MATH 8803 Algebraic Methods in Combinatorics (Spring '10)

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Home Assignment 1

Due date: 02/02/10

Please submit organized and well written solutions!

Problem 1. Let $P = P(x_1, \ldots, x_n)$ be a polynomial in n variables over a field $\mathbb{F}$. For every $1 \leq i \leq n$ let t_i be the degree of P as a polynomial in x_i, and let $S_i \subseteq \mathbb{F}$ be a set of at least $t_i + 1$ distinct members of $\mathbb{F}$. Show that if $P(a_1, \ldots, a_n) = 0$ for all n-tuples $(a_1, \ldots, a_n) \in S_1 \times S_2 \times \cdots \times S_n$, then $P \equiv 0$.

Problem 2. Let p be a prime and suppose $A \subseteq \mathbb{Z}_p$ is sum-free, that is, for any $a, b \in A$ we have $a + b \notin A$. Show that $|A| \leq \lceil p/3 \rceil$, and that this bound is tight for infinitely many values of p.

Problem 3. Use the Chevalley-Waring Theorem to prove the following: Suppose p is a prime and G is a graph with maximum degree $2p - 1$ and average degree strictly larger than $2p - 2$. Then G contains a p-regular subgraph.

Problem 4. Let $H_1, \ldots, H_m$ be a family of affine hyperplanes in $\mathbb{R}^n$ which cover all the points $\{0, 1\}^n$ besides $\mathbf{0}$, which is **not** covered. Show that $m \geq n$ and that this bound is tight. What happens if we allow $\mathbf{0}$ to be covered by $H_1, \ldots, H_m$?

Problem 5. Let p be a prime, suppose $k < p$ and let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be two subsets of $\mathbb{F}_p$ (of distinct elements). Show that there is an ordering $b_{\pi(1)}, \ldots, b_{\pi(k)}$ such that the sums $a_1 + b_{\pi(1)}, \ldots, a_k + b_{\pi(k)}$ are pairwise distinct.

Hint: You can use the fact that when $k < p$ the coefficient of $x_1^{k-1} \cdots x_k^{k-1}$ in the polynomial $\prod_{1 \leq i < j \leq k} (x_i - x_j)^2$ is non-zero over $\mathbb{F}_p$. You may wish to try and compute this coefficient explicitly. Also, what happens if we allow $k = p$?