Homework 5

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1. - Prove that $\sqrt{F}(\rho, \sum_i \lambda_i \sigma_i) \geq \sum_i \lambda_i \sqrt{F}(\rho, \sigma_i)$.
   - Prove that if the classical fidelity function F (over probability distributions) is concave, then so does the quantum fidelity (over density matrices).
   - Prove that $F(\rho, \sum_i \lambda_i \sigma_i) \geq \sum_i \lambda_i F(\rho, \sigma_i)$.

2. Prove that for every constant $\varepsilon > 0$, $Q_\varepsilon(IP) = \Omega(n)$.

In What follows you may us the following inequalities:

Sub-additivity : $S(AB) \leq S(A) + S(B)$, and,

Araki-Lieb inequality : $S(AB) \geq S(A) - S(B)$.

A fact about the Holevo quantity :

Suppose X is a random variable with $p_x = \Pr(X = x)$. Let ρ be any quantum encoding of X and denote $\rho_x = \rho(x)$. The Holevo quantity is $\chi(\rho) = S(\sum_x p_x \rho_x) - \sum_x p_x S(\rho_x)$. You may use the fact that if ρ is over quantum registers $A \otimes B$, then

$$\chi(\mathrm{Tr}_B(\rho)) \leq \chi(\rho).$$

3. Show an example where $I(A : B) > S(B)$. Prove that $I(A : B) \leq 2S(B)$.

4. Alice wants to communicate an arbitrary $x \in \{0,1\}^n$ to Bob. Alice and Bob communicate in rounds, in each round Alice (or Bob) applies a unitary transformation on his/her part and transmits a quibt to the other side, until at the end Bob measures his state and tries to infer x. The protocol is successful if for every $x \in \{0,1\}^n$ Bob succeeds with probability 1. Let n_A / n_B) be the number of messages sent by Alice / Bob respectively.

   (a) Show that for every n_A and n_B such that $n_A \geq \lceil n/2 \rceil$ and $n_A + n_B \geq n$ there exists a successful protocol with parameters n_A, n_B.

   (b) Show that in any successful protocol $n_A \geq \lceil n/2 \rceil$ and $n_A + n_B \geq n$.

   Hint: Follow how $S(\rho_i)$ and $\chi(\rho_i)$ change with i, where ρ_i is Bob's density matrix at round i.

   (c) Define a protocol as p-successful if for every $x \in \{0,1\}^n$ Bob succeeds with probability p. Prove a statement similar to item (b) for p-successful protocols.

5. Assume the same situation as Q4 except that now Alice and Bob may share unlimited pre-prepared entanglement. Prove that in any successful protocol $n_A \geq \lceil n/2 \rceil$.