheaf of $\mathcal{K}(C)$.
3. $\mathcal{F}$ is invertible.

And $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$ iff $D_1 \equiv D_2$.

Proof

$\underline{1 \Rightarrow 2}$: From definition of $\mathcal{O}(D)$.

$\underline{2 \Rightarrow 3}$: If $\mathcal{F}$ is fractional ideal and $P \in C$ then $\mathcal{F}_P \subseteq k(C)$ is a fractional ideal so $\mathcal{F}_P = t^d \mathcal{O}_{C,P}$. Taking $U \subseteq C$ s.t. $ord_Q(t) = 0$ for all $Q \in U \backslash \{P\}$ we get that $\mathcal{F}|_U = t^d \mathcal{O}_C|_U \cong \mathcal{O}_C|_U$ so $\mathcal{F}$ is invertible.

$\underline{3 \Rightarrow 1}$: Pick some non zero section $s \in \mathcal{F}(U)$ (U is arbitrary open subset of C). Let $D = div(s) = \sum n_P P$. We want to define an isomorphism $\varphi : \mathcal{O}(-D) \xrightarrow{\sim} \mathcal{F}$. Let $f \in \mathcal{O}(-D)(U')$ then $fs \in \mathcal{F}(U \cap U')$ has

$$ord_P(fs) = ord_P(f) + ord_P(s) = ord_P(f) + n_P \geq 0$$

for every $P \in U'$. So, by the lemma, there exist a unique $s' \in \mathcal{F}(U')$ with $s'|_{U \cap U'} = fs$, define $\varphi(f) = s'$. The easy verification that φ is a morphism of $\mathcal{O}_C$-modules is left as an exercise (one need to check that φ commutes with restriction maps and that $\varphi(U)$ is $\mathcal{O}_C(U)$-linear). We can check that it is an isomorphism on the level of stalks where φ_P is a non-constant morphism of rank 1 free $\mathcal{O}_{C,P}$-modules.

If $D_1 \equiv D_2$ then $D_1 + div(f) = D_2$ for some $f \in k(C)$ and then

$$\mathcal{O}_C(D_1) \to \mathcal{O}_C(D_2)$$
$$g \mapsto fg$$

is an isomorphism. Conversely, let $\varphi : \mathcal{O}_C(D_1) \xrightarrow{\sim} \mathcal{O}_C(D_2)$ and $f \in \mathcal{O}_C(D)(U)$ arbitrary non zero section. First, note that φ induces isomorphism on stalks and

$$\mathcal{O}_C(D)_P = t_P^{-ord_P(D)}\mathcal{O}_{C,P}$$

so

$$ord_P(f) + ord_P(D_1) = ord_P(\varphi(f)) + ord_P(D_2)$$

We can construct an isomorphism

$$\varphi' : \mathcal{O}_C \to \mathcal{O}_C(D_1 - D_2)$$
$$g \mapsto g\frac{f}{\varphi(f)}$$

where $g\frac{f}{\varphi(f)} \in \mathcal{O}_C(D_1 - D_2)(V)$ since

$$\begin{aligned} ord_P\left(g\frac{f}{\varphi(f)}\right) + ord_P(D_1 - D_2) = \\ ord_P(g) + ord_P(f) - ord_P(\varphi(f)) + ord_P(D_1) - ord_P(D_2) = \\ ord_P(g) + (ord_P(f) + ord_P(D_1)) - (ord_P(\varphi(f)) + ord_P(D_2)) = \\ ord_P(g) \geq 0 \end{aligned}$$

and this is an isomorphism since it induces isomorphism on stalks. Now, it is easy enough to see that $\mathcal{O}_C(D_1 - D_2) = \varphi'(1) \cdot \mathcal{O}_C$ so $D_1 - D_2 = div(\varphi'(1))$.

Principal parts and Cousin problem

Definitions

1. Let $\mathcal{F}$ be a coherent sheaf on C. Define $Rat(\mathcal{F}) = \varinjlim_{U \neq \emptyset} \mathcal{F}(U)$ and $\underline{Rat}(\mathcal{F})$ be the constant sheaf associated with $Rat(\mathcal{F})$.

2. Let $\mathcal{F}$ be an invertible (locally free coherent) sheaf on C and let $P \in C$. There is a natural embedding $\mathcal{F}_P \to Rat(\mathcal{F})$, define $Prin_P(\mathcal{F}) := {}^{Rat(\mathcal{F})}/_{\mathcal{F}_P}$ *the group of principal parts of* $\mathcal{F}$.

3. Define the sheaf $Prin(\mathcal{F})$ on C by

$$U \mapsto \bigoplus_{P \in U} Prin_P(\mathcal{F})$$

with obvious restriction maps.

Lemma There is an exact sequence of sheaves

$$0 \to \mathcal{F} \to \underline{Rat}\,(\mathcal{F}) \xrightarrow{\alpha} Prin\,(\mathcal{F}) \to 0$$

and $Rat\,(\mathcal{F})$, $Prin\,(\mathcal{F})$ are flabby.

Proof First, let us define α. For $f \in Rat\,(\mathcal{F})$ define $\alpha\,(U)\,(f) = \sum_{P \in U} (f \mod \mathcal{F}_P)$. This is well defined because $f \in \mathcal{F}_P$ on V, where $f \in \mathcal{F}\,(V)$ and $U \backslash V$ is finite. This is exact because it is exact on stalks. $\underline{Rat}\,(\mathcal{F})\,, Prin\,(\mathcal{F})$ are obviously flabby.

Definition Taking global sections of the above exact sequence we get the exact

$$0 \to \Gamma\,(\mathcal{F}) \to Rat\,(\mathcal{F}) \xrightarrow{\alpha(C)} \bigoplus_{P \in C} Prin\,(\mathcal{F})$$

a sheaf is called *ordinary* if $\alpha\,(C)$ is surjective.

Example

1. Every sheaf on an affine curve is ordinary.
2. The structure sheaf of a projective curve C is ordinary iff $C \cong \mathbb{P}^1$.

Proof

1. $\underline{Rat}\,(\mathcal{F})$ and $Prin\,(\mathcal{F})$ are quasi-coherent, so this is follows from the general fact about quasi-coherent sheaves on affine variety.
2. Let $C = \mathbb{P}^1$. It is sufficient to show that $Prin_P\,(\mathcal{O}_{\mathbb{P}^1}) \in im\left(\alpha\left(\mathbb{P}^1\right)\right)$ for every $P \in \mathbb{P}^1$ and by projective change of ccoordinates we can assume $P = [0:1]$. Let $s \in Prin_P\,(\mathcal{O}_{\mathbb{P}^1})$ and let $t = \frac{x_0}{x_1}$, then

   $$s = \sum_{1 \le i \le n} a_i t^i \left(\mod \mathcal{O}_{\mathbb{P}^1,P}\right)$$

   so $s = \alpha\left(\sum_{1 \le i \le n} a_i t^i\right)$.
   Conversely, let C be a curve s.t. $\mathcal{O}_C$ is ordinary. Let $P_0, P_1 \in C$ be 2 distinct points. By the ordinarity of $\mathcal{O}_C$, exist $f_0, f_1 \in Rat\,(\mathcal{F})$ s.t. $\alpha\,(f_i)$ $Prin_P\,(\mathcal{O}_C)$
   TODO: finish this part or ommit it.

Lemma Let $\mathcal{L} = \mathcal{O}\,(D)$ be an invertible sheaf on C. If for any effective divisor E

$$\dim_k\left(\Gamma\left(\mathcal{O}\left(D+E\right)\right)\right) = \dim_k\left(\Gamma\left(\mathcal{O}\left(D\right)\right)\right) + \deg E$$

then $\mathcal{L}$ is ordinary.

Proof $\Gamma\left({}^{\mathcal{L}(E)}/_{\mathcal{L}}\right)$ is isomorphic to a $\deg E$ dimensional subspace of $Prin\left(\mathcal{L}\right)$. α embeds ${}^{\Gamma(\mathcal{L}(E))}/_{\Gamma(\mathcal{L})}$ to it, so if

$$\dim_k \frac{\Gamma\left(\mathcal{L}\left(E\right)\right)}{\Gamma\left(\mathcal{L}\right)} = \dim_k \Gamma\left(\frac{\mathcal{L}\left(E\right)}{\mathcal{L}}\right) = \deg E$$

α is surjective.

Introduction to cohomology of curves

Definition Let $\mathcal{F}$ be a quasi-coherent sheaf over a variety X. Let

$$0 \to \mathcal{F} \to \mathcal{F}_1 \to \mathcal{F}_2 \to ...$$

a flabby resolution of $\mathcal{F}$ (i.e. all $\mathcal{F}_i$ are flabby and the sequence is exact). We will define $H^i\left(X, \mathcal{F}\right)$ to be the i-th homology of the chain complex

$$0 \to \Gamma\left(\mathcal{F}_1\right) \to \Gamma\left(\mathcal{F}_2\right) \to \ldots$$

Remark To make this definition well defined we need to show that it doesn't depend on the flabby resolution we pick and that such a flabby resolution always exist. We would postpone it to later.

Lemma For an invertible sheaf $\mathcal{F}$ on a complete smooth curve C:

1. $H^0\left(C, \mathcal{F}\right) \cong \Gamma\left(C, \mathcal{F}\right)$.
2. If $\mathcal{F}$ is flabby, $H^i\left(C, \mathcal{F}\right) = 0$ for all $i > 0$.
3. $H^i\left(C, \mathcal{F}\right) = 0$ for all $i > 1$.

Proof

1. Since $\Gamma\left(C, \cdot\right)$ is left exact, we get exact

   $$0 \to \Gamma\left(C, \mathcal{F}\right) \to \Gamma\left(C, \mathcal{F}_1\right) \to \Gamma\left(C, \mathcal{F}_2\right)$$

   so the zero homology of

   $$0 \to \Gamma\left(C, \mathcal{F}_1\right) \to \Gamma\left(C, \mathcal{F}_2\right)$$

   is $\Gamma\left(C, \mathcal{F}\right)$.

2. We have the flabby resolution

   $$0 \to \mathcal{F} \to \underline{Rat}\left(\mathcal{F}\right) \to Prin\left(\mathcal{F}\right) \to 0$$

   so $H^i\left(\mathcal{F}\right) = 0$ for $i > 1$.

3. We have the flabby resolution

$$0 \to \mathcal{F} \xrightarrow{Id} \mathcal{F} \to 0$$

so the chain complex is

$$0 \to \Gamma(\mathcal{F}) \xrightarrow{Id} \Gamma(\mathcal{F}) \to 0$$

and all its high homology groups vanish.

Definition We will construct a *canonical flabby resolution* by

$$C^0(\mathcal{F}) = \mathcal{F}$$

$$D^i(\mathcal{F}) = D\left(C^i(\mathcal{F})\right)$$

where $D(\mathcal{F})$ is the sheaf of discontinuous sections of $\mathcal{F}$ and C^{i+1} defined inductively as the cokernel in the exact sequence

$$0 \to C^i(\mathcal{F}) \to D^i(\mathcal{F}) \to C^{i+1}(\mathcal{F}) \to 0$$

then

$$0 \to \mathcal{F} \to D^0(\mathcal{F}) \to D^1(\mathcal{F}) \to \dots$$

is a flabby resolution.

Lemma Let

$$0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$$

exact sequence of sheaves, then exist a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r, "\alpha"] \arrow[d] & \mathcal{F}_2 \arrow[r, "\beta"] \arrow[d] & \mathcal{F}_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D(\mathcal{F}_1) \arrow[r, "D(\alpha)"] \arrow[d] & D(\mathcal{F}_2) \arrow[r, "D(\beta)"] \arrow[d] & D(\mathcal{F}_3) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1' \arrow[r, "\alpha'"] \arrow[d] & \mathcal{F}_2' \arrow[r, "\beta'"] \arrow[d] & \mathcal{F}_3' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns.

Proof $D(\alpha)$ and $D(\beta)$ are defined as composition with the induced map on stalks, i.e.

$$D(\alpha) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$$

$$(P \mapsto \sigma_P) \mapsto (P \mapsto \alpha_P(\sigma_P))$$

the commutativity of the first 2 rows is obvious. The exactness of the second row can be checked on the level of stalks which are the same as the stalks of the first row. The construction of α' is due to the universal property of co-kernels: If there exist a map $D(\mathcal{F}_1) \to \mathcal{F}_2'$ s.t. composed with $\mathcal{F}_1 \to D(\mathcal{F}_1)$ gives the zero morphism, then it factors uniquely through $D(\mathcal{F}_1) \to \mathcal{F}_1$. Finally, to prove exactness of the last row, consider the long induced sequence in homology induced by the short exact sequence of chain compleces

$$0 \to \mathcal{F}_\bullet \to D(\mathcal{F}_\bullet) \to \mathcal{F}_\bullet' \to 0$$

Proposition Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be an exact sequence of sheaves on X. Then we have the long exact sequence in cohomology

$$0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}) \to$$

$$\to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{F}) \to \dots$$

Proof Let

$$0 \to \mathcal{F}_i \to D^0(\mathcal{F}_i) \to D^1(\mathcal{F}_i) \to \dots$$

be the canonical flabby resolution of the sheaf $\mathcal{F}_i$. We get the commutative diagram of sheaves with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0] \arrow[d] & \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d] & D^0(\mathcal{F}_1) \arrow[r] \arrow[d] & D^1(\mathcal{F}_1) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{F}_2 \arrow[r] \arrow[d] & D^0(\mathcal{F}_2) \arrow[r] \arrow[d] & D^1(\mathcal{F}_2) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{F}_3 \arrow[r] \arrow[d] & D^0(\mathcal{F}_3) \arrow[r] \arrow[d] & D^1(\mathcal{F}_3) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where exactness follows from inductively applying the lemma. Taknig global sections we get the commutative diagram of $\mathcal{O}_X(X)$ modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0] \arrow[d] & \\
0 \arrow[r] & \Gamma\left(X, D^0(\mathcal{F}_1)\right) \arrow[r] \arrow[d] & \Gamma\left(X, D^1(\mathcal{F}_1)\right) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Gamma\left(X, D^0(\mathcal{F}_2)\right) \arrow[r] \arrow[d] & \Gamma\left(X, D(\mathcal{F}_2)\right) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Gamma\left(X, D^0(\mathcal{F}_3)\right) \arrow[r] \arrow[d] & \Gamma\left(X, D^1(\mathcal{F}_3)\right) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

since $D^j(\mathcal{F}_i)$ are flabby for all j the columns of this diagram are exact so, by the snake lemma, we get the desired long exact sequence in cohomology.