MATH 7018 - Probabilistic Combinatorics (Fall '09)

Instructor: Asaf Shapira

Home Assignment 3

Due date: 11/03/09

Please submit organized and well written solutions!

Problem 1. Show that the *Moment Method* is always at least as good as the *Chernoff Method*. That is, show that if X is a non-negative random-variable, then for any $t > 0$

$$\min_k \frac{\mathbf{E}[X^k]}{t^k} \leq \inf_{s>0} \frac{\mathbf{E}[e^{sX}]}{e^{st}}$$

Problem 2. Let $S = S(n,p)$ be a random subset of $[n] = \{1, \ldots, n\}$ constructed by putting every integer $x \in [n]$ in S independently with probability p. Find a function $p(n)$ such that if $p \gg p(n)$ then whp $S(n,p)$ contains a 3-term arithmetic progression, while if $p \ll p(n)$ then whp $S(n,p)$ does not contain a 3-term arithmetic progression.

Problem 3. Show that the condition $ep(d+1) \leq 1$ in the symmetric Lovász Local Lemma cannot be replaced by the weaker condition $pd \leq 2$.

Problem 4. Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a list $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that $c \in S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

Problem 5. Prove that there is a positive constant c so that every d-regular graph, where $d \geq 2$, contains a spanning subgraph in which every connected component is a star with at least $cd/\log d$ leaves.

Problem 6. A simple path of an even length $P = v_1, v_2, \ldots, v_{2k}$ in a graph $G = (V, E)$ with a vertex coloring $c : V \mapsto [r]$ is *periodic* if $c(v_j) = c(v_{k+j})$ for all $1 \leq j \leq k$. Prove that there is a finite r so that every graph G with maximum degree 3 admits a vertex coloring with r colors in which no simple path (of any even length) is periodic.

Problem 7. Show that there is a finite n_0 such that any directed graph on $n > n_0$ vertices in which each out-degree is at least $\log_2 n - \frac{1}{100} \log_2 \log_2 n$ contains an even simple directed cycle.