MATH 8803 Algebraic Methods in Combinatorics (Spring '10)

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Home Assignment 2

Due date: 03/02/10

Please submit organized and well written solutions!

Problem 1. For two non-empty subsets $A, B \subseteq \mathbb{F}_p$ let $A \oplus B = \{a + b \mid a \in A, b \in B, ab \neq 1\}$. Show that $|A \oplus B| \geq \min\{p, |A| + |B| - 3\}$. Show that this is sharp for $|A|, |B| \geq 2$.

Problem 2. Recall that for $A, B \subseteq \mathbb{F}_p$ we defined $A \dot{+} B = \{a + b \mid a \in A, b \in B, a \neq b\}$. Show that if $A \neq B$, then $|A \dot{+} B| \geq \min\{p, |A| + |B| - 2\}$.
Hint: Consider separately the cases $A \cap B = \emptyset$ and $A \cap B \neq \emptyset$. In the latter case, use the result we proved in class, stating that when $|A| < |B|$ we have $|A \dot{+} B| \geq \min\{p, |A| + |B| - 2\}$.

Problem 3. Let $D = (V, E)$ be an n vertex digraph and suppose there is a set of vertex disjoint cycles covering all its vertices. Show that for any choice of sets $S_1, \ldots, S_n \subseteq \mathbb{R}$, each of size two, there is a choice $x_1 \in S_1, \ldots, x_n \in S_n$, such that for every vertex $1 \leq v \leq n$ we have $\sum_{u:(v,u)\in E} x_u \neq 0$.

Problem 4. A vector $s \in \{*, 0\}^m$ is a *zero pattern* of a set of functions $f_1, \ldots, f_m$ in n variables, if there is $x \in \mathbb{R}^n$ such that for every $1 \leq i \leq m$ we have $f_i(x) = 0$ if and only if $s_i = 0$. Show that a set of m linear functions in n variables has at most $\sum_{i=0}^{n} \binom{m}{i}$ zero-patterns (assume that $m \geq n$).

Problem 5. Let $f(x, y)$ be a symmetric polynomial. We say that a graph $G = (V, E)$ on n vertices is an f-graph if there are $x_1, \ldots, x_n \in \mathbb{R}$ such that $(i, j) \in E(G)$ if and only if $f(x_i, x_j) = x_k$ for some $1 \leq k \leq n$. Show that for any fixed f, the number of n-vertex f-graphs is bounded from above by n^{cn}, where c depends only on the degree of f.

Problem 6. Let $P_1 = 0, \ldots, P_m = 0$ be a system of polynomial equations in n variables over $\mathbb{R}$, each of total degree at most d. Denote by S the set of solutions to this system. The *Milnor-Thom Theorem* asserts that S has at most $d(2d-1)^{n-1}$ connected components. Use this to prove the following:

Suppose $d \geq 2$ and S is the set of solutions of the polynomials equations $P_1 = 0, \ldots, P_m = 0$ and the polynomials inequalities $Q_1 \geq 0, \ldots, Q_h \geq 0$, where the polynomials are on the same n variables and are each of total degree at most d. Then S has at most $d(2d-1)^{n+h-1}$ connected components.