The Picard Group And some more about sheaves

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Outline

Coherence

The story Reminders Locally Free Quasi-Coherence Coherence

The Picard Group

Invertible sheaves The Picard group Divisors Examples

Bird’s view

We want to define invariants for algebraic varieties, to be able to test whether varieties are isomorphic. To do that we will study the intrinsic geometry of a variety; In this lecture we will study the line bundles of the variety, that give rise to an invariant called the Picard group. To define it for a genral variety we define coherent modules. In the next lecture we shall define a differential form on an algebraic variety, and use it to give an intrinsic definition of the tangent and cotangent bundles on a variety. Then constructions from differential geometry will lead to additional numeric invariants. The Picard group will also later be seen to be a particular case of a cohomology group.

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The story

X an affine variety, A = k[X]. For every A-module M, attach a module

M ̃

over the sheaf of rings O

x

. This defines an equivalence of the category of A-modules with the category of quasi-coherent O

x

-modules.

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Reminders and philosophical addenda

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, its étalé space, such that there is a local homeomorphism π : Y → X. Indeed, we define the topology on Y by the following basis. Note that an element of Y is a germ. For every U open in X and for every s ∈ F(U), take the set of all germs of s in U to be in the basis.

Reminders II

Let X be a topological space with a sheaf of rings O

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▶ F × F → F ▶ O

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(U)-module morphism = (pre-)sheaf morphism + module morphism on every open U ⊆ X. Examples:

▶ If X is a single point then this is just a usual module. ▶ If F is the constant sheaf Z we get a sheaf of abelian groups.

Why Bother?

For a reason to study O

x

-modules, consider P1 over C, the

Riemann sphere.

Next week’s tangent sheaf has sections given by tangent vectors at each point (varying ”nicely”). Then the right description of this sheaf is not just as a ring, since we can multiply by the corresponding regular functions; we want to capture its structure as also an that O x

-module. this sheaf To is locally

help motivate ∼

= O

X

further development, but not globally, since notice

it does not even have constant global sections by the hairy ball theorem (Every continuous tangent field on S2 has a zero). Thus this is an example of an invertible sheaf.

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with u ◦ i = 0 ”largest”: For every object Z there is a bijection g ↦→ i ◦ g between Hom(Z,Ker(u) → {f ∈ Hom(Z,X) : u ◦ f = 0}

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category] ▶ Letting Im(u) := Ker(Coker(u)),Coim(u) := Coker(ker(u)),

The path X → Coim(u) → Im(u) → Y is the same as u : X → Y . [abelian category]

Locally-Free Module

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-modules. (And every commutative ring with identity satisfies IBN, the invariant basis number.)

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Useful Preliminaries

We will use these later.

▶ Given a continuous function f : X → Y between topological

spaces, we have the pushforward of a sheaf f

∗

F(U) := F(f −1(U)) which is also a sheaf. ▶ Given a sheaf G on Y, we define the pullback by the

sheafification of f ∗G(U) = lim→G(V) over all V ⊃ f (U). Note this is necessary since f might not be an open map. ▶ in particular, stalks are given by F

x

= i∗ x

F for the embedding i

x

: {x} → X. ▶ Let p ∈ P1, then we define the ideal sheaf O(−p) of this

point to be the subsheaf of O given by:

O(−p)(U) = {f ∈ O(U)|f (p)=0}.

▶ More generally, the ideal sheaf of a closed subvariety of an

algebraic variety has the sections of the structure sheaf which vanish on the closed subset. ▶ The twisting sheaf O(d) is defined on Pn by

O(d)(U) = k[

U] ̃

d

= {

p q

|deg(p) − deg(q) = d,q|

U ̃

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Quasi-Coherent modules

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Affine Case II

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∗

j∗F = lim

→

f −nF, where j

∗

F(V) := F(U ∩ V). (Kempf uses ad-hoc notation for this.)

Projective Case

In the projective case, too, we have a global description of quasi-projective sheaves. For a graded module M, define a quasi-coherent sheaf U is (

M ̃

An+1

U))

̃

0

on Pn, U ̃

denoted

M ̃

Pn

, as follows. Its section on is the lift of U to the cone An+1 \ {0}. Theorem

▶ M ↦→

(

, where

M ̃

Pn

is an exact functor ▶ Every quasi-coherent sheaf F on Pn is of the form

M ̃

Pn

, and for coherent sheaves we may take M finitely generated. Moreover, concretely M = ⊕n ≥ 0Γ(F(n)).

Proof

Proof. Let F ∈ QCoh(Pn), and An+1 \ {0} → Pn are j,π respectively. One quasi-coherent may compute sheaf π

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n≥0

Γ(F(n)), then M /M has only negative degrees and its associated sheaf is (M M M ̃ ̃

M ̃

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Pn (U) =

= M ̃ = 0 Pn j (as . 8

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M ̃

for M finitely generated; pick d > the degrees of all generators, then M

≥d

is generated by M

d

, so

M ̃

≥d

=

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By def’n of finitely generated, A⊕k[−d] → M is surjective.

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Proof. If M is finitely generated then it is in particular finitely presented, so

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M ̃

is coherent. Cover X by principal M finitely ̃

f

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-module by itself, is also locally free of the same rank. (Since direct sum commutes with Hom) Let L be locally free of rank 1. Then in particular the same is true for L ∨. Proof. the map

L ∨ ⊗

O

X

L → O

X

given by (φ,m) ↦→ φ(m) is an isomorphism. Indeed, isomorphism is a local that L

property ∼

= O

X

∼ = so L it ∨ is enough to check and so we’re done.

locally, where we know

Ignore this I

We will now see in two steps that the dual of L is its inverse with respect to ⊗. Lemma 1. There is an isomorphism

L

∨

⊗

O

X

L

∼ = Hom

O

X

(L,L )

. Theorem (Giving lemma 1) Let F,G,H be O

X

-modules where F is locally-free, then there is an isomorphism

Hom

O

X

(F,G) ⊗

O

X

H → Hom

O

X

(F,G ⊗

O

X

H )

Ignore this II

Lemma 2. The canonical homomorphism

O

X

→ Hom

O

X

(L ,L)

that sends a section s ∈ O

X

(U) to scalar multiplication by F is an isomorphism. Proof. This is a local question, so we may assume L = O

X

Ignore this III

Lemma 1.

L

∨

⊗

O

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Together we obtain the desired:

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We define an invertible sheaf to be a locally-free sheaf of rank 1. This is justified since the converse is also true: if M ⊗

O

X

N

∼ = O

X

, then M is locally-free of rank 1. (Proof omitted. See

Link

For quasi-coherent sheaves, and 19.11 in

Link

for the general case.)

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Coherence

The story Reminders Locally Free Quasi-Coherence Coherence

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The Story I

We define the Picard Group of a general variety as the isomorphism classes of ”line bundles”.

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We define the Picard Group of a general variety as the isomorphism classes of ”line bundles”. The Picard group is an invriant of isomorphism, and so is useful to show that two varieties are not isomorphic. For example, it can be proven that the Picard group of a smooth cubic curve in P2 of a smooth cubic curve in P2 is nontrivial, and so it is never isomorphic to P1.

The Story II

We follow an idea from number theory to represent the Picard group as a quotient of invertible fractional ideals by its subgroup of principal ideals, using a short exact sequence.

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Invertible sheaf

Let L be an invertible sheaf. Set L ⊗−n := (L ∨)

n

. Then

L

⊗n

⊗

O

X

L

⊗m ∼

= L

⊗n+m

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X

modules form an abelian group under this operation, called the Picard group. The neutral element is the class of O

X

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be Let X be an irreducible variety. Let K

X

the constant sheaf equal to the field k(X) of rational functions on X,

k(X) =

⋃

U∈top(X)

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X

. (The definition of the sheaf of rational functions is less trivial for schemes, where there might be zero divisors.) A sheaf of fractional ideals I is a coherent subsheaf of K

X

. The invertible sheaves of fractional ideals form a group IFI(X) under multiplication. There is a subgroup P(X) of IFI consisting of the principal ideals f · O

X

. It is important because of the following theorem.

Exact sequence of Picard group

Theorem There is a short exact sequence

0 → P(X) → IFI(X) →ψ Pic(X) → 0,

where ψ sends an invertible fractional ideal to its isomorphism class.

Proof. ker(ψ) = P(X): By definition ψ((I)) = 0 iff I

∼ = O

X

. Let f be the image of 1 under Conversely f · O X

this ∼ = O

isomorphism, X

, so indeed then I = f · O

X

, so I ∈ P(X). ker(ψ) = P(X). All left is showing ψ is surjective. Let L be an invertible sheaf, then I ∼ = L. we construct Let σ ∈ L(V) an invertible for V open fractional and dense. ideal I (Note such we that

cannot simply take I(U) = {f V = X ∈ k(X)|f as there may be · σ = τ ∈ L(U)}

no ∼

= global L sections.) Then

via multiplication by σ.

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Motivation

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It is a classical question to determine the sets of zeros and poles of rational (”meromorphic”) functions on a given variety. We want to determine the local/global relationship: given zero/pole configurations on an open covering, are these configurations induced from a global rational function? (E.g. in complex analysis: Mittag-Leffler, Weierstrass product)

What is exactly a “configuration”?

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)× — u is a unit, so f and g should be thought of having the same zero/pole configuration. On smooth varieties, the two notions are equivalent.

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On singular varieties we need to distinguish between the two:

▶ codimension-1 subvarieties = Weil divisors ▶ varieties which can locally be defined by one equation =

Cartier divisors

We will also see a correspondence between divisors and invertible sheaves, which are by definition line bundles;

We will also see a correspondence between divisors and invertible sheaves, which are by definition line bundles; Thus much of algebraic geometry studies an arbitrary variety by analyzing its codimension-1 subvarieties and the corresponding line bundles.

Weil Divisors

Definition Let X be an irreducible variety. An irreducible Weil divisor on X is an irreducible subvariety D with dimD = dimX − 1. The group Div

W (X) divisors. is the free abelian group Thus a Weil divisor is generated of the form by D =

∑ the n i irreducible

D

i

for n

i

∈ Z. A divisor D is effective if n

i

≥ 0∀i.

Cartier Divisors

Let X be an irreducible variety. The Cartier divisor group, Div

C

(X), consists of subvarieties locally given by a nonzero rational function defined up to multiplication by a nonvanishing function. Definition An element of Div

C

(X) is given by an open cover U

i

together with rational functions f

i

= 0 on U

i

, such that on the intersection U

i

∩ U

j

we have f

i

= phi

ij

f

j

for some φ

ij

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Claim. The Cartier divisors are equivalent to the category of invertible sheaves. Indeed, match a divisor D to the sheaf defined locally by f

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which is invertible, and conversely an invertible sheaf has local data defining a divisor. Claim. There is an exact sequence Γ(K) → Div

C

(X) = IFI(X) → Pic(X). Indeed, the latter morphism is just L → [L, and it is surjective: Choose a local trivialization of a given isomorphism L sections of K∗/O∗ ⊗

O

x isomorphic K ∼ = line K. bundle Now for L to O X

, then we have an the kernel: it consists of , which is the set of nowhere zero rational functions.

Reduction in smooth case

We saw in a previous lecture that for smooth varieties the local rings O

X,x

are regular local rings. Theorem (Auslander, Buchsbaum, 1959) Any local regular ring (R,m) is UFD.

(Sketch to give a sense of the difficulty).

▶ Let k = R/m, then the completion of R is isomorphic to

k[[x

1

, ..,x

d

]] where d = dimR. ▶ For Noetherian local rings, if the completion is UFD then the

ring itself is UFD. ▶ k[[x

1

, ..,x

d

]] is UFD.

Theorem Let X be a locally factorial variety, meaning that all O

X,x

are regular local rings. Then Div

C

(X) = Div

W

(X).

Proof of Weil = Cartier

Consider the map Div

W

(X) given by D ↦→ O(−D) ⊆ O

X

(X) → Div

C ⊆ K, where O(−D) denotes the sheaf of functions vanishing on D.

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▶ O(−D) is locally principal. In a UFD, every prime ideal of

height one is principal: O(−D) is locally induced by a prime ideal of height 1 by definition, so when we pass to the stalk it is induced by (f

x

) and O(−D) only differ on components that do not pass x (as they agree on the stalk), which can only happen on finitely many other components, so after shrinking our local neighborhood we can have (f

x

) for some f

x

∈ K. Now (f

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) agreeing with O(−D) on some neighborhood. ▶ The map D ↦→ O(−D) is injective: enough to see that

nD ↦→ 0 does not hold for n > 0, but the image is O(−D)n = 0.

Proof Cont.

▶ The map is surjective: Assume first L ⊂ O, we want to find

a Weil divisor D ↦→ L . We can asssume that we know this for all bigger L , L ⊂ L ⊂ O

X

. Working locally, there is some L = (f ). Let D be an irreducible component of D(f ), O(−D) ⊃ L. Let (φ) = L, then φ−1L ⊃ L comes from some D , and we have D + D ↦→ L.

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. Working locally, there is some L = (f ). Let D be an irreducible component of D(f ), O(−D) ⊃ L. Let (φ) = L, then φ−1L ⊃ L comes from some D , and we have D + D ↦→ L. Finally, in the general case where we don’t assume L ⊂ O, we still have locally L first case we know that = there (

g h

) are for D some ↦→ α,D g,h ∈ ↦→ O(U). β, so By the

all in all D − D ↦→ L .

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Pic(An)

Example

Pic(A

n

)=0

. Indeed: Proposition. Let A be a UFD. Then every polynomial ring A[x

1

, ...,x

n

] is also a UFD, and for every multiplicative set the localization S−1A is a UFD. Thus An is locally factorial, and we conclude by noting that every variety of codimension 1 is given by a single polynomial, so is principal. Cn is a contractible manifold, and hence has no nontrivial topological vector bundles, which is analogous to this result.

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, ...,x

n

Pic(Pn)

Example

Pic(P

n

) = Z

. Concretely, this means that every codimension-1 subvariety of Pn is defined by the vanishing of a single homogeneous polynomial. Indeed, O(d

1

) ⊗ Z O(d

⊆ Pic(Pn) 2 )

∼ = O(d because +1+ d

we 2

). have

Moreover these are pairwise nonisomorphic: enough to show specific two are nonisomrphic, but O is not isomorphic to O(d) for negative d since there the global sections of O(d) vanish, as we saw. Now for the other inclusion, let D be of codimension 1, then there is homogeneous a homogeneous ideal polynomial of D, so O(−D)

p of degree ∼

= O(−d) d that by generates multiplication the

by p.