Definitions:

Let D be a dinisor and

- $\deg D = \sum n_i$
- If $\mathcal{F}$ is an invertible sheaf s.t. $\mathcal{F} \cong \mathcal{O}_{\mathcal{C}}(D)$ then define $\deg \mathcal{F} := \deg D$.
- If $\mathcal{F}$ is a coherent sheaf and $x \in V$ then we have a vector space $\mathcal{F}\mid_x = \mathcal{F}_x/m_x\mathcal{F}_x$.
- Define a sheaf k_P on C s.t. $k_P(U) = \begin{cases} k & , P \in U \\ 0 & , P \notin U \end{cases}$.

Lemma:

Sequence

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_C(D+P) \longrightarrow k_p \longrightarrow 0$$

Is exact.

Proof: Look at P :

$$0 \longrightarrow \mathcal{O}_C(D)_P \xrightarrow{\subseteq} \mathcal{O}_C(D+P)_P \longrightarrow k \longrightarrow 0$$

Here we have inclusion of $t^n\mathcal{O}$ in $t^{n-1}\mathcal{O}$
and at Q:

$$0 \longrightarrow \mathcal{O}_C(D)_Q \xrightarrow{\subseteq} \mathcal{O}_C(D+P)_Q \longrightarrow 0 \longrightarrow 0$$

Here we have nothing changed i.e. $t^n\mathcal{O}$

Lemma:

Let $\mathcal{F}$ be an invertible sheaf. Then $\dim \Gamma(C, \mathcal{F}) \le \deg \mathcal{F} + 1$

Proof:

Assume that$\Gamma(C, \mathcal{F}) \neq 0$. Then $\mathcal{F} \approx \mathcal{O}_{\mathcal{C}}(D')$ if D' is not an effective divisor then ther is $f \in \Gamma(C, \mathcal{F})$ s.t. $\mathrm{div} f + D' \ge 0$ and denote $\mathrm{div} f + D' = D$ but then we know $\mathcal{O}_{\mathcal{C}}(\mathrm{div} f + D') = \mathcal{O}_C(D')$ which means $\mathcal{F} \approx \mathcal{O}_{\mathcal{C}}(D)$ for an effective divisor D.

If $\deg \mathcal{F} = 0$ then $D = 0$, which means $\mathcal{F} \approx \mathcal{O}_C$. Hence $\Gamma(C, \mathcal{F}) = k$. And thus $\dim \Gamma(C, \mathcal{F}) = 1$.

If $\deg D > 0$. Then $D = P + D_1$ for some effective divisor D_1. Then we have exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}}(D_1) \longrightarrow \mathcal{O}_{\mathcal{C}}(D) \longrightarrow k_P \longrightarrow 0$$

Then in global sections we have an exact sequence:

$$0 \longrightarrow \Gamma\left(\mathcal{O}_{\mathcal{C}}\left(D_1\right)\right) \longrightarrow \Gamma\left(\mathcal{O}_{\mathcal{C}}\left(D\right)\right) \longrightarrow \Gamma\left(k_P\right)$$

Then $\dim \Gamma\left(\mathcal{O}_{\mathcal{C}}\left(D\right)\right) \leq \dim \Gamma\left(\mathcal{O}_{\mathcal{C}}\left(D_1\right)\right) + \dim \Gamma\left(k_p\right)$ but $\Gamma\left(k_p\right) = k$. And so we are done by induction.

□

Lemma:

If $\Gamma\left(C, \Omega_C \otimes M^{\vee}\right) = 0$ then $H^1\left(C, \mathcal{M}\right) = 0$.

Proof: Let $\mathcal{F} = \mathcal{M}\left(D\right)$ for some effective divisor D. Then

$$\Gamma\left(C, \Omega_C \otimes \mathcal{F}^{\vee}\right) = \Gamma\left(C, \left(\Omega \otimes \mathcal{M}^{\vee}\right)\left(-D\right)\right) \subset 0$$

Let $P \in C$. We have exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}\left(P\right) \longrightarrow \mathcal{F}\left(P\right)|_P \longrightarrow 0$ which gives us a differential

$$\delta_P : H^0\left(C, \mathcal{F}\left(P\right)|_P\right) \to H^1\left(C, \mathcal{F}\right)$$

We will look on how δ_c variates globally in C. Look at exact sequence:

$$0 \longrightarrow \pi_1^*\mathcal{F} \longrightarrow \pi_1^*\mathcal{F}\left(\Delta\right) \longrightarrow \pi_1^*\mathcal{F}\left(\Delta\right)|_{\Delta} \longrightarrow 0$$

Then taking π_{2*} we have differential:

$$\delta : \pi_{2*}\left(\pi_1^*\mathcal{F}\left(\Delta\right)|_{\Delta}\right) \to H^1\left(\mathcal{F}\right) \otimes_k \mathcal{O}_{\mathcal{C}}$$

We can see that the value of $\delta\left(c\right)$ is δ_c (because π_2 is isomorphism from Δ to C). Now $\mathcal{O}_{C\times C}\left(-\Delta\right)|_{\Delta} \approx \Omega_C$. Thus $\pi_{2*}\left(\pi_1^*\mathcal{F}\left(\Delta\right)|_{\Delta}\right)$ is just $\mathcal{F} \otimes_{\mathcal{O}_C} \Omega_C^{\otimes -1}$. Hence δ has dual

$$\delta^{\vee} : H^1\left(\mathcal{F}\right)^{\vee} \otimes_k \mathcal{O}_C \to \Omega_C \otimes_{\mathcal{O}_C} \mathcal{F}^{\vee}$$

which has to be zedo due to the lack of non-zero sections in $\Omega_C \otimes \mathcal{F}^{\vee}$.

By lemma (from the first hour) to show that $\mathcal{F}$ has zero first cohomology we need the following: By the previous paragraph we have exact sequence

$$0 \longrightarrow \Gamma\left(C, \mathcal{M}\left(E\right)\right) \longrightarrow \Gamma\left(C, \mathcal{M}\left(E\right)\right) \longrightarrow \Gamma\left(C, \mathcal{M}\left(E+P\right)|_P\right) \longrightarrow 0$$

Where E is an effective divisor. And hence we get that the restriction is quotient. Then we have $\dim \Gamma\left(C, \mathcal{F}\left(E+P\right)|_P\right) = 1$. And thus $\mathcal{M}$ is ordinary.

□

Proposition

- $H^1\left(C, \mathcal{F}\right)$ and $H^0\left(C, \mathcal{F}\right)$ are finite dimensional vector spaces over k.
- And $\dim H^0\left(C, \mathcal{F}\right) - \dim H^1\left(C, \mathcal{F}\right) = \deg\left(\mathrm{div}\mathcal{F}\right) + \left(1-g\right)$ where $g = \dim H^1\left(C, \mathcal{O}_{\mathcal{C}}\right)$ is genus of C.

Denote $\dim H^0\left(C, \mathcal{F}\right) - \dim H^1\left(C, \mathcal{F}\right) = \chi\left(\mathcal{F}\right)$ and call it Euler characteristic of $\mathcal{F}$.

Proof: For every invertible sheaf $\mathcal{F}$ there is a divisor D s.t. $\mathcal{M} \approx \mathcal{O}_{\mathcal{C}}(D)$.

Notice that $H^0(\mathcal{O}_{\mathcal{C}}(D)) = \Gamma(C, \mathcal{O}_{\mathcal{C}}(D))$ and then because of the inequality above we have that $H^0(\mathcal{O}_{\mathcal{C}}(D))$ is finite dimentional.

Now let $P \in C$ be some point.

Then we have an exact sequence of sheafs

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}}(D) \longrightarrow \mathcal{O}_{\mathcal{C}}(D+P) \longrightarrow k_P \longrightarrow 0$$

Which induces a long exact sequnce in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_{\mathcal{C}}(D)) \longrightarrow H^0(\mathcal{O}_{\mathcal{C}}(D+P)) \longrightarrow H^0(k_P)$$

$$\longrightarrow H^1(\mathcal{O}_{\mathcal{C}}(D)) \longrightarrow H^1(\mathcal{O}_{\mathcal{C}}(D+P)) \longrightarrow 0$$

The last cohomology is zero because $H^1(k_P) = 0$ becasue k_p is flabby (becasue id and zerp-morphism (to zero) are surjective).

Then $H^1(\mathcal{O}_{\mathcal{C}}(D))$ is finite dimentional iff $H^1(\mathcal{O}_{\mathcal{C}}(D+P))$ is fintie dimentional. Notice that in this case $\chi(\mathcal{O}_{\mathcal{C}}(D+P)) = \chi(\mathcal{O}_{\mathcal{C}}(D)) + 1$.

Then what do we have:

$$\chi(\mathcal{O}_{\mathcal{C}}(D)) = \deg D + \chi(\mathcal{O}_{\mathcal{C}}) =$$

$$= \deg \mathcal{O}_{\mathcal{C}}(D) + \dim \Gamma(C, \mathcal{O}_{\mathcal{C}}) - \dim H^1(C, \mathcal{O}_{\mathcal{C}}) =$$

$$= \deg D + 1 - g$$

Let $\mathcal{L}$ be such an invertible sheaf that $\deg \mathcal{L} > \deg \Omega_C$ then $\Gamma\left(C, \Omega_C \otimes_{\mathcal{O}_C} \mathcal{L}^{\otimes -1}\right) = 0$[1]

And by lemma ther exist a locally free coherent sheaf s.t. $H^1(C, \mathcal{M}) = 0$ and we are finished.

□

Serre duality:

$$H^1(\mathcal{O}_{\mathcal{C}}(D)) \cong H^0(\mathcal{O}_{\mathcal{C}}(K-D))$$

Riemann-Roch theorem:

Let $\mathcal{F}$ be an invertible sheaf then:

$$\dim H^0(\mathcal{O}_{\mathcal{C}}(D)) - \dim H^0(\mathcal{O}_{\mathcal{C}}(K-D)) = \deg(\mathrm{div}\mathcal{F}) + 1 - g$$

Which is an exact application of Serre's duality to the previous proposition.

[1] Because $\deg \Omega_C \otimes_{\mathcal{O}_C} \mathcal{L}^{\otimes -1} < 0$ and impling previous lemma.

Application 1:

- $\deg \Omega_C = 2g - 2$
- $\dim \Gamma(C, \Omega_C) = g$

Proof:

From Riemann-Roch we get

$$h^0(D) - h^0(K - D) = \deg D - g + 1$$

Now let $D = 0$ and then $h^0(K) = g$ i.e. $\dim \Gamma(C, \Omega_C) = g$.

Now letm $D = K$ and then $h^0(K) = \deg K - g + 2$ and thus $\deg K = 2 - 2g$

□

Application 2:

Let C be a smooth curve which is not complete. Then C is affine.

Proof:

Let $\overline{C}$ be the completion of C. $\overline{C} \backslash C$ is finite. Define a divisor $D = \sum_{i=1}^{m} n_i P_i$ we a free to choose n_i to be positive and large enough to promice existance of regular function $f : X \longrightarrow \mathbb{P}^1$ s.t. $f^{-1}(\infty) = \{P_1, \ldots, P_m\}$ and finite verywhere else. Why can we? Since n_i are large then $\deg D > 2g - 2$ we get $H^0\left(\overline{C}, \mathcal{L}(K - D)\right) = 0$. Then Riemann-Roch gives us $H^0\left(\overline{C}, \mathcal{L}(D)\right) = \sum n_i + 1 - g$. So there is some function of $\overline{C}$ which is meromorphic and has poles exactly at P_i of orders worst n_i and nowehere else.

Now we use the linear system defined by D to embed $\overline{C}$ into projective space. Then divisor D becomes a hyperplane section with n_i's being the intersection multiplicity. See Kempf 5.7.1 Then we have $C = \overline{C} \backslash \left(\overline{C} \cap H\right)$ for a hyperplane H. But removing hyperplane turns projective space to an affine space and we still habe X embedded as a closed subvariety.

□