Problem set 11

out: 13/01/15
due: 2/2/15

1. Give a log-space algorithm that given an undirected graph G outputs a spanning forest in G.

Notation for the next exercise: A BPL machine has:

- A read-only, multiple access input tape,
- A read/write work tape
- A read-only, read-once random tape
- A write-only output tape.

The space complexity is the maximal size used by the work tape. BPL is the class of languages solved by polynomial time, logarithmic space, bi-sided error probabilistic algorithm with a constant gap between the acceptance probability of yes and no instances. We now define a new class BPL^* which is the same except that the random tape (that is initialized uniformly as before) is *multiple access* (but still read only). ZPL^* is the same except that it is zero-error meaning that the machine may answer $\{yes, no, quit\}$, when it answers *yes* or *no* it is correct, and for every input the probability of answering *quit* is at most half.

2. (Nisan) Prove that $\mathsf{BPL} \subseteq \mathsf{ZPL}^*$.

3. A k-source is a distribution D with $H_\infty(D) \geq k$. Prove that if X is ε-far from any k-source, then there exists a subset A of cardinality at most 2^k such that $\Pr_{x \in X}[x \in A] \geq \varepsilon$.

Definition: $S_1, \ldots, S_m \subseteq \{1, \ldots, t\}$ is an (ℓ, a)–design if each set S_i $(1 \leq i \leq \ell)$ has cardinality ℓ, and every two different sets S_i and S_j in the collection have intersection size $|S_i \cap S_j|$ at most a.

4. Prove that for every $\ell, a \geq 1$, there exists an (ℓ, a) design $S_1, \ldots, S_m \subseteq [t]$ with $t = O(\ell^2/a)$ and $m = 2^{\Omega(a)}$.

5. Two norm one vectors $v_1, v_2 \in \mathbb{R}^d$ are almost orthogonal if $|\langle v_1 | v_2 \rangle| \leq \varepsilon$.

   - Show how to convert an (ℓ, a) design $S_1, \ldots, S_m \in [t]$ into:
     - A set of m nearly orthogonal norm 1 vectors.
     - A binary error correcting code of length t with m codewords and large distance.
   - How many norm 1 orthogonal vectors can one put into $\mathbb{R}^d$?
   - How many ε–almost, norm 1 orthogonal vectors can you put into $\mathbb{R}^d$? Upper bound? Lower bound? Is it tight?

6. Suppose $y, x_1, \ldots, x_m$ are norm 1 vectors in $\mathbb{R}^n$ such that

   - $\langle y, x_i \rangle > 0$ (i.e., y has sharp angle with each x_i), and,
   - $\langle x_i, x_j \rangle \leq -\Delta$ for all $i \neq j$ and some $\Delta > 0$ (i.e., the angle between any two x_i is strictly blunt)

   Prove $m \leq \frac{1}{\Delta} + 1$.

7. Suppose $y, x_1, \ldots, x_m \in \mathbb{R}^n$ such that

   - $\langle y, x_i \rangle > 0$ (i.e., y has sharp angle with each x_i), and,
   - $\langle x_i, x_j \rangle \leq 0$ for all $i \neq j$ (i.e., the angle between any two x_i is blunt)

   Prove $m \leq n$.

   Hint: If $m > n$ there is a non-trivial linear dependence between the vectors x_i. Prove this implies the origin $(0, \ldots, 0)$ is in the convex hull of the x_i.

8. (The Johnson's bound) Let C be a $[n, k, d = (\frac{1}{2} - \varepsilon)n]_2$ code. Set $\tau = \frac{1}{2} - \sqrt{\frac{\varepsilon}{2}}$. Prove C is $(\tau n, \frac{1}{4\varepsilon} + 1)$ list-decidable.

   Hint: Suppose $y \in \mathbb{F}_2^n$ is close to L words $x_1, \ldots, x_L \in \mathbb{F}_2^n$. I.e., y is close to each x_i, but the x_i are far from each other. Map $\mathbb{F}_2^n$ to $\mathbb{R}^n$ by applying $\phi(x) = (-1)^x$ to each coordinate. Manipulate the vectors so that you can use the previous questions.

9. Show $B(\lambda n, n) < 2^{H(\lambda)n}$ and $\lim_{n\to\infty} \frac{1}{n} \log B(\lambda n, n) = H(\lambda)$.
   Hint: Expand $(\lambda + (1 - \lambda))^n$ and use Stirling's formula.