Problem set 7 - More on expanders

In this exercise, for a normal operator A, $\lambda(A)$ is its second largest eigenvalue in absolute value.

1. Construct an explicit $[D^8, D^2, \frac{1}{8}]$ graph and its rotation map. (The goal is to get a feeling of how the recursive construction works).

2. Prove that if M and A are $n \times n$ symmetric matrices with the same first eigenvector, and if $M = \alpha A + \beta B$ for positive α and β, then $\lambda(M) \leq \alpha\lambda(A) + \beta\|B\|$.

3. Prove the Mixing Lemma: Let $G = (V, E)$ be a D-regular, undirected graph over N vertices and A its transition matrix. Let $\lambda = \lambda(A)$, $0 \leq \lambda \leq 1$. Then for any $S, T \subseteq V$:

$$|\frac{|E(S,T)|}{N \cdot D} - \rho(S)\rho(T)| \quad \leq \quad \lambda \cdot \sqrt{\rho(S)(1-\rho(S))\rho(T)(1-\rho(T))},$$

   where $E(s,t) = \{(a,b) \in E \mid a \in S, b \in T\}$.

   Compare with Homework 5, Question 2. Also compare the proof technique with Homework 6, question 7.

4. Let G be a D-regular, undirected graph over N vertices. Set $0 \leq \lambda \leq 1$ as before. Let $\alpha(G)$ denote the size of the largest independent set of G. Let $\chi(G)$ denote the minimal number of colors needed to color G. Prove that:

   - $\alpha(G) \leq \frac{\lambda}{1+\lambda} N$.
   - Prove that $\chi(G) \geq \frac{N}{\alpha(G)}$ and conclude that $\chi(G) \geq \frac{1+\lambda}{\lambda}$.

5. Prove Tanner inequality: Let $G = (V, E)$ be an undirected, D-regular graph over N vertices with $0 \leq \lambda \leq 1$ as before. For $A \subseteq V$ let $\Gamma(A) = \{w \in V \mid \exists v \in A \text{ s.t. } (v,w) \in E\}$. Then:

$$|\Gamma(A)| \geq |A| \cdot \frac{1}{\rho(A) \cdot 1 + (1-\rho(A))\lambda^2}.$$

   Assume G is Ramanujan. Conclude that there exists some constant $\alpha > 0$ such that all sets $A \subseteq V$ of density at most α (and this is still constant density) the vertex expansion $\frac{|\Gamma(A)|}{|A|}$ is at least $\frac{D}{4}$.

6. The fully explicit family $\{G_n\}$ we constructed in class was sparse. The following construction amends that: H is an $[D^8, D, \frac{1}{8}]$ graph. $G_1 = H^2$ and

$$G_t = (G_{\lceil t/2 \rceil} \; \otimes \; G_{\lfloor t/2 \rfloor})^2 ⓩ H.$$

   Prove (or sketch a proof) that the family is well defined, fully explicit, has regular degree D^2 and second normalized eigenvalue at most half.