Problem 1. Explain why for any $1 \leq k \leq n$ and $0 < x < 1$ we have

$$\left(\frac{n}{k}\right) x^k \leq (1 + x)^n \leq e^{xn}.$$  

Use this to show that $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$. Then modify your proof to show that in fact $\sum_{i=1}^{k} \binom{n}{i} \leq \left(\frac{en}{k}\right)^k$.

Problem 2. Use integration to prove the following bounds

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + \Theta(1)$$

$$\sum_{k=1}^{n} k^6 = (1 + o(1)) \frac{n^7}{7}$$

Problem 3. Let $q(n)$ denote the number of ordered sets of positive integers whose sum is $n$.

1. Calculate $q(n)$ using a direct counting argument.

2. How many ordered sets of $k$ positive integers are there whose sum is $n$? Use your answer to calculate $q(n)$ (in a less direct way).

Problem 4. Let $p(n)$ denote the number of unordered sets of positive integers whose sum is $n$.

Show that

$$p(n) \geq \max_{1 \leq k \leq n} \frac{(n-1)}{k!}$$

Hint: Recall item (2) from the previous question.

Deduce that there is an absolute constant $c > 0$ for which $p(n) \geq e^{c \sqrt{n}}$.

Problem 5. Let $\pi(m, n)$ denote the set of prime numbers in the interval $[m, n]$.

1. Show that $\prod_{p \in \pi(m+1, 2m)} p \leq \binom{2m}{m}$.

2. Use the previous item to show that $\prod_{p \in \pi(1, n)} p \leq 4^n$.

3. Deduce from the previous item that $|\pi(1, n)| = O(n/\log n)$.