Problem 1. Show that the probability that in the random graph $G(2k, 1/2)$ the maximum degree is at most $k - 1$ is at least $1/4^k$.

Problem 2. Let $G$ be a graph and let $P$ denote the probability that a random subgraph of $G$ obtained by picking each edge of $G$ with probability $1/2$, independently, is connected (and spanning). Let $Q$ denote the probability that in a random 2-coloring of $G$, where each edge is chosen, randomly and independently, to be either red or blue, the red graph and the blue graph are both connected (and spanning). Is $Q \leq P^2$? (Prove, or supply a counter-example).

Problem 3. Show that for any $\epsilon > 0$ there is a $C = C(\epsilon)$ such that every set $S$ of at least $\epsilon 3^n$ vectors in $\mathbb{Z}_3^n$ contains three vectors so that the Hamming distance between any pair of them is at least $n - C \sqrt{n}$. Hint: Use an appropriate martingale to show that more than $2/3$ of the vectors are within distance $C \sqrt{n}/2$ of $S$.

Problem 4. Prove that there exists a positive constant $\delta > 0$ and an integer $n_0$ so that for all $n > n_0$ and every collection $S_1, S_2, \ldots, S_m$, where $m \leq 2^{bn}$, of subsets of $[2n]$, satisfying $|S_i| = n$ for all $1 \leq i \leq m$, there is a function $f : [2n] \mapsto [n]$ so that $0.63n \leq |f(S_i)| \leq 0.64n$ for every $1 \leq i \leq m$. Hint: $0.63 < 1 - 1/e < 0.64$.

Problem 5. Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of $V$ chosen uniformly among all $2^{|V|}$ subsets of $V$. Let $H = G[U]$ be the induced subgraph of $G$ on $U$. Prove that

$$Pr[\chi(H) \leq 400] < 1/100.$$ 

Hint: Prove first that the expectation of $\chi(H)$ is at least 500.

Problem 6. Show that for every $\epsilon > 0$ there are constants $c = c(\epsilon)$ and $n_0 = n_0(\epsilon)$ such that for all $n > n_0$ we have

$$Pr \left[ |\chi(G(n, 1/2)) - E(\chi(G(n, 1/2)))| > c \sqrt{n}/ \log n \right] < \epsilon$$