MATH 7018 - Probabilistic Combinatorics (Fall '09)

Instructor: Asaf Shapira

Home Assignment 4

Due date: Not for submission

Please submit organized and well written solutions!

Problem 1. Show that the probability that in the random graph $G(2k, 1/2)$ the maximum degree is at most $k-1$ is at least $1/4^k$.

Problem 2. Let G be a graph and let P denote the probability that a random subgraph of G obtained by picking each edge of G with probability 1/2, independently, is connected (and spanning). Let Q denote the probability that in a random 2-coloring of G, where each edge is chosen, randomly and independently, to be either red or blue, the red graph and the blue graph are both connected (and spanning). Is $Q \le P^2$? (Prove, or supply a counter-example).

Problem 3. Show that for any $\epsilon > 0$ there is a $C = C(\epsilon)$ such that every set S of at least $\epsilon 3^n$ vectors in Z_3^n contains three vectors so that the Hamming distance between any pair of them is at least $n - C\sqrt{n}$. **Hint:** Use an appropriate martingale to show that more than 2/3 of the vectors are within distance $C\sqrt{n}/2$ of S.

Problem 4. Prove that there exists a positive constant $\delta > 0$ and an integer n_0 so that for all $n > n_0$ and every collection $S_1, S_2, \ldots, S_m$, where $m \le 2^{\delta n}$, of subsets of $[2n]$, satisfying $|S_i| = n$ for all $1 \le i \le m$, there is a function $f : [2n] \mapsto [n]$ so that $0.63n \le |f(S_i)| \le 0.64n$ for every $1 \le i \le m$. **Hint:** $0.63 < 1 - 1/e < 0.64$.

Problem 5. Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of V chosen uniformly among all $2^{|V|}$ subsets of V. Let $H = G[U]$ be the induced subgraph of G on U. Prove that

$$Pr[\chi(H) \le 400] < 1/100 .$$

Hint: Prove first that the expectation of $\chi(H)$ is at least 500.

Problem 6. Show that for every $\epsilon > 0$ there are constants $c = c(\epsilon)$ and $n_0 = n_0(\epsilon)$ such that for all $n > n_0$ we have

$$Pr\Big[\ |\chi(G(n,1/2)) - E(\chi(G(n,1/2)))| > c\sqrt{n}/\log n\Big] < \epsilon$$