Random Walks over Undirected Graphs – I Amnon Ta-Shma and Dean Doron

## 1 Undirected Graphs as Operators

**Definition 1.** Let G be a (possibly weighted) undirected graph over n vertices with an adjacency matrix  $A_G$ . The normalized adjacency matrix, or the transition matrix, is the matrix  $A = A_G D^{-1}$  where D is the diagonal degree matrix, i.e.,  $D[i, i] = \sum_{i} A_G[i, j]$  for every  $i \in [n]$ , and so

$$A[i,j] = \frac{1}{d(j)} A_G[i,j].$$

If G is d-regular then A is Hermitian, and is simply  $\frac{1}{d}A_G$ .

**Theorem 2.** Let G be an undirected graph over n vertices and let A be its normalized adjacency matrix. Let  $\lambda_n, \ldots, \lambda_1$  be the eigenvalues of A. Then:

- 1.  $\lambda_1, \ldots, \lambda_n$  are real.
- 2.  $\lambda_1 = 1$ .
- 3.  $\lambda_n \geq -1$ .
- 4.  $\lambda_1 = \ldots = \lambda_k = 1$  and  $\lambda_{k+1} < 1$  if and only if G has exactly k connected components.
- 5.  $\lambda_n = -1$  if and only if at least one of the connected components of G is bipartite.

Claim 3 (bipartite graphs). Let G be a d-regular undirected bipartite graph over n vertices and let  $A_G$  be its adjacency matrix. Then, the eigenvalues of  $A_G$  are symmetric around 0. That is, every positive eigenvalue  $\lambda_k$  has a negative eigenvalue  $-\lambda_k$  and vice versa.

## 2 Undirected graphs have non-negligible spectral gap

Throughout, we denote  $\overline{\lambda}(G) = \max_{i \neq 1} |\lambda_i|$ . In this section we will find an upper bound on  $\lambda_2$ and a lower bound on  $\lambda_n$ , thus bounding  $\overline{\lambda}$  from above. Throughout, we let G be a d-regular, non-bipartite, connected undirected graph (similar results also hold in the non-regular case). We let A be the normalized adjacency matrix of G, with eigenvalues  $\lambda_n \leq \ldots \leq \lambda_1$  and corresponding eigenvectors  $v_n, \ldots, v_1$ . We also let **1** be the all-ones vector, J be the all-ones matrix and  $\mathbf{1}_X$  is the vector which is 1 over some index set X and 0 elsewhere.

**Claim 4.** Let G be an undirected, regular graph over n vertices and let A be its normalized adjacency matrix. Then,  $\bar{\lambda}(G) = \|A - \frac{1}{n}J\|$ .

**Claim 5.** Let A be the normalized adjacency matrix of a regular undirected graph over n vertices and let  $\lambda_n \leq \ldots \leq \lambda_1$  be the eigenvalues of A. Then,

$$\lambda_2 = \max_{\substack{x \perp \mathbf{1}, x \neq 0}} \frac{x^{\dagger} A x}{x^{\dagger} x}$$
$$\lambda_n = \min_{\substack{x \neq 0}} \frac{x^{\dagger} A x}{x^{\dagger} x}$$

**Claim 6.** Let M be a symmetric  $n \times n$  operator, v a real length n vector. Let D be the  $n \times n$  diagonal matrix with  $D[i, i] = \sum_{j} M[i, j]$ . Then

$$\sum_{i,j} M[i,j](v_i - v_j)^2 = 2v^{\dagger}(D - M)v.$$

*Proof.* A straightforward computation shows that:

$$\begin{split} \sum_{i,j} M[i,j](v_i - v_j)^2 &= \sum_{i,j} M[i,j](v_i^2 + v_j^2) - 2\sum_{i,j} M[i,j]v_iv_j = 2\sum_{i,j} M[i,j]v_i^2 - 2\sum_i v_i \sum_j M[i,j]v_j \\ &= 2\sum_i v_i^2 \sum_j M[i,j] - 2\sum_i v_i (Mv)_i = 2v^{\dagger} Dv - 2\sum_i v_i (Mv)_i \\ &= 2v^{\dagger} Dv - 2v^{\dagger} Mv = 2v^{\dagger} (D - M)v. \end{split}$$

**Theorem 7** ([2]).  $\lambda_2 \leq 1 - \frac{1}{2dn^3}$ .

*Proof.* We first recall:

**Claim 8.** 
$$v^{\dagger}Av = v^{\dagger}v - \frac{1}{2}\sum_{i,j} A[i,j](v_i - v_j)^2$$
.

Thus,

$$\lambda_2 = \max_{x \perp 1} \frac{x^{\dagger} A x}{x^{\dagger} x} = 1 - \frac{1}{2d} \cdot \min_{x \perp 1, \|x\| = 1} \sum_{i, j: (i, j) \in E} (x_i - x_j)^2.$$

Let x be the non-zero vector with ||x|| = 1 and  $x \perp \mathbf{1}$  that is gives the above optimum. W.l.o.g.  $x_1 \geq \frac{1}{\sqrt{n}}$ . There exists  $j \neq 1$  for which  $x_j < 0$ . Clearly,  $|x_1 - x_j| \geq \frac{1}{\sqrt{n}}$ .

As G is connected, there is a path of length at most n from vertex 1 to vertex j, so by the triangle inequality there exists an edge  $\{\ell_1, \ell_2\} \in E$  on that path so that  $|x_{\ell_1} - x_{\ell_2}| \ge \frac{1}{n\sqrt{n}}$ . Hence:

$$\sum_{(i,j)\in E} (x_i - x_j)^2 \ge \frac{1}{n^3},$$

so overall  $\lambda_2 \leq 1 - \frac{1}{2dn^3}$ . **Theorem 9** ([3]).  $\lambda_n \geq -1 + \frac{1}{dn^2}$ . *Proof.* It holds that  $\lambda_n = \min_{x:||x||=1} x^{\dagger} A x$ . If G is simple (that is, no self loops), then for an x that minimizes  $x^{\dagger} A x$  we have

$$1 + \lambda_n = \frac{1}{d} \sum_i dx_i^2 + \frac{2}{d} \sum_{\{i,j\} \in E} x_i x_j = \frac{1}{d} \sum_{\{i,j\} \in E} (x_i^2 + x_j^2) + \frac{2}{d} \sum_{\{i,j\} \in E} x_i x_j = \frac{1}{d} \sum_{\{i,j\} \in E} (x_i + x_j)^2.$$

If G is not simple then

$$1 + \lambda_n \geq \frac{1}{d} \sum_{\{i,j\} \in E, i \neq j} (x_i + x_j)^2.$$

In any case,  $\lambda_n \ge -1 + \frac{1}{d} \sum_{\{i,j\} \in E, i \ne j} (x_i + x_j)^2$ . W.l.o.g., assume that  $x_1 \ge \frac{1}{\sqrt{n}}$ .

We partition the vertices to two sets,  $A = \{i : x_i \ge 0\}$  and  $B = [n] \setminus A$ . As G is not bipartite, there exists an edge  $\{i, j\} \in E$  such that either both coordinates  $x_i, x_j$  are non-negative or both of them are negative, i.e.,  $A \times A \cap E \neq \emptyset$  or  $B \times B \cap E \neq \emptyset$  (why?).

First, consider the case where  $\{i, j\} \in A \times A \cap E$  and take the shortest path from vertex 1 to *i* (assume it's shorter than the shortest path from 1 to *j*). Then, either the path  $1 \sim \ldots \sim i$  or  $1 \sim \ldots \sim i \sim j$  is of even length. Let *k* be the endpoint of the path of odd length (i.e.,  $k \in \{i, j\}$ ) and w.l.o.g. assume that the path is labeled by [k] (so *k* is even). We then have:

$$\sum_{\{i,j\}\in E} (x_i + x_j)^2 \geq \sum_{\ell=1}^{k-1} (x_\ell + x_{\ell+1})^2 \geq \frac{1}{k-1} \left( \sum_{\ell=1}^{k-1} |x_\ell + x_{\ell+1}| \right)^2.$$

As k is even, we can write

$$\left(\sum_{\ell=1}^{k-1} |x_{\ell} + x_{\ell+1}|\right)^2 \geq ((x_1 + x_2) + (-x_2 - x_3) + \dots + (-x_{k-2} - x_{k-1}) + (x_{k-1} + x_k))^2$$
$$= (x_1 + x_k)^2 \geq \frac{1}{n},$$

so  $\lambda_n \ge -1 + \frac{1}{dn(k-1)} \ge -1 + \frac{1}{dn^2}$ .

Next, we consider the case where  $\{i, j\} \in B \times B \cap E$ . Using the above reasoning, there exists a path  $1, \ldots, k$  where k is odd and  $x_k < 0$ . Similarly, we can write

$$\left(\sum_{\ell=1}^{k-1} |x_{\ell} + x_{\ell+1}|\right)^2 \geq ((x_1 + x_2) + (-x_2 - x_3) + \dots + (x_{k-2} + x_{k-1}) + (-x_{k-1} - x_k))^2$$
$$= (x_1 - x_k)^2 \geq \frac{1}{n},$$

and the same result holds.

**Corollary 10.** Let G be a regular, connected, non-bipartite, undirected graph over n vertices. Then,  $\bar{\lambda}(G) \leq 1 - \frac{1}{2n^4}$ .

In fact, tighter bonds are known for both the second and smallest eigenvalue:

**Corollary 11.** Let G be a d-regular, connected, non-bipartite, undirected graph over n vertices. Then,  $\bar{\lambda}(G) \leq 1 - \frac{1}{dn^2}$ .

## 3 Random walks over expanders mix fast

In a random walk over a graph G, we start with some initial vertex  $v_0$  and at each step we move from vertex v to an adjacent vertex in  $\Gamma(v)$  with probability proportional to the degree of v. Namely, if A is the normalized adjacency matrix of G, we move from vertex i to vertex j with probability A[j, i].

Suppose we start a random walk at a vertex chosen by a probability distribution p. After taking one step, the probability of being at vertex i is  $\sum_{j} p_{j}A[i, j]$  so the probability distribution after one step is described by Ap.

Iterating the above reasoning, we see that, after a *t*-step random walk whose initial vertex is chosen according to p, the last vertex reached is distributed according to  $A^t p$ . We say that  $\pi$  is a *stationary distribution* if  $A\pi = \pi$ , i.e., no further steps change the distribution.

Does every random walk over a graph approach some stationary distribution? If so, how fast? In the case of undirected regular expanders, the uniform distribution is the stationary distribution and we converge to it in a rate that depends on  $\bar{\lambda}(G)$ . Indeed,  $\bar{\lambda}(G^t) = \bar{\lambda}(G)^t$  so if  $\bar{\lambda}(G)$  is bounded away from 1,  $\bar{\lambda}(G^t)$  approaches 0 and  $A^t \to \lambda_1 v_1 v_1^{\dagger} = \frac{1}{n}J$ . Thus,  $A^t p \to \frac{1}{n}\mathbf{1}$  for every distribution p. Formally:

**Lemma 12.** Let G be a regular graph over n vertices with normalized adjacency matrix A. Then, for every distribution p over the vertices and integer t, we have

$$\left\|A^t p - \frac{1}{n}\mathbf{1}\right\| \leq \bar{\lambda}(G)^t.$$

*Proof.* Note that for every distribution,  $\frac{1}{n}Jp = \frac{1}{n}\mathbf{1}$  and recall that  $\bar{\lambda}(G)^t = \bar{\lambda}(G^t) = ||A^t - \frac{1}{n}J||$ . Denote by  $\lambda_n \leq \ldots \leq \lambda_1$  the eigenvalues of A and  $v_n, \ldots, v_1$  the corresponding eigenvectors. Recall that  $v_1 = \frac{1}{\sqrt{n}}\mathbf{1}$ ,  $\lambda_1 = 1$  and we can write  $A = \sum_i \lambda_i v_i v_i^{\dagger}$  and  $A^t = \sum_i \lambda_i^t v_i v_i^{\dagger}$ . Thus:

$$\left\|A^t p - \frac{1}{n}\mathbf{1}\right\| \leq \left\|A^t p - \frac{1}{n}Jp\right\| \leq \left\|A^t - \frac{1}{n}J\right\| \|p\| \leq \bar{\lambda}(G)^t.$$

We often measure the distance between distribution in the  $\ell_1$ -norm, as it is, up to a factor of 2, equivalent to the *total variation distance* between probability distributions – the maximum over all events of the difference between the probability of the event happening with respect to one distribution and the probability of it happening with respect to the other distribution. As  $||x||_1 \leq \sqrt{|\mathsf{Supp}(x)|} ||x||_2$ , we get:

**Corollary 13.** Let G be a regular graph over n vertices with normalized adjacency matrix A. Then, for every distribution p over the vertices and integer t, we have

$$\left\|A^t p - \frac{1}{n} \mathbf{1}\right\|_1 \leq \sqrt{n} \cdot \bar{\lambda}(G)^t.$$

Specifically, for  $t = \Omega\left(\frac{1}{1-\overline{\lambda}(G)}\ln\frac{n}{\varepsilon}\right)$  we have  $\left\|A^tp - \frac{1}{n}\mathbf{1}\right\|_1 \leq \varepsilon$ .

#### 3.1 Expanders have small diameter

The diameter of a graph is the maximum minimal distance between two vertices in the graph. For an undirected regular graph G, if  $\overline{\lambda}(G)$  is constant bounded away from 1, the graph's diameter is logarithmic. More generally:

**Lemma 14.** Let G be a d-regular undirected connected graph over n vertices. Then, the diameter of G is at most  $1 + \log_{\frac{1}{N(G)}} n$ .

The proof follows from  $\|A^t p - \frac{1}{n} \mathbf{1}\|_{\infty} \le \|A^t p - \frac{1}{n} \mathbf{1}\| \le \overline{\lambda}(G)^t$ .

## 4 Undirected connectivity in RL

Say we are given an undirected graph G and two vertices s, t and wish to decide whether s is connected to t. The fact that undirected graphs have non-negligible spectral gap together with the fact that spectral gap implies mixing, readily suggests the following algorithm:

- 1. Add self loops to G (at least one for each vertex) until it becomes regular. Note that it does not affect the graph's connectivity.
- 2. Start at vertex s.
- 3. For  $T = 4n^4 \ln n$  steps: Move to a random neighbor of the current vertex.
- 4. If at any point you come across t, halt and output "connected". Otherwise, output "not connected".

Theorem 15 ([1]). USTCONN  $\in \mathsf{RL}$ .

*Proof.* First, note that we only need to keep track of the current vertex, the number of steps and taking a random step. The overall space complexity is  $O(\log) + \log T = O(\log n)$ . If s is not connected to t, we always reject. We now move to the main part – analyzing the success probability in the case where s and t are connected. For the sake of analysis, we consider the connected component of s and t.

Since every vertex contains a self loop, the graph has no bipartite components. Let A be the normalized adjacency matrix of G (after adding the self loops), with eigenvalues  $\lambda_n \leq \ldots \leq \lambda_1$  and corresponding eigenvectors  $v_n, \ldots, v_1$ . Let  $x_T$  be the probability distribution over the vertices after T steps. Thus,  $x_T = A^T x_0$  where  $x_0$  is 1 on s and 0 elsewhere.

By Lemma 12 and Corollary 11, we have

$$\left\|x_T - \frac{1}{n}\mathbf{1}\right\|_{\infty} \leq \left\|x_T - \frac{1}{n}\mathbf{1}\right\| \leq \left(1 - \frac{1}{2n^4}\right)^T \leq e^{-\frac{T}{2n^4}} \leq \frac{1}{n^2},$$

so particularly  $x_T(t) \geq \frac{1}{2n}$ .

If we use the tighter bound for  $\overline{\lambda}(G)$ , it is sufficient to take  $T = 2dn^2 \ln n$  (where d is the degree of our "regularized" graph).

Think about which ingredients would fail if we consider directed graphs instead of undirected ones.

#### 4.1 Universal traversal sequences

**Definition 16.** A walk according to  $\sigma \in [d]^T$  that starts in  $v_1$  corresponds to taking the  $\sigma(i)$ -th neighbor of  $v_i$  at step *i*. A sequence  $\sigma$  is a universal traversal sequence for *d*-regular undirected graphs if for every such graph, every labeling of its edges and every two vertices *s*, *t*, *a* walk from *s* according to  $\sigma$  will visit *t* (provided they are connected).

Claim 17. There exists a UTS of length  $O(dn^5 \log n \log d)$ .

*Proof.* Fix  $T = 4n^4 \ln n$ , fix a *d*-regular undirected graph *G* and two vertices *s*, *t*. We proved that a uniform  $\sigma$  does not reach *t* with probability at most  $1 - \frac{1}{2n}$  (and it is true regardless of the vertex we start with!).

Thus, over a uniform  $\sigma$ , the probability that a random walk from s does not reach t within  $T' = 8Tn^3 \log d = O(n^7 \log n \log d)$  steps is at most  $\left(1 - \frac{1}{2n}\right)^{8n^3} \leq 2^{-4n^2 \log d}$ . The total number of labeled graphs is at most  $(d+1)^{n^2} \leq 2^{2 \log dn^2}$ , so overall, by the union-bound, the probability that a uniform  $\sigma$  is not a UTS is at most

$$n^2 \cdot 2^{2\log dn^2} \cdot 2^{-4n^2\log d} < 1.$$

so there exists such a UTS. The length of a UTS can be decreased to  $T' = O(dn^5 \log n \log d)$  by strengthening the bound on the spectral gap of undirected graphs (hence taking  $T = 2n^2 d \ln n$ ).  $\Box$ 

We will soon prove that USTCONN  $\in L$  – a major breakthrough in *derandomization*, due to Reingold. Finding an *explicit* UTS is still a major open problem in *pseudorandomness* and we will talk about it later on.

For *consistently labeled expander* graphs, an explicit UTS is known and we will see it in the exercise.

## 5 The Expander Mixing Lemma

We first show that an expander behaves like a random graph in the following sense: The number of edges between every two *large* subsets  $S, T \subseteq [n]$  is close to what we would have expected in a random graph of average degree d, i.e.,  $\frac{d}{n}|S||T|$ .

**Lemma 18** (Expander Mixing Lemma). Let G = (V = [n], E) be a d-regular graph and let  $S, T \subseteq [n]$ . Then,

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \bar{\lambda}(G) \cdot d\sqrt{|S|(1-|S|/n)|T|(1-|T|/n)}$$

where |E(S,T)| is the number of edges between the two sets.

*Proof.* Let A be the normalized adjacency matrix of G, so we have

$$|E(S,T)| = d \cdot \mathbf{1}_T^{\dagger} A \mathbf{1}_S.$$

We decompose  $\mathbf{1}_S$  and  $\mathbf{1}_T$  to a component parallel to  $\mathbf{1}$  (the 1-eigenvector of A) and a perpendicular component. Write  $\mathbf{1}_S = \frac{|S|}{n} \mathbf{1} + \frac{1}{n} \mathbf{1}_S^{\perp}$  where

$$\mathbf{1}_{S}^{\perp}[i] = \begin{cases} n - |S| & i \in S \\ -|S| & i \notin S. \end{cases}$$

and notice that  $\mathbf{1}_{S}^{\perp} \perp \mathbf{1}$ . Similarly we write  $\mathbf{1}_{T} = \frac{|T|}{n}\mathbf{1} + \frac{1}{n}\mathbf{1}_{T}^{\perp}$ . Then, using the fact that  $A\mathbf{1} = \mathbf{1}$ :

$$E(S,T) = d \cdot \left(\frac{|T|}{n}\mathbf{1} + \frac{1}{n}\mathbf{1}_T^{\perp}\right)^{\dagger} A\left(\frac{|S|}{n}\mathbf{1} + \frac{1}{n}\mathbf{1}_S^{\perp}\right)$$
$$= d \cdot \frac{|S||T|}{n^2}\mathbf{1}^{\dagger}\mathbf{1} + \frac{1}{n^2}\left(\mathbf{1}_T^{\perp}\right)^{\dagger} A\mathbf{1}_S^{\perp}.$$

As both  $\mathbf{1}_S$  and  $\mathbf{1}_S$  are perpendicular to the 1-eigenvector,

$$\left| \left( \mathbf{1}_T^{\perp} \right)^{\dagger} A \mathbf{1}_S^{\perp} \right| \leq \bar{\lambda}(G) \cdot \left\| \mathbf{1}_T^{\perp} \right\| \cdot \left\| \mathbf{1}_S^{\perp} \right\|.$$

A simple calculation shows that  $\|\mathbf{1}_{S}^{\perp}\| = \sqrt{n|S|(n-|S|)}$  and likewise for  $\|\mathbf{1}_{T}^{\perp}\|$ , so overall

$$\begin{aligned} \left| |E(S,T)| - \frac{d|S||T|}{n} \right| &\leq \bar{\lambda}(G) \cdot \frac{d}{n^2} \cdot \sqrt{n|S|(n-|S|)} \sqrt{n|T|(n-|T|)} \\ &= \bar{\lambda}(G) \cdot d \cdot \sqrt{|S|(1-|S|/n)} \sqrt{|T|(1-|T|/n)}, \end{aligned}$$

as desired.

**Corollary 19.** With respect to densities (dividing by dn), we can express the above result as

$$\left| \Pr_{e=(i,j)\in E} [i \in S \land j \in T] - \rho(S)\rho(T) \right| \leq \bar{\lambda}(G) \cdot \sqrt{\rho(S)(1-\rho(S))\rho(T)(1-\rho(T))},$$

where for  $A \subseteq B$  we denote  $\rho(A) = |A|/|B|$ .

#### 5.1 Expanders have no small cuts

An often desirable feature of a graph is that no deletion of few edges can cause the graph to be disconnected. It is indeed the case with expanders. Given an undirected *d*-regular graph G = (V, E) we define the *edge expansion* of a cut  $(S, V \setminus S)$  as

$$h(S) = \frac{|E(S, V \setminus S)|}{d \cdot \min\{|S|, |V \setminus S|\}}$$

and we let  $h(G) = \min_{S \subset V} h(S)$ .

**Exercise 20.** Let G = (V, E) be a d-regular undirected graph over n vertices. Use the expander mixing lemma to prove  $h(G) \geq \frac{1-\bar{\lambda}(G)}{2}$ .

We want to prove the stronger theorem:

**Theorem 21.** Let G = (V, E) be a d-regular undirected graph over n vertices and let  $\lambda_2$  be the second eigenvalue of its normalized adjacency matrix A. Then,  $h(G) \geq \frac{1-\lambda_2}{2}$ .

That is, for every  $S \subseteq [V]$  of cardinality at most  $\frac{n}{2}$ ,  $|E(S, V \setminus S)| \ge \frac{d(1-\lambda_2)}{2}|S|$ .

This theorem is one side of "Cheeger's Inequality". The other, harder, side is  $h(G) \leq \sqrt{2(1-\lambda_2)}$  and we will not prove it. Morally, Cheeger's Inequality tells us that algebraic expansion and edge expansion are equivalent up to some loss in parameters.

*Proof.* We need to prove that  $\lambda_2 \ge 1 - 2h(S)$  for every S with  $|S| \le \frac{n}{2}$ . Equivalently, we can find a  $v \perp \mathbf{1}$  for which  $\frac{v^{\dagger}Av}{v^{\dagger}v} \ge 1 - 2h(S)$ . Define a vector v such that:

$$v_i = \begin{cases} -n + |S| & i \in S \\ |S| & i \notin S. \end{cases}$$

First, notice that  $v \perp \mathbf{1}$ , as  $\sum_i v_i = |S|(-n+|S|) + |S|(n-|S|) = 0$ . Also, we have

$$v^{\dagger}v = |S|(-n+|S|)^2 + (n-|S|)|S|^2 = n|S|(n-|S|).$$

In our case,

$$\sum_{i,j} A[i,j](v_i - v_j)^2 = \frac{1}{d} \sum_{(i,j) \in E(S,\overline{S})} (|S| - (|S| - n))^2 = \frac{n^2}{d} 2|E(S,V \setminus S)|_{S}$$

so  $v^{\dagger}Av = v^{\dagger}v - \frac{n^2}{d}|E(S, V \setminus S)|$ , and

$$\frac{v^{\dagger}Av}{v^{\dagger}v} = 1 - \frac{n^{2}|E(S,V \setminus S)|}{d \cdot v^{\dagger}v} = 1 - \frac{n|E(S,V \setminus S)|}{d \cdot |S|(n-|S|)} \ge 1 - \frac{2|E(S,V \setminus S)|}{d \cdot |S|} = 1 - 2h(S).$$

# References

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