Problem set 5 - More pair-wise independence and walks on directed graphs

out: 27/11/14
due: 8/12/14

1. Suppose $X_1, \ldots, X_n$ are boolean, $k$-wise independent random variables.Let $X = \frac{1}{n}\sum X_i$ and $\mu = \mathbb{E}(X)$. Prove that
$$\Pr[|X-\mu| \geq \varepsilon] \;\;\leq\;\; (\frac{k}{2})^k(\frac{1}{n\varepsilon^2})^{k/2}.$$

2. Let $V = \{0,1\}^m$ and $H = \{h : V \to V\}$ a 2UFOHF (i.e., a two universal family of hash functions). Fix two sets $A, B \subseteq V$. Call a hash function $h \in H$ *$\varepsilon$–good for $A, B$* if
$$\left|\; \Pr_{x\in V}[x \in A \;\cap\; h(x) \in B] - \rho(A)\rho(B) \;\right| \;\;\leq\;\; \varepsilon,$$
where $\rho(C) = \frac{|C|}{|V|}$.
Prove that for any $A, B \subseteq V$, $\varepsilon > 0$,
$$\Pr_{h\in H}\;[h \text{ is not } \varepsilon\text{–good for } A, B] \;\;\leq\;\; \frac{\rho(A)\rho(B)(1-\rho(B))}{\varepsilon^2 \cdot |V|} \leq \frac{1}{\varepsilon^2|V|}.$$

3. For a set $C$, let $U_C$ denote the uniform distribution over $C$.
Let $H = \{h : \Lambda \to \Gamma\}$ be a 2UFHOF. For a distribution $D$ over $\Lambda$ let $(H, H(D))$ denote the distribution over $H \times \Gamma$ obtained by picking $d$ according to $D$, picking $h$ uniformly from $H$ and outputting $(h, h(d))$.
   - Prove that for any distribution $X$ over $C$, $\|X - U_C\|_2^2 = \|X\|_2^2 - \|U_C\|_2^2$.
   - Prove that for any distribution $X$, $\|X\|_2^2 = \Pr_{x_1,x_2\in X}[x_1 = x_2]$.
   - Prove that $\|(H, H(D))\|_2^2 \leq \|U_H\|_2^2 \cdot [\|U_\Gamma\|_2^2 + \|D\|_2^2]$.
   - Conclude that $\|(H, H(D)) - U_H \times U_\Gamma\|_2 \leq \|U_H\|_2 \cdot \|D\|_2$.
   - Prove that $\|(H, H(D)) - U_H \times U_\Gamma\|_1 \leq \sqrt{|\Gamma|} \cdot \|D\|_2$.

4. Let $G = (V, E)$ be a directed graph and $A$ its transition matrix. Let $\lambda \in \mathbb{C}$ be any eigenvalue of $A$. Prove that $|\lambda| \leq 1$.

5. In this question we want to demonstrate that an almost $\lambda$-eigenvector of a stochastic matrix $A$ (in the sense that it almost solves the eigenvector equation $Av = \lambda v$) might be very far from any true $\lambda$-eigenvector of $A$.
   - Let $G_1 = (V_1, E_2)$ be the following graph:
$$\begin{aligned} V_1 &= \{s, 0, 1, \ldots, k, t\} \\ E_1 &= \{(s,0), (k,t), (t,s)\} \cup \{(i, i+1) | 0 \leq i \leq k-1\} \cup \{(i, s) | 0 \leq i \leq k\}. \end{aligned}$$
   Let $A_1$ denote the transition matrix of $G_1$. Find a vector $\pi \in \mathbb{R}^n$ s.t. $A_1\pi = \pi$ and prove it is the unique eigenvector of $G_1$ with eigenvalue 1.

- Let $G_2$ be the graph $G_1$ with an additional vertex *sink* and two additional edges $(t, sink)$ and $(sink, sink)$. Prove that the only 1-eigenvector of this graph is the vector that is zero everywhere except for the *sink* coordinate.
- Find a stochastic operator $A \in M_n(\mathbb{R})$ that has a unique 1-eigenvector $v$ (By the Perron-Frobenius theorem $v$ has only non-negative coordinates and w.l.o.g. it is a probability distribution) and another probability vector $w$ such that:
  - $\|Aw - w\| \leq O(2^{-n})$ (i.e., $w$ "nearly" solves the equation $Aw = w$),
  - $w$ is very far from $v$, $|v - w|_1 = 1 - O(2^{-n})$ and $\langle v, w \rangle = 0$.