# MATH 8803 Algebraic Methods in Combinatorics (Spring '10)

**Instructor: Asaf Shapira**

## Home Assignment 4

**Due date: 04/16/10**

**Please submit organized and well written solutions!**

**Problem 1.** Consider the following variant of the Hadamard code. Given $a \in \{0,1\}^k$ let $f_a(x) = \sum_{i=1}^{k} a_i x_i$ and let the encoding of $a$ consist of the evaluation (over $\mathbb{F}_2$) of $f_a$ on all $x \in \mathbb{F}_2^k$. Let $v \in \{0,1\}^{2^k}$ be the encoding of some $x \in \{0,1\}^k$ an suppose $v'$ is obtained from $v$ by changing $\epsilon 2^k$ of the entries of $v$. Show that we can recover any bit of $x$ with probability at least $1 - 2\epsilon$ by querying only 2 entries of $v'$.

**Problem 2.** Let $p$ be an even integer, and suppose let $a_1, \ldots, a_m$ be vectors in $\mathbb{R}^n$ satisfying $\ell_p(a_i - a_j) = 1$ for all $i < j$. Show that $m \leq 1 + (p-1)n$.

**Problem 3.** Let $A_1, \ldots, A_m$ be subsets of $[n]$ and suppose that their symmetric differences attain only two sizes. Prove that $m \leq \frac{n(n+1)}{2} + 1$. Give an example of a collection of $\frac{n(n-1)}{2} + 1$ subsets of $[n]$ whose symmetric differences attain only two sizes.

**Problem 4.** Let $S$ be a subset of $Z_3^n$ and suppose that for every pair of distinct vectors $u, v \in S$ there is an index $i$ for which $v_i = u_i + 1 \pmod 3$. Show that $|S| \leq 2^n$.

**Problem 5.** A vector $s \in \{*, 0\}^m$ is a *zero pattern* of a set of functions $f_1, \ldots, f_m$ in $n$ variables, if there is $x \in \mathbb{R}^n$ such that for every $1 \leq i \leq m$ we have $f_i(x) = 0$ if and only if $s_i = 0$. Suppose $f_1, \ldots, f_m$ are polynomials of degrees $d_1, \ldots, d_m$ in $n$ variables. Show that they have at most $\binom{n + \sum_{i=1}^{m} d_i}{n}$ zero-patterns.