MATH 8803 Algebraic Methods in Combinatorics (Spring '10)

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Home Assignment 3

Due date: 03/23/10

Please submit organized and well written solutions!

Problem 1. Let $\mathcal{P} = \{P_1, \dots, P_m\}$ be m polynomials in n variables $x_1, \dots, x_n$, each of total degree at most d. Let S be a set of points representing distinct sign patterns of $\mathcal{P}$ (that is for $s \neq s' \in S$ there is at least one i for which $P_i(s) > 0$ while $P_i(s') < 0$ or vise versa). Set $\epsilon = \min\{P_i(s) \ : \ 1 \leq i \leq m, s \in S\}$ and define

$$Q(x_1, \dots, x_n, x_{n+1}) = -x_{n+1}^2 - \frac{1}{2}\epsilon^{2m} + \prod_{1 \leq i \leq m} P_i^2(x_1, \dots, x_n)$$

- Show that for every $s = (s_1, \dots, s_n) \in S$ there is s_{n+1} such that $Q(s_1, \dots, s_n, s_{n+1}) = 0$.
- Show that if $s \neq s' \in S$, then the two solutions in the previous item belong to distinct connected components of the variety $Q = 0$.
- Conclude that $\mathcal{P}$ has at most $(2md)^{n+1}$ sign patterns.

Problem 2. Let $f(n, k)$ denote the number of $n \times n$ sign (that is, ± 1) matrices of rank at most k. Show that $f(n, k) \leq (Cn/k)^{2nk}$, where C is an absolute constant.
Hint: Recall that $rank(M) \leq k$ iff there are two $k \times n$ matrices U, V satisfying $U^T V = M$.

Problem 3. Show that for infinitely many values of n, one can *explicitly* construct an $n \times n$ bipartite graph containing no copy of $K_{t,t}$ or its complement, where $t = \sqrt{n}$?

Problem 4. Suppose $R_1, \dots, R_m \subseteq [n]$ is a collection of clubs with the property that $|R_i|$ are all even, while $|R_i \cap R_j|$ are all odd. Show that if n is odd then $m \leq n$, and if n is even $m \leq n - 1$. Show that these two bounds are tight.

Problem 5. Suppose $R_1, \dots, R_m \subseteq [n]$ is a collection of clubs satisfying $|R_i| \neq 0 \pmod 6$ for every i, while $|R_i \cap R_j| = 0 \pmod 6$ for every $i \neq j$. Show that $m \leq 2n$.

Problem 6. Let G be an n-vertex graph and suppose every pair of vertices has an odd number of common neighbors. Prove that n is odd.
Hint: Start by showing that all vertices of G have an even degree.

Problem 7. Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of an n-element set with the property that $|A \cap B|$ is odd for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $|\mathcal{A}||\mathcal{B}| \leq 2^{n-1}$ and find $\mathcal{A}$ and $\mathcal{B}$ matching this bound.