MATH 7018 - Probabilistic Combinatorics (Fall '09)

Instructor: Asaf Shapira

Home Assignment 2

Due date: 10/20/09

Please submit organized and well written solutions!

Problem 1. Prove that every 3-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least $2n^{3/2}/\sqrt{27m}$.

Problem 2. Show that if a CNF formula is 2-locally-satisfiable then there is an assignment that satisfies at least a ϕ-fraction of its clauses, where $\phi = \frac{\sqrt{5}-1}{2}$.

Problem 3. A *topological* K_r in a graph is composed of r vertices that are connected by vertex disjoint paths. The *Hajós number* of a graph is the largest r for which the graph contains a topological K_r.

- Show that the *Hajós number* of $G(n, 0.5)$ is $\Theta(\sqrt{n})$ with high probability.
- Show that there is a graph whose chromatic number is (much) larger than its Hajós number.

Problem 4. Prove that $R(4, k) = \Omega(k^2/\log^2 k)$.

Problem 5. Let $T_k(n)$ be the smallest integer such that every n by n bipartite graph with $T_k(n)$ edges contains a copy of the complete k-by-k bipartite graphs. Prove that for every constant $k \geq 2$ there is a $c = c(k) > 0$ such that $T_k(n) \geq cn^{2-2/(k+1)}$.

Hint: Start by showing that $T_k(n) \geq cn^{2-2/k}$ and then try to improve this.

Problem 6. Prove that for every set X of at least $4k^2$ distinct residue classes modulo a prime p, there is an integer a such that the set $\{ax \pmod{p} : x \in X\}$ intersects every interval of length at least p/k in $\{0, 1, ..., p-1\}$.

Hint: Pick random residues a and b and consider $\{ax + b \pmod{p} : x \in X\}$.

Problem 7. Let $M(n)$ be the set of all integers which can be written as a product of two integers from $\{1, \ldots, n\}$. That is $M(n) = \{a \cdot b \ : \ 1 \leq a, b \leq n\}$. Show that $|M(n)| = o(n^2)$.