MATH 7016 Combinatorics (Spring '09)

Instructor: Asaf Shapira

Home Assignment 5

Due date: 04/09/09

Please submit organized and well written solutions!

Problem 1. Let P be a partial order on elements $x_1, \ldots, x_n$, and let $G = G(P)$ be the directed graph on $x_1, \ldots, x_n$ in which (x_i, x_j) is a directed edge if and only if $x_i < x_j$ in P.

- Let $LP(G)$ denote the length of the longest path in G. Show that $\chi(G) = LP(G)$.
- Let $PC(G)$ denote the smallest number of directed paths that cover all the vertices of G. Use the previous item, along with a (seemingly unrelated) theorem we saw in class, to prove *Dilworth's Theorem* which states that $PC(G) = \alpha(G)$

Problem 2. Let $R_1, \ldots, R_m$ be a club satisfying $|R_i| = k$ for every i, while $|R_i \cap R_j| = \ell$ for every $i \neq j$. Give a *direct* proof (without using Fischer's Inequality) that $m \leq n$.

Problem 3. Suppose there are m red clubs $R_1, \ldots, R_m \subseteq [n]$ and m blue clubs $B_1, \ldots, B_m \subseteq [n]$ such that $|R_i \cap B_i|$ is odd for every i, while $|R_i \cap B_j|$ is even for every $i \neq j$. Show that $m \leq n$. Can you weaken the second requirement to $i < j$?

Problem 4. Suppose $R_1, \ldots, R_m$ is a club satisfying $|R_i| \neq 0 \pmod 6$ for every i, while $|R_i \cap R_j| = 0 \pmod 6$ for every $i \neq j$. Show that $m \leq 2n$.

Problem 5. An $n \times n$ matrix A is said to be *diagonally dominated* if for every $1 \leq i \leq n$ it satisfies $|A_{i,i}| > \sum_{j \neq i} |A_{i,j}|$.

- Show that a diagonally dominated matrix has full rank.
- Use the previous item to show that the upper bound on the size of 2-distance sets holds even if all the pairwise distances are very close to one of two distances.