Problem 1. Prove that the number of surjective (i.e. onto) mappings from \([n]\) to \([k]\) is given by 
\[ \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \].
Use this to deduce that:
\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n = n! \].
\[ \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n = 0 \text{ when } k > n \].
\[ S(n, k) = \frac{1}{n!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \], where \( S(n, k) \) are the Stirling numbers of the second kind.

Problem 2. Consider the number of ways of coloring the integers \(\{1, \ldots, 2n\}\) using the colors red/blue in such a way that if \(i\) is colored red then so does \(i-1\). Deduce the identity
\[ \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} 2^{2n-2k} = 2n + 1 \]

Problem 3. Let \(\varphi(n)\) denote the Euler totient function.

1. Show that if \(m, n\) are coprime then \(\varphi(mn) = \varphi(m)\varphi(n)\) (you can use the formula for \(\varphi(n)\)).
2. Derive the formula of \(\varphi(n)\) from the assumption that \(\varphi(mn) = \varphi(m)\varphi(n)\) for coprime \(m, n\).
3. Use item (1) to prove that \(\sum_{d|n} \varphi(d) = n\) (Hint: induction on the number of prime divisors).

Problem 4. Let \(A_1, \ldots, A_n\) be a family of \(n\) sets. Show that
\[ |\bigcup_{i=1}^{n} A_i| \geq \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \]
and
\[ |\bigcup_{i=1}^{n} A_i| \leq \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \]