MATH 4032 Combinatorial Analysis (Spring '11)

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Home Assignment 2

Due date: 03/01/11

Please submit organized and well written solutions!

Problem 1. Explain why for any $1 \le k \le n$ and $0 < x < 1$ we have

$$\binom{n}{k} x^k \le (1+x)^n \le e^{xn} .$$

Use this to show that $\binom{n}{k} \le (\frac{en}{k})^k$. Then modify your proof to show that in fact $\sum_{i=1}^{k} \binom{n}{i} \le (\frac{en}{k})^k$.

Problem 2. Use integration to prove the following bounds

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + \Theta(1)$$

$$\sum_{k=1}^{n} k^6 = (1+o(1))\frac{n^7}{7}$$

Problem 3. Let $q(n)$ denote the number of **ordered** sets of positive integers whose sum is n.

1. Calculate $q(n)$ using a direct counting argument.
2. How many **ordered** sets of k positive integers are there whose sum is n? Use your answer to calculate $q(n)$ (in a less direct way).

Problem 4. Let $p(n)$ denote the number of **unordered** sets of positive integers whose sum is n. Show that

$$p(n) \ge \max_{1 \le k \le n} \frac{\binom{n-1}{k-1}}{k!}$$

Hint: Recall item (2) from the previous question.
Deduce that there is an absolute constant $c > 0$ for which $p(n) \ge e^{c\sqrt{n}}$.

Problem 5. Let $\pi(m,n)$ denote the set of prime numbers in the interval $[m,n]$.

1. Show that $\prod_{p \in \pi(m+1,2m)} p \le \binom{2m}{m}$.
2. Use the previous item to show that $\prod_{p \in \pi(1,n)} p \le 4^n$.
3. Deduce from the previous item that $|\pi(1,n)| = O(n/\log n)$.