0368-4283: Space-Bounded Computation 27/3/2018 – Lecture 4

Reingold's Algorithm

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1 USTCONN $\in \mathsf{L}$ via the Zig-Zag Product

We will now re-prove USTCONN $\in \mathsf{L}$, this time using the zig-zag product. This is actually the original, breakthrough, proof, and was given by Omer Reingold [2].

We start with an undirected graph $G = (V, E)$ with $|V| = N$ vertices and two fixed vertices $s, t \in V$. The overall idea is to turn our original graph to an expander using the zig-zag product. Once the connected component of s (or, more accurately, the vertex that corresponds to s in the new graph) has logarithmic diameter, we can simply go over all possible paths.

Step 1: We want to construct an undirected graph $G_0 = (V_0, E_0)$ and $s_0, t_0 \in V_0$ so that:

- s_0 and t_0 are connected in G_0 iff s and t are connected in G.
- Every vertex in G_0 has a self loop.
- G_0 is D^2-regular for D large enough so there exists a $(D^4, D, \frac{1}{4})$ graph, call it H.

The way we do it is by replacing every vertex $v \in V$ with degree d_v by a cycle having d_v vertices and connect the inter-cycle edges accordingly. This gives a 3-regular graph, so we just add self loops to every vertex until we reach D^2.

Clearly, we can choose s_0 to be any vertex in the cycle of s and likewise for t_0. Let $|V_0| = N_0$ and note that $N_0 = \text{poly}(N)$.

Step 2: Fix some $K = \Theta(\log N_0)$. For $k \in [K]$, define

$$G_k = G_{k-1}^2 \text{ⓩ} H.$$

Let s_k be any vertex in the cloud corresponding to s_{k-1} and likewise for t_k. Let C_k be the connected component of s_k in G_k and note that the our graph product acts separately within connected components, i.e., $C_k = C_{k-1}^2 \text{ⓩ} H$.

Claim 1. *C_K is a $(N_0 \cdot D^{4K}, D^2, \frac{17}{18})$-graph.*

Proof. Let $\gamma_0 \geq \frac{1}{D^2 N_0^2}$ be the spectral gap of G_0. For $k \geq 1$, C_k is a $(N_0 D^{4k}, D^2, \lambda_k = 1 - \gamma_k)$-graph, where

$$\gamma_k \;\geq\; \left(\frac{3}{4}\right)^2 (1 - (1 - \gamma_{k-1})^2) \;=\; \frac{9}{16}(1 - \lambda_{k-1}^2) \;=\; \frac{9}{16}(1 - \lambda_{k-1})(1 + \lambda_{k-1}).$$

Now, if $\lambda_{k-1} \geq \frac{17}{18}$ then $\gamma_k \geq \frac{9}{16}\gamma_{k-1}\left(1 + \frac{17}{18}\right) = \frac{35}{32}\gamma_k$. If $\lambda_{k-1} < \frac{17}{18}$ then $\gamma_k \geq \frac{9}{16}(1 - \frac{17}{18}^2) > \frac{1}{18}$. Thus, $\gamma_{k+1} \geq \min\left\{\frac{1}{18}, \frac{35}{32}\gamma_k\right\}$. Setting $K = \frac{1}{\log(35/32)} \log(D^2 N_0^2)$, the proof is complete. □

Corollary 2. *Let R_K be the diameter of C_K. Then, $R_K = O(\log N)$.*

The following claim holds since the squaring and zig-zag product preserves connectivity (why?).

Claim 3. *s_K is connected to t_K in G_K iff s is connected to t in G.*

Step 3: For every $(i_1, \ldots, i_{R_k}) \in [D^2]^{R_K}$, compute $s_k[i_1] \ldots [i_{R_K}]$. If one of them is t_K, accept. Otherwise, reject.

To complete the proof, we will prove in class (using in-place computation):

Lemma 4. *For every $v \in [N_0 \cdot D^{4K}]$ and $(i, j) \in [D^2]$, $Rot_{G_K}(v, (i, j))$ can be computed in $\mathsf{DSPACE}(O(\log N))$.*

References

[1] Michal Koucky. *On traversal sequences, exploration sequences and completeness of Kolmogorov random strings*. PhD thesis, Rutgers University, 2003.

[2] Omer Reingold. Undirected connectivity in log-space. *J. ACM*, 55(4), 2008.