MATH 7016 Combinatorics (Spring '09)

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Home Assignment 4

Due date: 03/26/09

Please submit organized and well written solutions!

Problem 1. Let $K_{n,n}$ be the complete bipartite graph on $n = \binom{2k-1}{k}$ vertices. Think of every vertex v in each partition class of $K_{n,n}$ as representing a subset $S_v \subseteq [2k-1]$ of size k, and assign v the list of colors S_v. Show that there is no legal coloring of G from the lists.

Problem 2. A homomorphism from a graph H to a graph G is a mapping $\phi : V(H) \mapsto V(G)$ which maps edges to edges, that is, if $(u, v) \in E(H)$ then $(\phi(u), \phi(v)) \in E(G)$. Show that there is an infinite sequence of graphs $G_1, G_2, \ldots$ such that for every $i < j$ there is no homomorphisms from G_i to G_j or from G_j to G_i.

Hint: Observe that if there is a homomorphism from H to G then $\chi(H) \leq \chi(G)$.

Problem 3. A *topological* K_r in a graph is composed of r vertices that are connected by vertex disjoint paths. The *Hajós number* of a graph is the largest r for which the graph contains a topological K_r.

- Show that the *Hajós number* of $G(n, 0.5)$ is $\Theta(\sqrt{n})$ with high probability.
- Show that there is a graph whose chromatic number is (much) larger than its Hajós number.

  Hint: use the previous item.

Problem 4. Suppose X is a set of n elements, and $S_1, \ldots, S_m$ are m subsets of X of average size at least n/w. Show that if $m \geq kw^k$ then there are k distinct sets $S_{i_1}, \ldots, S_{i_k}$ satisfying $|S_{i_1} \cap \cdots \cap S_{i_k}| \geq n/4w^k$.

Problem 5. Let H be a 3-partite 3-uniform hyper-graph with partition classes of size n each. Show that if H has more than $20n^{2.75}$ edges then it has a copy of the complete 3-partite 3-uniform hyper-graph with each partition class of size 2.

Problem 6. Show that for every ϵ there is an $n_0(\epsilon)$ such that if G is a graph on $n \geq n_0(\epsilon)$ vertices and G contains $\epsilon\binom{n}{3}$ triangles then G contains a copy of K_3^2, that is, the complete 3-partite graph with 2 vertices in each of the three partition classes.

Hint: use the previous problem.