MATH 4032 Combinatorial Analysis (Spring '11)

Instructor: Asaf Shapira

Home Assignment 6

Due date: 04/28/11

Please submit organized and well written solutions!

Problem 1. Let π be a permutation that can be generated by m transpositions. Show that $(-1)^m = (-1)^t$ where t is the number of inversion of π.

Problem 2. Compute the number of spanning trees of the graph obtained from K_n by removing one edge.

Problem 3. Suppose the functions $f, g, h : \mathbb{N} \mapsto \mathbb{N}$ satisfy the relation

$$g(n) = \sum_{d|n} f(d)h(\frac{n}{d}) \ .$$

Show that if g and h are multiplicative then so is f (where by multiplicative we mean $f(mn) = f(m)f(n)$ for coprime m, n).

Problem 4. Prove the Binet-Cauchy Theorem using the Lindstrom-Gessel-Viennot Lemma.

Hint: Given $r \times s$ and $s \times r$ matrices A and B, consider the 3-layer graph on vertex sets $A = \{a_1, \ldots, a_r\}$, $B = \{b_1, \ldots, b_s\}$ and $C = \{c_1, \ldots, c_r\}$ where the weight of the edge (a_i, b_j) is $A_{i,j}$ and the weight of the edge (b_i, c_j) is $B_{i,j}$.

Problem 5. Let $s(n, k)$ and $S(n, k)$ denote the Stirling numbers of the first and second kind.

- Show that
$$x^n = \sum_{k=1}^{n} \left(\sum_{m=k}^{n} S(n, m)s(m, k) \right) x^k \ .$$
- Let A, B be the $n \times n$ matrices whose (i, j) entries are $s(i, j)$ and $S(i, j)$ respectively. Deduce from the previous item that $B = A^{-1}$.