MATH 7016 Combinatorics (Spring '09)

Instructor: Asaf Shapira

Home Assignment 1

Due date: 01/20/09

Please submit organized and well written solutions!

Problem 1. Prove that for every r, ℓ and c there is an integer $N = N(r, \ell, c)$ such that for every $n > N$, every c-coloring of all the subsets of $\{1, \ldots, n\}$ of size **at most** ℓ, there exists a subset $S \subseteq \{1, \ldots, n\}$ of size r such that for every $1 \leq i \leq \ell$, all the i-element subsets of S are colored with the same color.

Problem 2. Prove that for every $k \geq 2$, there is an integer $n = n(k)$, such that for every coloring of the integers $1, \ldots, n$ using k colors, there always exist three distinct numbers x, y, z that are colored with the same color and satisfy $x \cdot y = z$.

Problem 3. A tournament on n vertices is an orientation of the complete graph on n vertices. That is, for each pair (i, j) we have either an edge directed from i to j or an edge directed from j to i. A tournament T is said to be transitive if we can number its vertices $1, \ldots, n$ such that T has an edge from i to j if and only if $i < j$.

- Show that every tournament on n vertices, contains a transitive tournament on $\lfloor \log_2 n \rfloor$ vertices.
- Show that there exists a tournament on n vertices that does not contain a transitive tournament on $2\log_2 n + 2$ vertices.

Problem 4. The *Abbott product* of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, denoted $G \otimes H$, is the graph obtained by replacing every vertex $v \in V_G$ with an entire copy of H, denoted H_v, and where two such copies H_v, H_u have either all the edges between them, if $(u, v) \in E_G$, and none of them if $(u, v) \notin E_G$. We let G^ℓ denote $G \otimes G \otimes \cdots \otimes G$ (ℓ times). Let $\omega(G)$ denote the size of the largest clique in G, and $\alpha(G)$ denote the size of the largest independent set in G.

- Prove that $\omega(G \otimes H) = \omega(G) \cdot \omega(H)$ and $\alpha(G \otimes H) = \alpha(G) \cdot \alpha(H)$.
- Prove that in time $m^{O(\log m)}$ we can construct an m-vertex graph, with no clique or independent set of size $2 \log m$.
- Use the above two facts in order to construct in polynomial time an n-vertex graph with no clique or independent set of size $2^{O(\sqrt{\log n} \log \log n)}$.