Solution for Exercise no.1 Quantum Computation. Fall semester 2002.
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1

Let V, W be vector spaces. The tensor product $V \otimes W$ is defined as $SPAN\{v \otimes w \mid v \in V,\ w \in W\}$, where $\otimes(x, y) = x \otimes y$ is a product mapping linear in each of its two arguments.
The definition as given lacked the fact that all pairs are different, except if they fall into the linearity condition. That is, $v_1 \otimes w \neq v_2 \otimes w$, for $v_1 \neq v_2$ and $w \neq 0$.

1.1

$V \otimes W$ is a vector space as it is defined as a SPAN of $\{v \otimes w \mid v \in V,\ w \in W\}$. i.e. a span of a non-empty group.
We denote the addition neutral element as $0_{V \otimes W}$ and think of it as:
$0_{V \otimes W} = \sum_{i,j} 0 \cdot (v_i \otimes w_j)\ (= 0_V \otimes 0_W)$
It is immediate that for each $v \in V \otimes W,\ v + 0_{V \otimes W} = v$

1.2

Let $A = \{a_1, ..., a_n\}$ be a basis for V, and $B = \{b_1, ..., b_m\}$ a basis for W.
Let $\sum_k \gamma_k \cdot v_k \otimes w_k$ be a vector in $V \otimes W$, where $v_k \in V, v_k = \sum_{i=1}^n \alpha_{k_i} a_{k_i}$ and $w_k \in W, w_k = \sum_{j=1}^m \beta_{k_j} b_{k_j}$. Thus,

$$\begin{aligned} \sum_k \gamma_k \cdot v_k \otimes w_k &= \sum_k \gamma_k \cdot (\sum_{i=1}^n \alpha_{k_i} a_{k_i}) \otimes (\sum_{j=1}^m \beta_{k_j} b_{k_j}) = \\ &= \sum_k \gamma_k \sum_{i,j} \alpha_{k_i} \beta_{k_j} \cdot a_{k_i} \otimes b_{k_j} \end{aligned}$$

and we can conclude that $\{a_i \otimes b_j\}$ spans $V \otimes W$.
Assume by contradiction that there exist $\{\gamma_{i,j}\}$ not all zero such that

$$\sum_{i,j} \gamma_{i,j} \cdot a_i \otimes b_j = 0_{V\otimes W}$$

This implies that for each i_0:

$$a_{i_0} \otimes \sum_j \gamma_{i_0,j} \cdot b_j + \sum_{i \neq i_0, j} \gamma_{i,j} \cdot a_i \otimes b_j = 0_{V\otimes W}$$

Since A is a basis for V, the second sum have no elements of the form $a_{i_0} \otimes b_j$, so it must be that the first sum is equal to zero, namely, that $\sum_j \gamma_{i_0,j} \cdot b_j = 0_W$. Thus all $\gamma_{i_0,j}$ are zero as the b_j are linearly independent. We conclude that all $\{\gamma_{i,j}\}$ must be zero, yielding that all $\{a_i \otimes b_j\}$ are linearly independent. Altogether we get that $\{a_i \otimes b_j\}$ is a basis for $V \otimes W$.

1.3

Assuming V, W are inner product vector spaces we define

$$\langle v_1 \otimes w_1 | v_2 \otimes w_2 \rangle = \langle v_1 | v_2 \rangle \cdot \langle w_1 | w_2 \rangle$$

A natural extension of the definition, to include linear combinations from $V \otimes W$ is:

$$\langle (\sum_i \alpha_i (a_i \otimes b_i)) | (\sum_j \beta_j (a_j \otimes b_j)) \rangle = \sum_{i,j} \alpha_i^* \cdot \beta_j \langle a_i | a_j \rangle \langle b_i | b_j \rangle$$

Linearity in the second argument is immediate from the definition:

$$\begin{aligned} \langle v_1 \otimes w_1 | (\sum_j \beta_j (a_j \otimes b_j)) \rangle &= \sum_j \beta_j \langle v_1 | a_j \rangle \langle w_1 | b_j \rangle = \\ &= \sum_j \beta_j \langle v_1 \otimes w_1 | a_j \otimes b_j \rangle \end{aligned}$$

For anti symmetry, we use the anti symmetry of the inner product in V, W :

$$\begin{aligned}
\langle\sum_i \alpha_i \cdot v_i \otimes w_i|\sum_j \beta_j \cdot v_j \otimes w_j\rangle &= \sum_{i,j} \alpha_i^* \beta_j \cdot \langle v_i|v_j\rangle\langle w_i|w_j\rangle = \\
&= \sum_{i,j}(\alpha_i^* \beta_j)^* \cdot \langle v_j|v_i\rangle^* \langle w_j|w_i\rangle^* = \\
&= (\sum_{i,j}(\alpha_i^* \beta_j) \cdot \langle v_j|v_i\rangle\langle w_j|w_i\rangle)^* = \\
&= (\langle v_2 \otimes w_2|v_1 \otimes w_1\rangle)^*
\end{aligned}$$

For non negativity we look at:

$$\langle\sum_i \alpha_i \cdot a_i \otimes b_i | \sum_i \alpha_i \cdot a_i \otimes b_i\rangle$$

where $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_m\}$ are orthonormal bases of V, W. We can assume such a representation as we saw that if $\{a_1, ..., a_n\}$, $\{b_1, ..., b_m\}$ are bases of V, W then $\{a_i \otimes b_j\}$ is basis for $V \otimes W$.

$$\begin{aligned}
\langle\sum_i \alpha_i \cdot a_i \otimes b_i | \sum_i \alpha_i \cdot a_i \otimes b_i\rangle &= \sum_{i,j} \alpha_i^* \alpha_j \cdot \langle a_i|a_j\rangle\langle b_i|b_j\rangle = \\
&= \sum_i \mid \alpha_i \mid^2 \cdot \langle a_i|a_i\rangle\langle b_j|b_j\rangle = \sum_i \mid \alpha_i \mid^2 \geq 0
\end{aligned}$$

The above is equal to 0 if and only if $\alpha_i = 0$ for each i, implying that $\sum_i \alpha_i \cdot a_i \otimes b_i = 0_{V \otimes W}$

1.4

Assume that A, B are orthonormal basis. consider the inner product:

$$\langle a_i \otimes b_j | a_{i'} \otimes b_{j'}\rangle = \langle a_i|a_{i'}\rangle\langle b_j|b_{j'}\rangle = 1$$

yielding that $\{a_i \otimes b_j\}$ is an orthonormal group of vectors.

2

T is a linear transformation on $V \otimes W$. T_1, T_2 are two linear operators on V, W respectively, such that:

$$T(v \otimes w) = T_1(v) \otimes T_2(w)$$

Let $A = \{a_1, ..., a_n\}, B = \{b_1, ..., b_m\}$ be the basis for V, W respectively. Also let M_A be the matrix representing T_1 in basis A and M_B the matrix representing T_2 in basis B.

2.1

We want to express $M_{A\otimes B}$, the matrix representation of T in the basis $\{a_i \otimes b_j\}$. Specifically, the j column of $M_{A\otimes B}$ should express the coordinates of T applied on the j vector in a $V \otimes W$ basis. Since we express the basis of $V \otimes W$ as $\{a_i \otimes b_j\}$, it would be convenient to enumerate $M_{A\otimes B}$ columns using double indices, namely (k, l), where $k = 1...n$, $l = 1...m$ rather then a single index j. The (k, l) column is the $(k-1) * m + l$ column in the actual matrix representation $M_{A\otimes B}$. In a similar way we will enumerate $M_{A\otimes B}$ lines using a double index (i, j), where $i = 1...n, j = 1...m$.
Note that the dimensions of $M_{A\otimes B}$ implied by the indexing method above are nm x nm, which comply with the dimension of $V \otimes W$. To describe $M_{A\otimes B}$, we find its $(i, j), (k, l)$ coordinate.
By definition $M_{A\otimes B}(a_k \otimes b_l) = M_A(a_k) \otimes M_B(b_l)$. M_A, M_B are given in the bases A, B respectively, thus

$$\begin{aligned} M_A(a_k) \otimes M_B(b_l) &= \sum_{i=1}^{n}(M_A)_{i,k} \cdot a_i \otimes \sum_{j=1}^{m}(M_B)_{j,l} \cdot b_j = \\ &= \sum_{i=1..n,j=1..m} (M_A)_{i,k}(M_B)_{j,l} \cdot a_i \otimes b_j \end{aligned} \tag{1}$$

On the other hand the (k, l) column of $M_{A\otimes B}$ should express the coordinates of applying T on $a_k \otimes b_l$, thus

$$M_{A\otimes B}(a_k \otimes b_l) = \sum_{i=1..n,j=1..m} (M_{A\otimes B})_{(i,j)(k,l)} \cdot a_i \otimes b_j \tag{2}$$

So it must be that

$$(M_{A\otimes B})_{(i,j)(k,l)} = (M_A)_{i,k} \cdot (M_B)_{j,l} \tag{3}$$

Another way to represent $M_{A\otimes B}$, known as the **Kronecker product,** is to visualize $M_{A\otimes B}$ as a n x n block matrix, where each block is of size m x m, as follows. The (i, k) block is of the form $(M_A)_{i,k} \cdot M_B$. To see that this is consistent with (3) consider the (i', k') block, whose coordinates are $(M_{A\otimes B})_{(i',j)(k',l)}$, where j, l run both over $1..m$. According to (3) these coordinates are exactly $(M_A)_{i',k'} \cdot M_B$.

As an example consider the X, Y Pauli matrices.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$X \otimes Y = \begin{pmatrix} 0 \cdot Y & 1 \cdot Y \\ 1 \cdot Y & 0 \cdot Y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

2.2

T_1 eigenvectors' are $v_1, ..., v_n$ with the eigenvalues $\lambda_1, ..., \lambda_n$, and T_2 eigenvectors' are $w_1, ..., w_m$ with the eigenvalues $\mu_1, ..., \mu_m$. It is easy to validate that $v_i \otimes w_j$ is an eigenvector of T with the eigenvalue $\lambda_i \mu_j$. This is immediate from the definition of the way T operates:

$$T(v_i \otimes w_j) = T_1(v_i) \otimes T_2(w_j) = \lambda_i \cdot v_i \otimes \mu_j \cdot w_j = \lambda_i \mu_j \cdot v_i \otimes w_j$$

3

Let $\Phi = |00\rangle + |11\rangle$

3.1

A general vector u in $\{v \otimes w | v, w \in V_2\}$ is of the form:

$$u = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha'|0\rangle + \beta'|1\rangle) = \alpha\alpha'|00\rangle + \alpha\beta'|01\rangle + \beta\alpha'|10\rangle + \beta\beta'|11\rangle$$

For u to be equal to Φ, it must be that $\beta\alpha' = 0$ and that $\alpha\beta' = 0$. In other words one of the following is true:

- $\beta = 0$ and $\alpha = 0$, yielding that also $\alpha\alpha' = 0$, there is no $|00\rangle$ addend in u, so $u \neq \Phi$
- $\alpha' = 0$ and $\alpha = 0$, again $\alpha\alpha' = 0$, and $u \neq \Phi$
- $\beta = 0$ and $\beta' = 0$, there in no $|11\rangle$ addend in u, and $u \neq \Phi$
- $\alpha' = 0$ and $\beta' = 0$,again $\beta\beta' = 0$ and $u \neq \Phi$

3.2

$$
\begin{aligned}
\Phi^{\otimes^2} &= \Phi \otimes \Phi = (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) = \\
&= |0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle
\end{aligned}
$$

$$
\begin{aligned}
\Phi^{\otimes^3} = \Phi^{\otimes^2} \otimes \Phi &= (|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle) \otimes (|00\rangle + |11\rangle) = \\
&= |000000\rangle + |000011\rangle + |001100\rangle + |001111\rangle + \\
&+ |110000\rangle + |110011\rangle + |111100\rangle + |111111\rangle
\end{aligned}
$$

4

We take a look at matrix tensor product.
We make use of the fact the operation of $A \otimes B$ on $V \otimes W$, is defined as

$$A \otimes B(v \otimes w) = A(v) \otimes B(w)$$

4.1

To see that the $\otimes$ operator is not commutative consider the Pauli X, Y operators mentioned in question 2.

$$X \otimes Y = \begin{pmatrix} 0 \cdot Y & 1 \cdot Y \\ 1 \cdot Y & 0 \cdot Y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$Y \otimes X = \begin{pmatrix} 0 \cdot X & -i \cdot X \\ i \cdot X & 0 \cdot X \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

Clearly $X \otimes Y \neq Y \otimes X$.
Say that A, B, C are operators on V, W, U respectively. For each $v \in V$, $w \in W$ and $u \in U$:

$$
\begin{aligned}
& (A \otimes B) \otimes C(v \otimes w \otimes u) = \\
= \ & (A \otimes B)(v \otimes w) \otimes C(u) = A(v) \otimes B(w) \otimes C(u) = \\
= \ & A(v) \otimes (B \otimes C)(w \otimes u) = \\
= \ & A \otimes (B \otimes C)(v \otimes u \otimes w)
\end{aligned}
$$

4.2

Note that:

$(M_A \otimes M_B)^\dagger_{(i,j)(k,l)} = (M_A \otimes M_B)^*_{(k,l)(i,j)} = (M_A)^*_{k,i} \cdot (M_B)^*_{l,j} = (M_A)^\dagger_{i,k} \cdot (M_B)^\dagger_{j,l}$

where the second equality holds due to (3). So altogether due to (3) we get that $(M_A \otimes M_B)^\dagger = M_A^\dagger \otimes M_B^\dagger$

4.3

Say that A, B are operators on V, C operator on W. For each $v \in V$, $w \in W$:

$$\begin{aligned}((A+B)\otimes C)(v\otimes w) &= (A+B)(v)\otimes C(w) = \\ &= (A(v)+B(v))\otimes C(w) = A(v)\otimes C(w) + B(v)\otimes C(w) = \\ &= (A\otimes C)(v\otimes w) + (B\otimes C)(v\otimes w)\end{aligned}$$

where the third equality uses distributivity of $V \otimes W$, which is immediate from the tensor product linearity.
Altogether we get that $(A+B)\otimes C = A\otimes C + B\otimes C$.
Similarly $A\otimes(B+C) = A\otimes B + A\otimes C$.

4.4

A_{n1Xk1}, C_{k1Xm1}, B_{n2Xk2}, D_{k2Xm2} are matrices. For each v from dimension $m1$ and w from dimension $m2$ we get that:

$$\begin{aligned}(A\otimes B)(C\otimes D)(v\otimes w) &= (A\otimes B)(C(v)\otimes D(w)) = \\ &= (A(C(v))\otimes(B(D(w)) = (AC)(v)\otimes(BD)(w) \\ &= (AC)\otimes(BD)(v\otimes w)\end{aligned}$$

Altogether we get that $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$

4.5

A, B are regular, so we can define $D = A^{-1}\otimes B^{-1}$.
We now note that:

$$(A\otimes B)D = (A\otimes B)(A^{-1}\otimes B^{-1}) = (AA^{-1})\otimes(BB^{-1}) = I_A\otimes I_B = I_{A\otimes B}$$

Similarly $D(A\otimes B) = I_{A\otimes B}$ and so we get $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$.

4.6

A, B are unitary matrices. Thus $AA^\dagger = I_A$, $BB^\dagger = I_B$.
So for $A \otimes B$ we get:

$$(A \otimes B)(A \otimes B)^\dagger = (A \otimes B)(A^\dagger \otimes B^\dagger) = AA^\dagger \otimes BB^\dagger = I_{A\otimes B}$$

Altogether we get that $A \otimes B$ is unitary.

4.7

Suppose that the dimensions of A, B are $n\ Xn$, $m\ Xm$ respectively.

$$\begin{aligned} Trace(A \otimes B) &= \sum_{i=1..n,j=1..m} (A \otimes B)_{(i,j),(i,j)} = \sum_{i,j} A_{i,i} B_{j,j} \\ = \sum_i A_{i,i} \sum_j B_{j,j} &= Trace(A)Trace(B) \end{aligned}$$

4.8

$Rank(A)$ is the dimension of the vector space $Im(A)$.
$Rank(B)$ is the dimension of the vector space $Im(B)$.
Thus the dimension of the space $Im(A) \otimes Im(B)$ is $Rank(A)Rank(B)$, implying that $Rank(A \otimes B) = Rank(A)Rank(B)$

5

5.1

Operating A before B is equivalent to operating:

$$(I \otimes B)(A \otimes I) = (I \cdot A) \otimes (B \cdot I) = A \otimes B$$

Similarly, operating B before A is equivalent to:

$$(A \otimes I)(I \otimes B) = (A \cdot I) \otimes (I \cdot B) = A \otimes B$$

5.2

Operating A, C, B, D in the different orders is equivalent to:

$$(C \otimes I)(A \otimes I)(I \otimes D)(I \otimes B) = (CA \otimes I)(I \otimes DB) = CA \otimes DB$$

$$(C \otimes I)(I \otimes D)(A \otimes I)(I \otimes B) = (CI \otimes ID)(AI \otimes IB) = CA \otimes DB$$
$$(I \otimes D)(C \otimes I)(I \otimes B)(A \otimes I) = (IC \otimes DI)(IA \otimes BI) = CA \otimes DB$$
$$(I \otimes D)(I \otimes B)(C \otimes I)(A \otimes I) = (I \otimes DB)(CA \otimes I) = CA \otimes DB$$

Note that in any case A must operate before C and B before D.

5.3

We look at what the circuit does to the $V_2^{\otimes 2}$ base states.

$$\begin{aligned} |x,\ y\rangle - 1^{st}CNOT \ &\rightarrow \ |x,\ x \oplus y\rangle - 2^{nd}CNOT \rightarrow |x \oplus y \oplus x,\ x \oplus y)\rangle = \\ &= \ |y,\ x \oplus y\rangle - 3^{rd}CNOT \rightarrow |y,\ y \oplus x \oplus y)\rangle = |y,\ x\rangle \end{aligned}$$

and note that the circuit swaps the input qubits.