Problem set 3 - Random walks on graphs

out: 11/11/14
due: 24/11/14

1. An undirected graph is *simple* if it has no multiple edges or self loops. We saw in class that for every simple, undirected d–regular graph G with transition matrix A, and every norm 1 vector $x \in \mathbb{R}^n$, $\sum_{(i,j)\in E}(x_i - x_j)^2 = d(1 - \langle Ax, x\rangle)$. Show the same for arbitrary undirected d–regular graphs. Conclude that for any such graph G with transition matrix A, $\lambda_2(A) \leq 1 - \frac{1}{n^2 d}$.

2.
   - Show that if $A \in M_n(\mathbb{R})$ (i.e., A is an $n \times n$ matrix over $\mathbb{R}$) and symmetric then A has an ortho-normal basis of *real* eigenvectors.
   - Let A be the transition matrix of the undirected n-cycle. Prove that $\{\chi_k\}_{k=0}^{n-1}$ is an eigenvector basis of A, where $\chi_k(i) = w^{ki}$ and w is a primitive n'th root of unity.
   - Find a *real* ortho-normal basis for A.

3. The diameter of a graph is the maximum minimal distance between two vertices in the graph. Let G be an undirected D regular graph over N vertices.

   (a) Prove $diam(G) \geq \log_{D-1}(N-1) - 2$.

   (b) $diam(G) \leq \log_{\frac{D}{\overline{\lambda}(G)}}(N) + 1$.

   $\overline{\lambda}(G)$ denotes the second largest eigenvalue in absolute value of the adjacency matrix of G.

4. Give a *LogSpace* algorithm that checks whether a given undirected graph is a connected tree or not, without using the $SL = L$ result.

5. Let us call a directed graph *balanced* if every vertex has the same indegree as outdegree.
   - Prove that in a balanced graph each strongly connected component is isolated.
   - Omer Reingold showed how to solve s, t connectivity in undirected graphs. Assuming this result, give a *LogSpace* algorithm solving s,t connectivity in balanced directed graphs.