Problem set 6 - Norms, tensors, rotation maps

out: 3/12/14
due: 15/12/14

This homework has many questions, but many of them are simple and shouldn't take much time. You may skip any question you feel you already know.

1. Let $A(x;y)$ be a probabilistic algorithm with success probability $\frac{1}{2}+\varepsilon$. A uses m random coins and takes T time. Show how to amplify the success probability to $1-\delta$ using $m+O(\frac{\log(1/\delta)}{\varepsilon^2})$ random coins and taking $T \cdot \text{poly}(m, \log(\delta^{-1}), \frac{1}{\varepsilon})$ time.

2. (RVW) Let $\mathbb{F}$ be a field with q elements. We defined the graph $G=(V=\mathbb{F}\times\mathbb{F}, E)$ where $((a,b),(c,d)) \in E$ iff $(a,b) \in \ell_{c,d}$ and $\ell_{c,d} = \{(x,y) \mid y = cx - d\}$. We also defined the map $Rot : V \times [q] \to V \times [q]$ by

$$Rot((a,b),t) \;=\; \begin{cases} ((\frac{t}{a}, t-b), t) & \text{If } a \neq 0, t \neq 0 \\ ((t,-b),a) & \text{Otherwise} \end{cases}$$

Prove this defines a rotation map.

Norms and tensors

For a vector $z \in \mathbb{C}^n$ we denote $\|v\|_p = (\sum_{i=1}^n |z_i|^p)^{1/p}$ and $\|z\|_\infty = \max_i |z_i|$.

3. (The spectral norm) Let A be a matrix. Define $\|A\| = sup_{v\neq 0}\frac{\|Av\|_2}{\|v\|_2}$. Prove:
   - $\|A+B\| \leq \|A\| + \|B\|$
   - $\|cA\| = |c|\|A\|$, $\|A\| = 0$ iff $A=0$
   - $\|AB\| \leq \|A\|\|B\|$
   - If if A is normal then $\|A\| = \lambda_1(A)$.

4. Let A be a matrix over the complex field $\mathbb{C}$. Define $\|A\|_{row} = sup_{v\neq 0}\frac{\|Av\|_\infty}{\|v\|_\infty}$. Prove:
   - $\|A\|_{row} = max_i \|A_i\|_1$, where A_i is the i'th row of A.
   - $\|A+B\|_{row} \leq \|A\|_{row} + \|B\|_{row}$
   - $\|cA\|_{row} = |c|\|A\|_{row}$, $\|A\|_{row} = 0$ iff $A=0$
   - $\|AB\|_{row} \leq \|A\|_{row}\|B\|_{row}$
   - If A is the transition matrix of an undirected graph then $\|A\|_{row} = 1$.

5. Recall that if $A \in M_{n_1,m_1}(F)$ and $B \in M_{n_2,m_2}(F)$ then $A\bigotimes B \in M_{n_1 n_2 \times m_1 m_2}(F)$ and is defined by $A\bigotimes B[(i_1,i_2),(j_1,j_2)] = A[i_1,j_1]\cdot B[i_2,j_2]$. Show that:

(a) $A\bigotimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & & & \\ & \ddots & & & \\ & & a_{i,j}B & & \\ & & & \ddots & \\ & & & & a_{n,n}B \end{pmatrix}$

(b) Prove that $(A\bigotimes B)^T = A^T\bigotimes B^T$, $(A\bigotimes B)^\dagger = A^\dagger\bigotimes B^\dagger$

(c) Prove that the tensor product of two projections is a projection.

(d) Prove that the tensor product of two unitary matrices is unitary.

(e) Prove that $(A\bigotimes B)\cdot(C\bigotimes D) = (AC)\bigotimes(BD)$, whenever the dimensions fit.

(f) Prove that $\mathrm{Tr}\ (A\bigotimes B) = \mathrm{Tr}\ (A)\cdot \mathrm{Tr}\ (B)$.

6. Let $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. $\bigotimes^n H = H\bigotimes\ldots\bigotimes H$ denotes the operator obtained by tensoring H n times. IP is the inner product matrix: $IP[s,t] = (-1)^{\langle s,t\rangle}$, for any $s,t \in \{0,1\}^n$.

   (a) Prove $\bigotimes^n H = IP$.

   (b) Compute the eigenvalues of IP.

   (c) Conclude that $\|IP\| = 2^{n/2}$ and $\sqrt{\frac{1}{2^n}}IP$ is unitary.

7. Look at the matrix IP. For two sets $S,T \subseteq [N = 2^n]$ the *combinatorial rectangle* $S\times T$ is the minor of the matrix when restricted to $S\times T$. The *discrepancy* $\Delta(S,T)$ of the combinatorial rectangle is $|t_1 - t_0|$ where t_b is the number of times $(-1)^b$ appears in the combinatorial rectangle $S\times T$. Prove that for any two sets S,T, $\Delta(S,T) \le \sqrt{N\cdot|S|\cdot|T|}$.

The eigenvalues and eigenvectors of Cayley graphs

A character of G is a function $\chi : G \to \mathbb{C}^* = \mathbb{C}\setminus 0$ such that for every $g_1, g_2 \in G$ it holds that $\chi(g_1\cdot g_2) = \chi(g_1)\cdot\chi(g_2)$ (where the first multiplication is in G and the second in $\mathbb{C}$). A finite Abelian group has exactly $|G|$ different characters.

Suppose G has n elements, $G = \{g_1,\ldots,g_n\}$ and $f : G \to \mathbb{C}$. We let $\vec{f}$ denote the vector

$$\vec{f} = \begin{pmatrix} f(g_1) \\ f(g_2) \\ \vdots \\ f(g_n) \end{pmatrix}.$$

8. For an integer k, let $\mathbb{Z}_k$ denote the $\{0,\ldots,k-1\}$ with addition mod k. Also, let $w = e^{\frac{2\pi i}{k}}$, i.e., w is a primitive root of unity of order k. Define $\chi_1,\ldots,\chi_k$ by $\chi_i(j) = w^{ij}$.

   (a) Prove that $\chi_1,\ldots,\chi_k$ are characters of $\mathbb{Z}_k$.

   (b) Prove that $\{\vec{\chi_1},\ldots,\vec{\chi_k}\}$ is an orthogonal basis of $\mathbb{C}^k$.

9. Let G_1, G_2 be two groups, $f_1 : G_1 \to \mathbb{C}$ and $f_2 : G_2 \to C$. Define $f_1\bigotimes f_2 : G_1\times G_2 \to \mathbb{C}$ by $f_1\bigotimes f_2(g_1,g_2) = f_1(g_1)\cdot f_2(g_2)$.

- Prove that if χ_1 is a character of G_1 and χ_2 a character of G_2 then $\chi_1 \bigotimes \chi_2$ is a character of $G_1 \times G_2$.
- Find all the characters of $\mathbb{Z}_2^n$. Prove their corresponding vectors are orthogonal. How does this compare to Question 6c?

10. Suppose $A \in M_n(\mathbb{C})$ is such that $A_{i,j} = f(g_i g_j^{-1})$ for some function $f : G \to \mathbb{C}$. Prove that $\vec{\chi}$ is an eigenvector of A with eigenvalue $\langle \vec{\chi}, \vec{f} \rangle = \sum_j f(j)\chi(j^{-1})$.

11. Let G be a group and $S \subseteq G$. The Caylely graph $C(G, S)$ is the graph $C(V, E)$ with $V = G$ and $(a, b) \in E$ iff $a = bs^{-1}$ for some $s \in S$. Identify the following graphs:

    - $C(\mathbb{Z}_n, \{1\})$
    - $C(\mathbb{Z}_n, \{1, -1\})$
    - $C(\mathbb{Z}_2^n, \{e_1, , e_n\})$, where e_i has 1 in the i'th coordinate and 0 otherwise.
    - $C(\mathbb{Z}_n \times Z_n, \{(0, 1), (0, -1), (1, 0), (-1, 0)\})$.

12. Calculate the eigenvalues and the spectral gap of the Cayley graphs given in question 11.