MATH 4022 Graph Theory (Fall '10)

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Home Assignment 4

Due date: 11/18/10

Please submit organized and well written solutions!

Problem 1. Suppose $(u, v) \notin E(G)$ and $d(u) + d(v) \geq n$. Show that G is Hamiltonian if and only if $G + (u, v)$ is.

Problem 2. Show that if G is Hamiltonian, then removing from G any $\emptyset \neq S \subseteq V(G)$ creates at most $|S|$ connected components.

Problem 3. Let G be a network with capacity 1 on each edge. Show that if G has a flow of value k then there are k edge-disjoint paths from s to t.

Hint: Start by showing that there is one path.

Problem 4. A directed graphs is *strongly-connected* if for any pair of vertices u, v we have a directed path from u to v and a directed path from v to u. A directed graphs is *cut-connected* if for any $S \subseteq V(G)$ there is at least one edge from S to $V(G) \setminus S$ (assuming $S \neq \emptyset, V(G)$). Show that G is strongly-connected if and only if it is cut-connected.

Problem 5. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$ we two sequences of non-negative integers satisfying $\sum_i a_i = \sum_j b_j$. We say that (a, b) is a *realizable* pair if there is a bipartite graph on vertex sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ where $d(x_i) = a_i$ and $d(y_j) = b_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. In what follows we assume (wlog) that $b_1 \geq b_2 \geq \ldots \geq b_m$.

1. Show that the following condition is necessary for (a, b) to be realizable:

$$\sum_{i=1}^{n} \min(a_i, k) \geq \sum_{j=1}^{k} b_j \qquad \text{for every } 1 \leq k \leq m$$

2. Given (a, b) let $N(a, b)$ be the network on vertices $s, t, x_1, \ldots, x_n, y_1, \ldots, y_m$ and the following edges: for every $1 \leq i \leq n$ and $1 \leq j \leq m$ there is a directed edge (x_i, y_j) with capacity 1. For every $1 \leq i \leq n$ we have a directed edge (s, x_i) of capacity a_i and for every $1 \leq j \leq m$ we have a directed edge (y_j, t) of capacity b_j.

   Show that (a, b) is realizable if and only if $N(a, b)$ has a flow of value $\sum_i a_i$.

3. Use item (2) to prove that the necessary condition from item (1) is also sufficient.