MATH 7016 Combinatorics (Spring ’09)
Instructor: Asaf Shapira

Home Assignment 3
Due date: 03/5/09

Please submit organized and well written solutions!

Problem 1. Let $E$ be a homogenous linear equation $\sum_{i=1}^{k} a_i x_i = 0$ and denote by $R_E(n)$ the size of the largest subset of $[n]$ containing a solution to $E$ with all $x_i$ being distinct.

- Use the removal lemma for directed graphs to show that if the coefficients of $E$ satisfy $\sum_{i=1}^{k} a_i = 0$ then $R_E(n) = o(n)$.
- Show that if the coefficients of $E$ satisfy $\sum_{i=1}^{k} a_i \neq 0$ then $R_E(n) = \Omega(n)$.

Problem 2. Let $E$ be a linear equation of the form $\sum_{i=1}^{k} a_i x_i = (\sum_{i=1}^{k} a_i)x_{k+1}$, where the $a_i$s are positive integers. A solution of such an equation is trivial if $x_1 = x_2 = \ldots = x_{k+1}$, otherwise it is non-trivial. Prove that there is a constant $C$ (that depends on $a_1, \ldots, a_k$) such that for every large enough $n$, there exists a subset $X \subseteq [n]$ of size $n/C\sqrt{\log n}$ that contains no non-trivial solution of $E$.

Problem 3. For integers $g, k \geq 3$ let $n = n(g, k)$ be the smallest integer for which there exists a graph on $n$ vertices with girth at least $g$ and chromatic number at least $k$. Show that $n(g, k) = k^{\Theta(g)}$, that is, that there are absolute constant $c, C$ such that $k^{cg} \leq n(g, k) \leq k^{Cg}$.

Problem 4. A kernel in a directed graph is an independent set of vertices $D$ such that for every vertex $v \notin D$ there is at least one edge pointing from $v$ to one of the vertices of $D$.

- Show that every strongly connected directed graph without odd directed cycles is bipartite.
- Use the previous item to show that every strongly connected directed graph without odd directed cycles has a kernel.
- Use the previous item to show that every directed graph without odd directed cycles has a kernel (this is Richardson’s Theorem).