Problem set 8 - Derandomized squaring on directed, regular graphs

out: 17/12/14
due: 29/12/14

We assume for every $D \geq 3$ and constant $\lambda < 1$ there exist explicit UTS for the family of undirected, D-regular graphs with normalized second largest eigenvalue at most λ. Such a construction was given by Hoory and Wigderson.

1. Prove the existence of LogSpace-Explicit UTS for undirected, consistently labeled 3-Regular graphs. I.e.,

   - Describe a Turing machine that on input 1^n outputs a sequence $\sigma = (\sigma_1, \ldots, \sigma_T) \in \{1, 2, 3\}^T$.
   - Prove it outputs a UTS for the above family of graphs, and,
   - Prove that it runs in LogSpace.
   - If you are apt to it, write a program implementing the machine. If possible, it would be nice to be able to see the internal transitions of the machine.

2. In this question we assume a LogSpace-Explicit UTS for undirected, consistently labeled 3-Regular graphs, as constructed in the previous question.

   Let $D \geq 3$ be a constant. Prove the existence of LogSpace explicit UES (universal *exploration* sequence) for the family of undirected D-Regular graphs.

The next questions are from the work of Rozenman and Vadhan on derandomized squaring. Questions 3-5 are adaptations of things we saw in the regular, undirected case to the *regular*, directed case.

3. We say a *directed* graph G is an $[N, D]$ graph if G has N vertices and is D-regular (i.e., the in-degree and out-degree of each vertex is exactly D). Let A be the transition matrix of an $[N, D]$ directed graph.

   - Prove that $\|A\| \leq 1$.
   - Prove that the all-one vector is a 1-eigenvector of A.

   Let G and A be as above, and u the all-one vector. Let $\lambda(A) = \max_{x \perp u, \|x\|=1} \|Ax\|$. We say G is an $[N, D, \lambda]$ graph if $\lambda(A) \leq \lambda$.

4. The SVD (singular value decomposition) states that every $n \times n$ matrix A over $\mathbb{C}$ can be expressed as $A = UDV$ where U, V are unitary and D is diagonal with non-negative real values σ_i over the main diagonal of D. The values $\sigma_1 \geq \ldots \geq \sigma_n$ are called the *singular* values of A and are uniquely determined by A. When A is normal $U = V$ and the singular values coincide with the eigenvalues of A.

   - Prove that for every A, $\|A\| = \sigma_1(A)$.
   - Prove that for every A, $\sigma_i(A) = \sqrt{|\lambda_i(A^\dagger A)|}$, where for a normal matrix M, $\lambda_i(M)$ is the i'th largest eigenvalue in absolute value.

- Let G be a directed, connected, D-regular graph with at least one self-loop on each vertex. Let A be its transition matrix. Prove that the following three conditions are equivalent:
  (a) G is strongly connected.
  (b) G is weakly connected.
  (c) $\lambda(A) < 1$.
- Let G and A be as above. Prove that if G is connected then $\lambda(A) \leq 1 - \frac{1}{2D^2N^2}$.

5. Prove that if H is a $[N, D, \lambda]$ directed graph with transition matrix A, then $A = (1-\lambda)J + \lambda C$ for some matrix C with $\|C\| \leq 1$. (We proved a similar lemma in class for the undirected case, but the proof for the directed regular case is slightly more intricate).

6. Assume G_1 is a directed $[N_1, D_1, \lambda_1]$ labeled graph with labeling ℓ_1, and H a directed $[D_1, D_2, \lambda_2]$ labeled graph with labeling ℓ_2. The *derandomized square graph* $G Ⓢ H$ has N_1 vertices, out-degree $D_1 \cdot D_2$ and the following labeling function: $\ell : [N_1] \times [D_1] \times [D_2] \to [N_1]$ defined by

$$\ell(v; a, b) \quad = \quad \ell_1(\ell_1(v; a); \ell_2(a; b)).$$

We write in short: $v[a, b] = v[a][a[b]]$.

- Give an example where G and H are connected but $G Ⓢ H$ is not connected.
- Give an example where G and H are undirected and consistently labeled, but $G Ⓢ H$ is *not* undirected.
- Give an example where $G Ⓢ H$ is *not* regular.
- Prove that if G is consistently labeled then $G Ⓢ H$ is $D_1 D_2$ regular and consistently labeled.

7. Assume G is consistently labeled with transition matrix A. Let M be the transition matrix of $G Ⓢ H$.

- Prove that $M = (1 - \lambda_2)A^2 + \lambda_2 D$ for some matrix D with $\|D\| \leq 1$. does this imply that if G and H are connected then so does $G Ⓢ H$?
- Conclude that $G Ⓢ H$ is an $[N_1, D_1 D_2, f(\lambda_1, \lambda_2)]$ graph for $f(\lambda_1, \lambda_2) \leq (1 - \lambda_2)\lambda_1^2 + \lambda_2$.
- Conclude that if G and H are connected then so does $G Ⓢ H$.

8. Consider the following sequence. $G_0 = G$ is a directed $[N, D_1, \lambda = 1 - \gamma]$ graph. H_i is a $[N_i = D_1 D_2^i, D_2, \frac{1}{100}]$ graph. Define

$$G_{i+1} \quad = \quad G_i Ⓢ H_i,$$

and notice that it is well defined for all $i \geq 0$.

- Prove that G_k is a $[N, N_i, 1 - (\frac{3}{2})^k \gamma]$ graph.
- Assume $\{H_i\}$ is fully explicit. Show how to compute the labeling function $\ell_k(v, a)$ of G_k in *Space*$(O(\log N), k)$.