MATH 4032 Combinatorial Analysis (Spring '11)

Instructor: Asaf Shapira

Home Assignment 3

Due date: 03/15/11

Please submit organized and well written solutions!

Problem 1. Show that there is an integer n_0 such that for all $n \geq n_0$, in every 9-coloring of the integers $\{1, 2, \ldots, n\}$, one of the 9 color classes contains 4 integers a, b, c, d satisfying $a + b + c = d$. Can you show that if $n \geq n'_0$, then we can guarantee 4 **distinct** integers a, b, c, d as above?

Problem 2. Show that every tournament on n vertices, contains a transitive tournament on $\lfloor \log_2 n \rfloor$ vertices. Also, show that there exists a tournament on n vertices that does not contain a transitive tournament on $2 \log_2 n + 2$ vertices.

Problem 3. Show that if an n-vertex graph $G = (V, E)$ has no copy of $K_{2,t}$ then we have

$$|E| \leq \frac{1}{2}(\sqrt{t-1}\; n^{3/2} + n)\ .$$

Problem 4. Suppose $S_1, \ldots, S_n \subseteq [n]$ have the property that $|S_i \cap S_j| \leq 1$ for all $1 \leq i < j \leq n$. Show that in this case $\frac{1}{n} \sum_i |S_i| = O(\sqrt{n})$.

Problem 5. Let $P = \{p_1, \ldots, p_n\}$ be n distinct points in the plain. Define a graph G on P by connecting p_i to p_j if and only if the distance between them is 1. Show that G has no copy of $K_{2,3}$ and deduce that any set of n points in the plain determines $O(n^{3/2})$ unit distances.

Problem 6. Let $P = \{p_1, \ldots, p_n\}$ be n distinct points in the plain and $L = \{\ell_1, \ldots, \ell_n\}$ be n distinct lines in the plain. We say that a pair (i, j) is *good* if p_i lies on ℓ_j. Show that for any P and L the number of good pairs is $O(n^{3/2})$.