Problem 1. For two non-empty subsets $A, B \subseteq \mathbb{F}_p$ let $A \oplus B = \{a + b \mid a \in A, b \in B, ab \neq 1\}$. Show that $|A \oplus B| \geq \min\{p, |A| + |B| - 3\}$. Show that this is sharp for $|A|, |B| \geq 2$.

Problem 2. Recall that for $A, B \subseteq \mathbb{F}_p$ we defined $A \triangle B = \{a + b \mid a \in A, b \in B, a \neq b\}$. Show that if $A \neq B$, then $|A \triangle B| \geq \min\{p, |A| + |B| - 2\}$.

Hint: Consider separately the cases $A \cap B = \emptyset$ and $A \cap B \neq \emptyset$. In the latter case, use the result we proved in class, stating that when $|A| < |B|$ we have $|A \triangle B| \geq \min\{p, |A| + |B| - 2\}$.

Problem 3. Let $D = (V, E)$ be an $n$ vertex digraph and suppose there is a set of vertex disjoint cycles covering all its vertices. Show that for any choice of sets $S_1, \ldots, S_n \subseteq \mathbb{R}$, each of size two, there is a choice $x_1 \in S_1, \ldots, x_n \in S_n$, such that for every vertex $1 \leq v \leq n$ we have $\sum_{u : (v, u) \in E} x_u \neq 0$.

Problem 4. A vector $s \in \{\ast, 0\}^m$ is a zero pattern of a set of functions $f_1, \ldots, f_m$ in $n$ variables, if there is $x \in \mathbb{R}^n$ such that for every $1 \leq i \leq m$ we have $f_i(x) = 0$ if and only if $s_i = 0$. Show that a set of $m$ linear functions in $n$ variables has at most $\sum_{i=0}^n \binom{m}{i}$ zero-patterns (assume that $m \geq n$).

Problem 5. Let $f(x, y)$ be a symmetric polynomial. We say that a graph $G = (V, E)$ on $n$ vertices is an $f$-graph if there are $x_1, \ldots, x_n \in \mathbb{R}$ such that $(i, j) \in E(G)$ if and only if $f(x_i, x_j) = x_k$ for some $1 \leq k \leq n$. Show that for any fixed $f$, the number of $n$-vertex $f$-graphs is bounded from above by $n^c$, where $c$ depends only on the degree of $f$.

Problem 6. Let $P_1 = 0, \ldots, P_m = 0$ be a system of polynomial equations in $n$ variables over $\mathbb{R}$, each of total degree at most $d$. Denote by $S$ the set of solutions to this system. The Milnor-Thom Theorem asserts that $S$ has at most $d(2d - 1)^{n-1}$ connected components. Use this to prove the following:

Suppose $d \geq 2$ and $S$ is the set of solutions of the polynomials equations $P_1 = 0, \ldots, P_m = 0$ and the polynomials inequalities $Q_1 \geq 0, \ldots, Q_h \geq 0$, where the polynomials are on the same $n$ variables and are each of total degree at most $d$. Then $S$ has at most $d(2d - 1)^{n+h-1}$ connected components.