Problem 1. Let $\pi$ be a permutation that can be generated by $m$ transpositions. Show that $(-1)^m = (-1)^t$ where $t$ is the number of inversion of $\pi$.

Problem 2. Compute the number of spanning trees of the graph obtained from $K_n$ by removing one edge.

Problem 3. Suppose the functions $f, g, h : \mathbb{N} \rightarrow \mathbb{N}$ satisfy the relation

$$g(n) = \sum_{d|n} f(d)h\left(\frac{n}{d}\right).$$

Show that if $g$ and $h$ are multiplicative then so is $f$ (where by multiplicative we mean $f(mn) = f(m)f(n)$ for coprime $m, n$).


**Hint:** Given $r \times s$ and $s \times r$ matrices $A$ and $B$, consider the 3-layer graph on vertex sets $A = \{a_1, \ldots, a_r\}$, $B = \{b_1, \ldots, b_s\}$ and $C = \{c_1, \ldots, c_r\}$ where the weight of the edge $(a_i, b_j)$ is $A_{i,j}$ and the weight of the edge $(b_i, c_j)$ is $B_{i,j}$.

Problem 5. Let $s(n, k)$ and $S(n, k)$ denote the Stirling numbers of the first and second kind.

- Show that

$$x^n = \sum_{k=1}^{n} \left( \sum_{m=k}^{n} S(n, m)s(m, k) \right) x^k.$$

- Let $A, B$ be the $n \times n$ matrices whose $(i, j)$ entries are $s(i, j)$ and $S(i, j)$ respectively. Deduce from the previous item that $B = A^{-1}$. 