**Exercise no. 1 Quantum Computation. Fall semester 2002.**
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Out: 20/10/2002
In: 24/10/2002. 10:00.
No late submissions.

**This time: Tensor products.**

1. Let $V, W$ be vector spaces. We define an outer product

$$V \otimes W \quad = \quad \text{SPAN}\{v \otimes w \mid v \in V, w \in W\}$$

   where $\otimes(x, y) = x \otimes y$ is linear in each of its two arguments.

   - Prove $V \otimes W$ is a vector space.
   - Let $A = \{a_1, \ldots, a_n\}$ be a basis for $V$ and $B = \{b_1, \ldots, b_m\}$ a basis for $W$. Show that $\{a_i \otimes b_j\}$ is a basis for $V \otimes W$.
   - Assume $V$ and $W$ are Hilbert spaces (i.e., they have an inner product). Define $\langle v_1 \otimes w_1 | v_2 \otimes w_2 \rangle = \langle v_1 | v_2 \rangle \cdot \langle w_1 | w_2 \rangle$ and extend it in the natural way. Prove this defines an inner product function on $V \otimes W$.
   - Show that if $A$ and $B$ above are orthonormal bases, then so does $\{a_i \otimes b_j\}$.

2. Let $T$ be a linear transformation on $V \otimes W$, and assume there are two linear operators $T_1, T_2$ on $V, W$ respectively, such that $T(v \otimes w) = T_1(v) \otimes T_2(w)$. Let $A$ be a basis for $V$, $B$ for $W$. Let $M_A$ be the matrix representing $T_1$ in the basis $A$, and $M_B$ representing $T_2$ in the basis $B$.

   - Express the matrix representing $T$ in the basis $\{a_i \otimes b_j\}$ by $M_A$ and $M_B$.
   - Assume $T_1$ has eigenvectors $v_1, \ldots, v_n$ with eigenvalues $\lambda_1, \lambda_n$, and $T_2$ has eigenvectors $w_1, \ldots, w_m$ with eigenvalues $\mu_1, \ldots, \mu_m$. What are the eigenvectors and eigenvalues of $T$ ?

3. Let $\phi = |00\rangle + |11\rangle$.

   - Prove $\phi \in V_2 \otimes V_2$ but $\phi \notin \{v \otimes w \mid v, w \in V_2\}$. We call such vectors entangled.
   - Write $\phi^{\otimes 2}$ and $\phi^{\otimes 3}$ explicitly.

4. We now prove some basic laws about matrix tensor products.

   - Check that the operator $\otimes$ is non-commutative and associative.
   - What is $(A \otimes B)^\dagger$ ?
   - Prove distributive laws for $\otimes$ and matrix sum.
   - Let $A_{n_1 \times k_1}, C_{k_1 \times m_1}, B_{n_2 \times k_2}, D_{k_2 \times m_2}$ be matrices. Prove that $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$.

- Prove that if $A$ and $B$ are regular than so does $A \otimes B$. What is $(A \otimes B)^{-1}$ ?
- Prove that if $A$ and $B$ are unitary than so is $A \otimes B$.
- What is $Trace(A \otimes B)$ ?
- What is $Rank(A \otimes B)$ ?

5. (Quantum Circuits)

- Prove that in the circuit of Figure 1 it does not matter which gate is executed first.
- Prove that in the circuit of Figure 2 it does not matter which gate is executed first (e.g., $A, C$ can be executed before $B, D$).
- What does the circuit of Figure 3 do?.

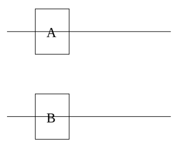


Figure 1: two parallel gates

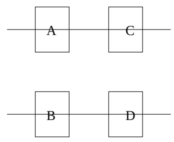


Figure 2: two pairs of parallel gates

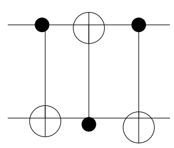

Figure 3: three cnot gates