

Two Erdős–Hajnal-type Theorems in Hypergraphs

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Abstract

The Erdős–Hajnal Theorem asserts that non-universal graphs, that is, graphs that do not contain an induced copy of some fixed graph H , have homogeneous sets of size significantly larger than one can generally expect to find in a graph. We obtain two results of this flavor in the setting of r -uniform hypergraphs.

A theorem of Rödl asserts that if an n -vertex graph is non-universal then it contains an almost homogeneous set (i.e. one with edge density either very close to 0 or 1) of size $\Omega(n)$. We prove that if a 3-uniform hypergraph is non-universal then it contains an almost homogeneous set of size $\Omega(\log n)$. An example of Rödl from 1986 shows that this bound is tight.

Let $R_r(t)$ denote the size of the largest non-universal r -graph \mathcal{G} so that neither \mathcal{G} nor its complement contain a complete r -partite subgraph with parts of size t . We prove an Erdős–Hajnal-type stepping-up lemma, showing how to transform a lower bound for $R_r(t)$ into a lower bound for $R_{r+1}(t)$. As an application of this lemma, we improve a bound of Conlon–Fox–Sudakov by showing that $R_3(t) \geq t^{\Omega(t)}$.

1 Introduction

Let us say that a set of vertices in a graph (or hypergraph) is *homogeneous* if it spans either a clique (i.e. a complete graph) or an independent set (i.e. an empty graph). Ramsey’s theorem states that every graph contains a homogeneous set of size $\frac{1}{2} \log_2 n$, and Erdős proved that in general, one cannot expect to find a homogeneous set of size larger than $2 \log_2 n$ (see [10]). Since Erdős’s example uses random graphs, and random graphs are universal (with high probability), that is, they contain an induced copy of every fixed graph H , it is natural to ask what happens if we assume that G is non-universal, or equivalently, that it is induced H -free for some fixed H . A theorem of Erdős and Hajnal [8] states that in this case we are guaranteed to have a homogeneous set of size $2^{\Omega(\sqrt{\log n})}$, that is, a significantly larger set than in the worst case. The notorious Erdős–Hajnal Conjecture states that one should be able to go even further and improve this bound to n^c , where $c = c(H)$. We refer the reader to [3] for more background on this conjecture and related results.

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Conlon, Fox and Sudakov [6] and Rödl and Schacht [16] have recently initiated the study of problems of this type in the setting of r -uniform hypergraphs (or r -graphs for short). Our aim in this paper is to obtain two results of this flavor which are described in the next two subsections.

1.1 Almost homogeneous sets in non-universal hypergraphs

Our first result is motivated by a theorem of Rödl [15]. Let us say that a set of vertices W in a graph is η -homogeneous if W either contains at least $(1 - \eta)\binom{|W|}{2}$ or at most $\eta\binom{|W|}{2}$ edges. It is a standard observation that Erdős's lower bound for Ramsey's theorem (mentioned above), actually shows that some (actually, most) graphs of order n do not even contain $\frac{1}{4}$ -homogeneous¹ sets of size $O(\log n)$. In other words, in the worst case relaxing 0-homogeneity to $\frac{1}{4}$ -homogeneity does not make the problem easier. Rödl's [15] surprising theorem then states that if G is non-universal then for any $\eta > 0$, it contains an η -homogeneous set of size $\Omega(n)$, where the hidden constant depends on η .

It is natural to ask if a similar² result holds also in hypergraphs. Random 3-graphs show that, in the worst case, the largest $\frac{1}{4}$ -homogeneous set in a 3-graph might be of size $O(\sqrt{\log n})$. Our first theorem in this paper shows that, as in graphs, if we assume that a 3-graph is non-universal then we can find a much larger almost homogeneous set.

Theorem 1.1. *For every 3-graph \mathcal{F} and $\eta > 0$ there is $c = c(\mathcal{F}, \eta) > 0$ such that every induced \mathcal{F} -free 3-graph on n vertices contains an η -homogeneous set of size $c \log n$.*

Rödl [15] found an example of a non-universal 3-graph in which the largest $\frac{1}{4}$ -homogeneous set has size $O(\log n)$. Hence, the bound in Theorem 1.1 is tight up to the constant c . We will describe in Section 5 (see Proposition 5.2) a generalization of Rödl's example, giving for every $r \geq 3$ an example of a non-universal r -graph in which the size of the largest $\frac{1}{4}$ -homogeneous set is $O((\log n)^{1/(r-2)})$. It seems reasonable to conjecture that this upper bound is tight, that is, that for every $r \geq 3$ every non-universal r -graph has an almost homogeneous set of size $\Omega((\log n)^{1/(r-2)})$.

Let K_k denote the complete graph on k vertices and let $K_k^{(3)}$ denote the complete 3-graph on k vertices. It is easy to see that up to a change of constants, a set of vertices has edge density close to 0/1 (i.e. is η -homogeneous for some small η), if and only if it has K_k -density close to 1 either in the graph or in its complement. The same applies to 3-graphs. An interesting feature of the proof of Theorem 1.1 is that instead of gradually building a set of vertices with very large/small edge density, we find it easier to build such a set with large $K_k^{(3)}$ -density either in \mathcal{G} or its complement. The way we gradually build such a set is by applying a variant of a greedy embedding scheme used by Rödl and Schacht [16] in order to give an alternative proof of an elegant theorem of Nikifirov [12] (this alternative proof is also implicit in [6]). To get this embedding scheme 'started' we prove a lemma saying that if a 3-graph \mathcal{G} is non-universal then there is a graph G on a subset of $V(\mathcal{G})$ such that either almost all or almost none of the K_k 's of G are also $K_k^{(3)}$'s in \mathcal{G} . This latter statement is proved via the hypergraph regularity method.

¹One can easily replace the $\frac{1}{4}$ with any constant smaller than $\frac{1}{2}$. We will stick with the $\frac{1}{4}$ in order to streamline the presentation.

²We of course say that a set of vertices W in an r -graph is η -homogeneous if W either contains at least $(1 - \eta)\binom{|W|}{r}$ or at most $\eta\binom{|W|}{r}$ edges

1.2 Complete partite sets in non-universal hypergraphs

Determining the size of the largest homogeneous set in a 3-graph is still a major open problem, see [5]. The best known lower and upper bounds are of order $\log \log n$ and $\sqrt{\log n}$ respectively. It is thus hard to formulate a 3-graph analogue of the Erdős–Hajnal Theorem since it is not clear which bound one is trying to beat. At any rate, as of now, we do not even know if a non-universal 3-graph contains a homogeneous set of size $\omega(\log \log n)$ (see Section 5 for further discussion on this problem). This motivated the authors of [6, 16] to look at the following related problem. Let $K_{t,t,t}^{(3)}$ denote the complete 3-partite 3-graph with each part of size t . It is a well known fact [7] that every 3-graph of positive density contains a copy of $K_{t,t,t}^{(3)}$ with $t = \Omega(\sqrt{\log n})$. This immediately means that for every 3-graph \mathcal{G} , either \mathcal{G} or its complement contains a $K_{t,t,t}^{(3)}$ with $t = \Omega(\sqrt{\log n})$. As evidenced by random 3-graphs, this bound is tight. A natural question, which was first addressed by Conlon, Fox and Sudakov [6] and by Rödl and Schacht [16] is whether one can improve upon this bound when \mathcal{G} is assumed to be non-universal.

It will be more convenient to switch gears at this point, and let $R_{3,\mathcal{F}}(t)$ denote the size of the largest induced \mathcal{F} -free 3-graph \mathcal{G} , so that neither \mathcal{G} nor $\overline{\mathcal{G}}$ contain a copy of $K_{t,t,t}^{(3)}$. So the question posed at the end of the previous paragraph is equivalent to asking if for every fixed \mathcal{F} we have $R_{3,\mathcal{F}}(t) \leq 2^{o(t^2)}$, and the results of [6, 16] establish that this is indeed the case³. Conlon, Fox and Sudakov [6] also found an example of a 3-graph \mathcal{F} for which $R_{3,\mathcal{F}}(t) \geq 2^{\Omega(t)}$. Our second result improves their lower bound as follows.

Theorem 1.2. *There is a 3-graph \mathcal{F} for which $R_{3,\mathcal{F}}(t) \geq t^{\Omega(t)}$.*

As discussed in [6], it is natural to consider the corresponding problem in general r -graphs. Letting $K_{t,\dots,t}^{(r)}$ denote the complete r -partite r -graph with parts of size t , we define $R_{r,\mathcal{F}}(t)$ to be the size of the largest induced \mathcal{F} -free r -graph \mathcal{G} , so that neither \mathcal{G} nor $\overline{\mathcal{G}}$ contain a copy of $K_{t,\dots,t}^{(r)}$. It follows from [7], which establishes that in every r -graph \mathcal{G} on $2^{\Omega(t^{r-1})}$ vertices with density $1/2$ we can find a $K_{t,\dots,t}^{(r)}$, that

$$R_{r,\mathcal{F}}(t) \leq 2^{O(t^{r-1})}. \quad (1)$$

It was shown in [6] that there is an r -graph \mathcal{F} satisfying

$$R_{r,\mathcal{F}}(t) \geq 2^{\Omega(t^{r-2})}. \quad (2)$$

An alternative proof of (2) follows from Proposition 5.2.

The famous step-up lemma of Erdős and Hajnal (see [10]) allows one to transform a construction of an r -graph without a large monochromatic set into an exponentially larger $(r+1)$ -graph without a large monochromatic set (of roughly the same size). Observe that both (1) and (2) suggest that if 2^{t^α} is the size of the largest non-universal r -graph \mathcal{G} , so that neither \mathcal{G} nor $\overline{\mathcal{G}}$ contain $K_{t,\dots,t}^{(r)}$, then the corresponding bound for $(r+1)$ -graphs is $2^{t^{\alpha+1}}$. The following theorem establishes one side of this relation, by proving an Erdős–Hajnal-type step-up lemma for the problem of bounding $R_{r,\mathcal{F}}(t)$.

³While the proof in [16] obtained the bound $R_{3,\mathcal{F}}(t) \leq 2^{t^{2/f(t)}}$ with $f(t)$ an inverse Ackermann-type function (on account of using the hypergraph regularity lemma), the proof in [6] gave the improved bound $R_{3,\mathcal{F}}(t) \leq 2^{t^{2-c}}$ where $c = c(\mathcal{F})$ is a constant that depends only on \mathcal{F} (but goes to zero with $|V(\mathcal{F})|$).

Theorem 1.3. *The following holds for every $r \geq 4$. For every $(r - 1)$ -graph \mathcal{F} there is an r -graph \mathcal{F}^+ and a constant $c = c(r, \mathcal{F}) > 0$, so that*

$$R_{r, \mathcal{F}^+}(t) \geq (R_{r-1, \mathcal{F}}(ct))^{ct} .$$

Theorem 1.3 implies that any improvement of (2) for $r = 3$ immediately implies a similar improvement of (2) for arbitrary $r \geq 3$. In particular, as a corollary of Theorems 1.2 and 1.3 we obtain the following improvement of (2).

Corollary 1.4. *For every $r \geq 3$ there is an r -graph \mathcal{F} satisfying $R_{r, \mathcal{F}}(t) \geq t^{\Omega(t^{r-2})}$.*

To prove Theorem 1.3 we need to overcome two hurdles. First, we need a way to construct the r -graph \mathcal{F}^+ given the $(r - 1)$ -graph \mathcal{F} . An important tool for this step will be an application of a theorem of Alon, Pach and Solymosi [1], which is a hypergraph extension of a result of Rödl and Winkler [17]. The second hurdle is how to construct an r -graph avoiding a $K_{t, \dots, t}^{(r)}$ given an $(r - 1)$ -graph avoiding a large $K_{t', \dots, t'}^{(r-1)}$. Here we will apply a version of a very elegant argument from [5], which is a variant of the Erdős–Hajnal step-up lemma. While this variant of the step-up lemma is not as efficient as the original one⁴, it is strong enough for our purposes.

It would be very interesting to further narrow the gap between (1) and Corollary 1.4. The most natural question is if one can extend the results of [6, 16] by showing that $R_{r, \mathcal{F}}(t) \leq 2^{o(t^{r-1})}$ for every $r \geq 4$ (the case $r = 3$ was handled in [6, 16]). We should mention that [6] conjectured that in fact $R_{r, \mathcal{F}}(t) \leq 2^{t^{r-2+o(1)}}$. Observe that by Theorem 1.3, showing that $R_{3, \mathcal{F}}(t) \geq 2^{t^{1+c}}$ for some 3-graph \mathcal{F} and $c > 0$ would disprove this conjecture for all $r \geq 3$.

1.3 Paper overview

The rest of the paper is organized as follows. In Section 2 we give the proof of Theorem 1.1, deferring the proof of one of the key lemmas to Section 3. The proof of Theorems 1.2 and 1.3 is given in Section 4. Section 5 contains some concluding remarks including the statement and proof of Proposition 5.2 which gives a generalization of Rödl’s example, establishing that the bound in Theorem 1.1 is asymptotically tight.

We are following the common practice of ignoring rounding issues for a better proof transparency. Throughout the paper, \log denotes the natural logarithm, \mathbb{N} stands the set of positive integers, and for $n \in \mathbb{N}$ we write $[n]$ for the set of integers $\{1, \dots, n\}$.

2 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1, save for one lemma that is proved in Section 3. The proof of Theorem 1.1 will rely on Lemmas 2.2 and 2.3 stated below. We start with a few definitions. All *graphs* (or 2-graphs) will be simple and undirected, and will be denoted by capital letters e.g. G, H , while *r-graphs* will be denoted by script letters e.g. \mathcal{G}, \mathcal{H} . For a graph G and a 3-graph \mathcal{G} , defined on the same vertex set, we say that G *underlies* \mathcal{G} if every edge of \mathcal{G} is a triangle in G . We

⁴Observe that step-up lemmas with an exponential blowup are *not* useful in our setting since (1) and (2) tell us that the gap between $R_{r-1, \mathcal{F}}(t)$ and $R_{r, \mathcal{F}^+}(t)$ is *not* exponential.

use $\mathcal{N}_k(G)$ and $\mathcal{N}_k(\mathcal{G})$ to denote the number of cliques K_k and hypercliques $K_k^{(3)}$ in a graph G and a 3-graph \mathcal{G} , respectively. For an integer $k \geq 3$, we say that a graph (resp. 3-graph) is k -partite on disjoint vertex classes V_1, \dots, V_k if each edge of the graph (resp. 3-graph) has at most one vertex in each of the k vertex sets. Observe that if a k -partite G on vertex sets V_1, \dots, V_k underlies \mathcal{G} , then \mathcal{G} is also k -partite with respect to these vertex sets.

Definition 2.1 ((ϵ, η) -dense graph). *Suppose G is a k -partite graph on disjoint vertex sets V_1, \dots, V_k . We say that G is (ϵ, η) -dense with respect to some 3-graph \mathcal{G} if*

1. G underlies \mathcal{G} .
2. $\mathcal{N}_k(G) \geq \epsilon \prod_i |V_i|$.
3. $\mathcal{N}_k(\mathcal{G}) \geq (1 - \eta)\mathcal{N}_k(G)$.

We are now ready to state the two key lemmas needed to prove Theorem 1.1, and then show how to derive it from them. In what follows we use $G[W]$ (or $\mathcal{G}[W]$) to denote the graph (or 3-graph) induced by a set of vertices W . As usual, we use $\bar{\mathcal{G}}$ to denote the complement of a 3-graph \mathcal{G} .

Lemma 2.2. *For every 3-graph \mathcal{F} and for any $k \in \mathbb{N}$ and $0 < \eta < 1$ there exist $c' = c'(\mathcal{F}, k, \eta) > 0$ and $\epsilon = \epsilon(\mathcal{F}, k, \eta) > 0$ as follows. If \mathcal{G} is an induced \mathcal{F} -free 3-graph on n vertices, then there are k disjoint vertex sets $V_1, \dots, V_k \subseteq V(\mathcal{G})$ of equal size at least $c'n$ and a k -partite graph G on V_1, \dots, V_k which is (ϵ, η) -dense either with respect to a subgraph of \mathcal{G} or with respect to a subgraph of $\bar{\mathcal{G}}$.*

Lemma 2.3. *For every $\epsilon > 0$ and integer $k \geq 3$ there exists $c'' = c''(\epsilon, k) > 0$ as follows. Suppose G is a k -partite graph on vertex sets V_1, \dots, V_k of size m each, and G is (ϵ, η) -dense with respect to some 3-graph \mathcal{G} . Then there are subsets $W_j \subseteq V_j$ for $j = 1, \dots, k$ of equal size at least $c'' \log m$ such that setting $W = \bigcup_i W_i$ we have*

$$\mathcal{N}_k(\mathcal{G}[W]) \geq (1 - 2^{\binom{k}{2}}\eta) \prod_{\ell=1}^k |W_\ell|. \quad (3)$$

Proof of Theorem 1.1. Given \mathcal{F} and η , set $k = 10/\eta$ and let $c' = c'(\mathcal{F}, k, \eta/2^{k^2})$ and $\epsilon = \epsilon(\mathcal{F}, k, \eta/2^{k^2})$ be the constants from Lemma 2.2. If G is induced \mathcal{F} -free then by Lemma 2.2 we can find k disjoint vertex sets V_1, \dots, V_k of equal size at least $c'n$ and a k -partite graph on these sets which is $(\epsilon, \eta/2^{k^2})$ -dense with respect to either a subgraph of \mathcal{G} or a subgraph $\bar{\mathcal{G}}$. Suppose, without loss of generality, that the former case holds. By Lemma 2.3 we can find subsets $W_1 \subseteq V_1, \dots, W_k \subseteq V_k$, each of size $c'' \log |V_i| \geq c'' \log(c'n) \geq c \log n$ satisfying⁵ $\mathcal{N}_k(\mathcal{G}[W]) \geq (1 - \eta/2) \prod_{\ell=1}^k |W_\ell|$.

Setting $W = \bigcup_i W_i$ we now claim that $e(\mathcal{G}[W]) \geq (1 - \eta) \binom{|W|}{3}$. To see this we first observe that since $k = 10/\eta$ and all sets W_i are of the same size, then at most $\frac{1}{2}\eta \binom{|W|}{3}$ of the 3-tuples in W do not belong to 3 distinct sets W_i . We also observe that for each triple of distinct sets W_p, W_q, W_r , the 3-graph \mathcal{G} has at least $(1 - \eta/2)|W_p||W_q||W_r|$ edges with one vertex in each of the sets. Indeed, if this is not the case then \mathcal{G} cannot contain the number of copies of $K_k^{(3)}$ asserted at the end of the previous paragraph. It is finally easy to see that the above two observations imply that $e(\mathcal{G}[W]) \geq (1 - \eta) \binom{|W|}{3}$. \square

⁵Note that this inequality would in fact hold for a subgraph of \mathcal{G} , i.e. the one we got from Lemma 2.2. By monotonicity we can work with \mathcal{G} itself.

The rest of this section is devoted to the proof of Lemma 2.3. The proof of Lemma 2.2 appears in the next section. The proof of Lemma 2.3 is carried out via repeated application of Lemma 2.5 stated below. For its proof, we will need the following special case of the classical Kóvári-Sós-Turán Theorem [11] (similar lemmas were used in [6, 12, 16]). For completeness we include the proof.

Lemma 2.4. *Suppose that $\epsilon > 0$ and G is a bipartite graph between sets A and B with $e(G) \geq \epsilon|A||B|$. Suppose further that $|A| \leq \frac{1}{2} \log |B|$ and $s, t > 0$ satisfy $s = \epsilon|A|$ and $t = \sqrt{|B|}$. Then G contains a copy of $K_{s,t}$ with s vertices in A and t vertices in B .*

Proof. Write $d(v)$ for the degree of v in G . Then the number of copies of $K_{s,1}$, with s vertices in A and one vertex in B is just $\sum_{v \in B} \binom{d(v)}{s}$. By convexity of $f(z) = \binom{z}{s}$ we have

$$\sum_{v \in B} \binom{d(v)}{s} \geq |B| \binom{\frac{1}{|B|} \sum_{v \in B} d(v)}{s} = |B| \binom{e(G)/|B|}{\epsilon|A|} \geq |B|.$$

Since A has fewer than $2^{|A|}$ subsets of size s , by the pigeonhole principle some set $U \subset A$ of size s appears in at least $2^{-|A|}|B| \geq \sqrt{|B|} = t$ of these copies of $K_{s,1}$. It is clear that U and the t vertices from B participating in these t copies of $K_{s,1}$ form a $K_{s,t}$. \square

If U, V are two disjoint vertex sets in a graph G , we use $G[U, V]$ to denote the bipartite subgraph of G induced by these sets, that is, the graph containing all edges with one vertex in each set. More generally, for disjoint vertex sets V_1, \dots, V_k we use $G[V_1, \dots, V_k]$ to denote the k -partite subgraph containing all edges of G connecting two distinct sets V_i, V_j . Similarly, if \mathcal{G} is a 3-graph then we use $\mathcal{G}[V_1, \dots, V_k]$ to denote the k -partite 3-graph containing all edges of \mathcal{G} connecting three distinct sets.

Lemma 2.5. *Suppose that $\eta > 0$, $0 < \epsilon < 1$, and $0 < \gamma \leq 1/2$ are constants, V_1, \dots, V_k are disjoint vertex sets in some 3-graph \mathcal{G} , and G is a k -partite graph on V_1, \dots, V_k which is (ϵ, η) -dense with respect to \mathcal{G} . If for some pair of indices $1 \leq i < j \leq k$ we have $|V_j| = m$ and $|V_i| \geq \gamma \log m$, then there exist subsets $S_i \subseteq V_i$ and $S_j \subseteq V_j$ such that*

- $|S_i| = \epsilon\gamma \log m$ and $|S_j| = m^{1/2}$,
- $G[S_i, S_j]$ is complete bipartite,
- $G[V_1, \dots, S_i, \dots, S_j, \dots, V_k]$ is $(\epsilon/4, 2\eta)$ -dense with respect to $\mathcal{G}[V_1, \dots, S_i, \dots, S_j, \dots, V_k]$.

Proof. Observe that by the conditions of the lemma, for any choice of S_i, S_j we have that the k -partite graph $G[V_1, \dots, S_i, \dots, S_j, \dots, V_k]$ underlies the k -partite 3-graph $\mathcal{G}[V_1, \dots, S_i, \dots, S_j, \dots, V_k]$, so we will not need to worry about item 1 of Definition 2.1.

For each $1 \leq p \neq q \leq k$ we use shorthand $G_{p,q}$ for the bipartite graph $G[V_p, V_q]$. For an edge $e \in E(G)$ let $\mathcal{N}_k(G, e)$ and $\mathcal{N}_k(\mathcal{G}, e)$ stand for the number of k -cliques in G and \mathcal{G} respectively, containing the vertices of e . Call an edge $e \in E(G_{i,j})$ *good* if $\mathcal{N}_k(\mathcal{G}, e) \geq (1 - 2\eta)\mathcal{N}_k(G, e)$, otherwise the edge is *bad*. Since G is (ϵ, η) -dense with respect to \mathcal{G} we have

$$\sum_{e \text{ good}} \mathcal{N}_k(G, e) + \sum_{e \text{ bad}} (1 - 2\eta)\mathcal{N}_k(G, e) \geq \mathcal{N}_k(\mathcal{G}) \geq (1 - \eta)\mathcal{N}_k(G),$$

implying that $\sum_{e \text{ bad}} \mathcal{N}_k(G, e) \leq \frac{1}{2} \mathcal{N}_k(G)$. This, and the assumption that G is (ϵ, η) -dense, imply that

$$\sum_{e \text{ good}} \mathcal{N}_k(G, e) \geq \frac{1}{2} \mathcal{N}_k(G) \geq \frac{\epsilon}{2} \prod_{\ell=1}^k |V_\ell|. \quad (4)$$

Let F'' be the subgraph of $G_{i,j}$ consisting of all good edges, and let G'' be the graph formed by F'' and $G_{p,q}$ for all $\{p, q\} \neq \{i, j\}$. By (4), we have $\mathcal{N}_k(G'') \geq \frac{\epsilon}{2} \prod_{\ell=1}^k |V_\ell|$. Let $F \subseteq F''$ be the set of all edges e such that $\mathcal{N}_k(G, e) \geq \frac{\epsilon}{4} \prod_{\ell \notin \{i, j\}} |V_\ell|$. Note that we must have $e(F) \geq \frac{\epsilon}{4} |V_i| |V_j|$, as otherwise

$$\mathcal{N}_k(G'') < \frac{\epsilon}{4} |V_i| |V_j| \cdot \prod_{\ell \notin \{i, j\}} |V_\ell| + |V_i| |V_j| \cdot \frac{\epsilon}{4} \prod_{\ell \notin \{i, j\}} |V_\ell| = \frac{\epsilon}{2} \prod_{\ell=1}^k |V_\ell|,$$

which would contradict the lower bound on $\mathcal{N}_k(G'')$ mentioned above.

Since $|V_i| \geq \gamma \log m$, by averaging there exists a subset $A \subseteq V_i$ with $|A| = \gamma \log m$ and

$$e(F[A, V_j]) \geq \frac{\epsilon}{4} |A| |V_j|. \quad (5)$$

By (5), we can apply Lemma 2.4 to the graph $F[A, V_j]$ to obtain subsets $S_i \subseteq A \subseteq V_i$ and $S_j \subseteq V_j$ with $|S_i| = \epsilon \gamma \log m$ and $|S_j| = m^{1/2}$, such that $G[S_i, S_j]$ is complete bipartite.

We now look at $G' = G[V_1, \dots, S_i, \dots, S_j, \dots, V_k]$. Since $G[S_i, S_j]$ is complete bipartite and for every edge $e \in E(G[S_i, S_j])$ we have that $\mathcal{N}_k(G, e) \geq \frac{\epsilon}{4} \prod_{\ell \notin \{i, j\}} |V_\ell|$, it follows that

$$\mathcal{N}_k(G') \geq \frac{\epsilon}{4} |S_i| |S_j| \prod_{\ell \notin \{i, j\}} |V_\ell|.$$

Moreover, since every $e \in E(G[S_i, S_j])$ is good, we have

$$\mathcal{N}_k(\mathcal{G}[V_1, \dots, S_i, \dots, S_j, \dots, V_k]) \geq (1 - 2\eta) \mathcal{N}_k(G'),$$

implying that G' is indeed $(\epsilon/4, 2\eta)$ -dense with respect to $\mathcal{G}[V_1, \dots, S_i, \dots, S_j, \dots, V_k]$. \square

Proof of Lemma 2.3. Set $\gamma = \frac{1}{2}$ and apply Lemma 2.5 with $i = 1$ and $j = 2$ to obtain subsets $S_1^1 \subseteq V_1$ and $S_2 \subseteq V_2$ with $|S_1^1| = \epsilon \gamma \log m$ and $|S_2| = m^{1/2}$, and with the properties stated in Lemma 2.5. Note now that the conditions of Lemma 2.5 are satisfied in the graph $G[S_1^1, S_2, V_3, \dots, V_k]$, with $2\eta, \epsilon/4, \epsilon \gamma, S_1^1$ and V_3 playing the role of $\eta, \epsilon, \gamma, V_i$ and V_j respectively, so Lemma 2.5 can be applied again to obtain subsets $S_1^2 \subseteq S_1^1$ and $S_3 \subseteq V_3$ with properties as stated therein. Continuing for $i = 1$ and (consecutively) $j = 2, \dots, k$ we obtain a set $S_1^{k-1} =: U_1 \subseteq V_1$ of size $c_1 \log m$ for a constant $c_1 = c_1(\epsilon, k)$, and sets S_2, \dots, S_k of size $m^{1/2}$ each, so that all graphs $G[U_1, S_j]$ are complete bipartite and $G[U_1, S_2, \dots, S_k]$ is $(\epsilon/4^{k-1}, 2^{k-1}\eta)$ -dense with respect to $\mathcal{G}[U_1, S_2, \dots, S_k]$.

Next, since $\log(m^{1/2}) = (1/2) \log m$, the above procedure can be applied again for $i = 2$ and $j = 3, \dots, k$ to obtain $U_2 \subseteq V_2$ of size $c_2 \log m$ and (abusing notation) sets $S_3 \subseteq V_3, \dots, S_k \subseteq V_k$ of size $m^{1/4}$ each, with analogous properties. Continuing in the same way for each i yields sets $U_i \subseteq V_i$ for $i = 1, \dots, k-1$ with $|U_i| = c_i \log m$ and $U_k = S_k \subseteq V_k$ (the output of the last application of Lemma 2.5) with $|U_k| = m^{1/2^{k-1}} \geq c_k \log m$ such that $G^* = G[U_1, \dots, U_k]$ is a complete k -partite

graph. Moreover, by Lemma 2.5 (which we applied $\binom{k}{2}$ times), G^* is $(\epsilon/4\binom{k}{2}, 2\binom{k}{2}\eta)$ -dense with respect to $\mathcal{G}[U_1, \dots, U_k]$. In particular,

$$\mathcal{N}_k(\mathcal{G}[U_1, \dots, U_k]) \geq (1 - 2\binom{k}{2}\eta)\mathcal{N}_k(G^*).$$

However, since (crucially) G^* is complete k -partite, this means that

$$\mathcal{N}_k(\mathcal{G}[U_1, \dots, U_k]) \geq (1 - 2\binom{k}{2}\eta) \prod_{\ell=1}^k |U_\ell|.$$

To complete the proof we just need to pick sets of equal size. Putting $c'' = \min\{c_1, \dots, c_k\}$, by averaging, there exist subsets $W_\ell \subseteq U_\ell$ for $\ell = 1, \dots, k$ of equal size $c'' \log m$, so that setting $W = \bigcup_i W_i$ we have

$$\mathcal{N}_k(\mathcal{G}[W]) \geq \mathcal{N}_k(\mathcal{G}[W_1, \dots, W_k]) \geq (1 - 2\binom{k}{2}\eta) \prod_{\ell=1}^k |W_\ell|,$$

as we had to show. □

3 Near-homogeneous multipartite hypergraphs

In this Section we prove Lemma 2.2. In the first subsection we will discuss some tools from the hypergraph regularity method that are needed for the proof of the lemma. We will follow the definitions from [13] and [14]. The proof itself appears in the second subsection.

3.1 A primer on hypergraph regularity

We first recall the definition of a (Szemerédi)-regular bipartite graph.

Definition 3.1. *Let $d_2, \delta_2 > 0$. A bipartite graph G with the bipartition $V(G) = X \sqcup Y$ is called (δ_2, d_2) -regular if for all subsets $X' \subseteq X$ and $Y' \subseteq Y$ we have*

$$|e(G[X', Y']) - d_2|X'||Y'|| \leq \delta_2|X||Y|.$$

Extending this notion to 3-graphs, we need the following definition of a regular 3-partite 3-graph [14, Definition 3.1]. For a graph G write $T(G)$ for the set of triangles in G .

Definition 3.2. *Let $d_3, \delta_3 > 0$. A 3-graph $\mathcal{H} = (V, E_H)$ is called (δ_3, d_3) -regular with respect to a tripartite graph $P = (X \sqcup Y \sqcup Z, E_P)$ with $V \supseteq X \cup Y \cup Z$, if for every subgraph $Q \subseteq P$ we have*

$$||E_H \cap T(Q)| - d_3|T(Q)|| \leq \delta_3|T(P)|. \tag{6}$$

We say that \mathcal{H} is δ_3 -regular with respect to P if it is (δ_3, d_3) -regular for an unspecified d_3 . Also, define

$$d(\mathcal{G}|P) := \frac{|E(\mathcal{G}) \cap T(P)|}{|T(P)|}$$

to be the *relative density* of \mathcal{G} with respect to $T(P)$. By putting $Q = P$ in (6), we obtain:

Remark 3.3. *If \mathcal{H} is (δ_3, d_3) -regular with respect to some graph P , then*

$$d_3 - \delta_3 \leq d(\mathcal{G}|P) \leq d_3 + \delta_3.$$

Next, we will state the regularity lemma for 3-graphs. Informally speaking, it is similar in spirit to the classical Szemerédi regularity lemma for 2-graphs, in that a large hypergraph is being partitioned into a bounded number of fragments, almost all of which are regular. One major difference between the 2-graph and 3-graph cases is that, due to various technical issues, for 3-graphs we partition not just $V(\mathcal{H})$ into subsets $V_0 \cup V_1 \cup \dots \cup V_t$, but also the (2-uniform) edge sets of the complete bipartite graphs $K(V_i, V_j)$ between V_i and V_j into sparser bipartite graphs. This will naturally give rise to a number of tripartite 2-graphs (‘triads’); the lemma states then that \mathcal{H} will be regular with respect to most of them.

To give a precise formulation, we shall be using the following version of the regularity lemma for 3-graphs [14, Theorem 3.2].

Proposition 3.4 (Regularity Lemma). *For all $\delta_3 > 0$, $\delta_2 : \mathbb{N} \rightarrow (0, 1]$ and $t_0 \in \mathbb{N}$ there exist $T_0, n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every 3-graph \mathcal{H} with $|V(\mathcal{H})| = n$ the following holds.*

There are integers $t_0 \leq t \leq T_0$ and $\ell \leq T_0$ such that there exists a partition $V_0 \cup V_1 \cup \dots \cup V_t = V(\mathcal{H})$, and for all $1 \leq i < j \leq t$ there exists a partition of $K(V_i, V_j)$

$$\mathcal{P}^{ij} = \{P^\alpha{}^{ij} = (V_i \cup V_j, E_\alpha^{ij}) : 1 \leq \alpha \leq \ell\},$$

satisfying the following properties:

- (i) $|V_0| \leq \delta_3 n$ and $|V_1| = \dots = |V_t|$,
- (ii) for all $1 \leq i < j \leq t$ and $\alpha \in [\ell]$ the bipartite graph P_α^{ij} is $(\delta_2(\ell), 1/\ell)$ -regular, and
- (iii) \mathcal{H} is δ_3 -regular w.r.t. $P_{\alpha\beta\gamma}^{hij}$ for all but at most $\delta_3 t^3 \ell^3$ triads

$$P_{\alpha\beta\gamma}^{hij} = P_\alpha^{hi} \cup P_\beta^{hj} \cup P_\gamma^{ij} = (V_h \cup V_i \cup V_j, E_\alpha^{hi} \cup E_\beta^{hj} \cup E_\gamma^{ij}),$$

with $1 \leq h < i < j \leq t$ and $\alpha, \beta, \gamma \in [\ell]$.

To state the Counting Lemma for 3-graphs (which typically complements the Regularity Lemma in the regularity method), consider the following setup.

Setup A. Let $k, m \in \mathbb{N}$ and $\delta_3, d_2, \delta_2 > 0$ be given. Suppose that

1. $V = V_1 \cup \dots \cup V_k, |V_1| = \dots = |V_k| = m$, is a partition of V .
2. $P = \bigcup_{1 \leq i < j \leq k} P^{ij}$ is a k -partite graph, with vertex set V and k -partition above, where all $P^{ij} = P[V_i, V_j]$, $1 \leq i < j \leq k$, are (δ_2, d_2) -regular.
3. $\mathcal{H} = \bigcup_{1 \leq h < i < j \leq k} \mathcal{H}^{hij} \subseteq T(P)$ is a k -partite 3-graph, with vertex set V and k -partition above, where all $\mathcal{H}^{hij} = \mathcal{H}[V_h, V_i, V_j]$, $1 \leq h < i < j \leq k$, are (δ_3, d_{hij}) -regular with respect to the triad $P^{hi} \cup P^{ij} \cup P^{hj}$, for some constant $0 \leq d_{hij} \leq 1$.

Now, first recall the counting lemma for 2-graphs in the following version.

Proposition 3.5 (Counting lemma for graphs). *For every integer $k \geq 3$ and positive constants $\gamma > 0, d_2 > 0$ there exist $\delta_2 = \delta_2(k, \gamma, d_2) > 0$ and $m_0 = m_0(k, \gamma, d_2, \delta_2) \in \mathbb{N}$ such that with these constants if a graph P is as in 1. and 2. of Setup A and $m \geq m_0$, then*

$$(1 - \gamma)d_2^{\binom{k}{2}} m^k \leq \mathcal{N}_k(P) \leq (1 + \gamma)d_2^{\binom{k}{2}} m^k .$$

Now, we state the Counting Lemma for 3-graphs [13, Corollary 2.3].

Proposition 3.6 (Counting Lemma for 3-graphs). *For every integer $k \geq 4$, and positive constants $\gamma, d_3 > 0$ there exists $\delta_3 = \delta_3(k, \gamma, d_3) > 0$ so that for all $d_2 > 0$ there exist $\delta_2 = \delta_2(k, \gamma, d_3, \delta_3, d_2) > 0$ and $m_0 = m_0(k, \gamma, d_3, \delta_3, d_2, \delta_2) \in \mathbb{N}$ so that with these constants, if \mathcal{H} and P are as in Setup A, with $m \geq m_0$ and $d_{hij} \geq d_3$ for all $1 \leq h < i < j \leq k$, then*

$$\mathcal{N}_k(\mathcal{H}) \geq (1 - \gamma)d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} m^k .$$

Combining Propositions 3.6 and 3.5 gives

Corollary 3.7. *For every $k \geq 4$, and positive constants $\gamma, d_3 > 0$ there exists $\delta_3 = \delta_3(k, \gamma, d_3) > 0$ so that for all $d_2 > 0$ there exist $\delta_2 = \delta_2(k, \gamma, d_3, \delta_3, d_2) > 0$ and $m_0 = m_0(k, \gamma, d_3, \delta_3, d_2, \delta_2) \in \mathbb{N}$ so that with these constants, if \mathcal{H} and P are as in Setup A, with $m \geq m_0$ and $d_{hij} \geq d_3$ for all $1 \leq h < i < j \leq k$, then*

$$\mathcal{N}_k(\mathcal{H}) \geq (1 - \gamma)d_3^{\binom{k}{3}} \mathcal{N}_k(P).$$

Proof. Let $\delta_3 = \delta_3(k, \gamma/2, d_3)$ be the constant from Proposition 3.6, and take δ_2 to be the minimum of $\delta_2(k, \gamma/2, d_2)$ from Proposition 3.5 and $\delta_2(k, \gamma/2, d_3, \delta_3, d_2)$ from Proposition 3.6. Let m_0 be the larger m_0 from these two propositions. Then, for $m \geq m_0$ we have

$$\mathcal{N}_k(\mathcal{H}) \geq (1 - \frac{\gamma}{2})d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} m^k \geq (1 - \gamma)d_3^{\binom{k}{3}} (1 + \frac{\gamma}{2})d_2^{\binom{k}{2}} m^k \geq (1 - \gamma)d_3^{\binom{k}{3}} \mathcal{N}_k(P).$$

□

As another corollary of Proposition 3.6, we obtain the following criterion for finding an *induced* copy of a fixed 3-graph \mathcal{F} .

Corollary 3.8 (Induced embedding lemma). *For every 3-graph \mathcal{F} with $|V(\mathcal{F})| = k \geq 4$, and every constant $1 > d_3 > 0$ there exists $\delta_3 = \delta_3(\mathcal{F}, d_3) > 0$ so that for all $d_2 > 0$ there exist $\delta_2 = \delta_2(\mathcal{F}, d_3, \delta_3, d_2) > 0$ and $m_0 = m_0(\mathcal{F}, d_3, \delta_3, d_2, \delta_2) \in \mathbb{N}$ so that, with these parameters, if \mathcal{H} and P are as in Setup A, with $m \geq m_0$ and $d_3 \leq d_{hij} \leq 1 - d_3$ for all $1 \leq h < i < j \leq k$, then \mathcal{H} contains vertices v_1, \dots, v_k with $v_i \in V_i$ for all $1 \leq i \leq k$, such that*

- $P[v_1, \dots, v_k]$ is a k -clique, and
- $\mathcal{H}[v_1, \dots, v_k]$ is an induced copy of \mathcal{F} .

Proof. Choose an arbitrary labeling $\varphi : [k] \rightarrow V(\mathcal{F})$, and apply Proposition 3.6 to the 3-graph \mathcal{H}_* defined as follows

- $V(\mathcal{H}_*) = V(\mathcal{H})$, and
- $\mathcal{H}_* = \bigcup_{1 \leq h < i < j \leq k} \mathcal{H}_*^{hij}$ where all $\mathcal{H}_*^{hij} = \mathcal{H}_*[V_h, V_i, V_j]$, $1 \leq h < i < j \leq k$ are defined by $\mathcal{H}_*^{hij} = \mathcal{H}^{hij}$ if $\{\varphi(h), \varphi(i), \varphi(j)\} \in \mathcal{F}$ and $\mathcal{H}_*^{hij} = \overline{\mathcal{H}}^{hij} = T(P[V_h, V_i, V_j]) \setminus E(\mathcal{H})$ otherwise.

Observe that $\overline{\mathcal{H}}^{hij}$ is $(\delta_3, 1 - d_{hij})$ -regular with respect to its triad. Therefore \mathcal{H}_* satisfies all the requirements of Proposition 3.6, implying that it contains a copy of $K_k^{(3)}$, which corresponds to an induced copy of \mathcal{F} in \mathcal{H} . \square

In what follows, we write $d(\mathcal{G}) = e(\mathcal{G})/\binom{|\mathcal{G}|}{3}$ for the *density* of a 3-graph \mathcal{G} . We shall also need the following rudimentary estimate for 3-graph Turán densities.

Claim 3.9. *If $d(\mathcal{G}) > 1 - \binom{k}{3}^{-1}$ then \mathcal{G} contains a k -clique $K_k^{(3)} \subseteq \mathcal{G}$.*

Proof. Consider a random k -vertex subset of $V(\mathcal{G})$. By the union bound, the probability that one of the $\binom{k}{3}$ edges is missing is less than 1. \square

3.2 Proof of Lemma 2.2

Proof. Given \mathcal{F} , k and η , put $f := |V(\mathcal{F})|$, and let us pick s and λ satisfying

$$s = R^{(3)}(k, k, f) \quad \text{and} \quad (1 - \lambda) \binom{k}{3}^{+1} = 1 - \eta, \quad (7)$$

where $R^{(3)}(k, k, f)$ is the 3-color Ramsey⁶ function of 3-graphs. Let us now set the stage for an application of Proposition 3.4 (the 3-graph regularity lemma) by defining the following parameters.

$$\delta_3 = \min \left\{ \frac{\lambda}{3}, \frac{s^{-3}}{10}, \delta_3(k, \lambda, 1 - \lambda), \delta_3(\mathcal{F}, \frac{\lambda}{3}), \right\}, \quad (8)$$

where $\delta_3(k, \lambda, 1 - \lambda)$ is as stated in Corollary 3.7, and $\delta_3(\mathcal{F}, \lambda/3)$ is as stated in Corollary 3.8. We define the function $\delta_2 : \mathbb{N} \rightarrow (0, 1]$ to satisfy for every integer ℓ

$$\delta_2(\ell) = \min\{\delta_2(k, \lambda, 1 - \lambda, \delta_3, 1/\ell), \delta_2(\mathcal{F}, \lambda/3, \delta_3, 1/\ell)\},$$

where $\delta_2(k, \lambda, 1 - \lambda, \delta_3, 1/\ell)$ and $\delta_2(\mathcal{F}, \lambda/3, \delta_3, 1/\ell)$ are as stated in Corollary 3.7 and Corollary 3.8, respectively. Finally, we set

$$t_0 = s.$$

Given an induced \mathcal{F} -free 3-graph \mathcal{G} , we apply Proposition 3.4 with the above constants/function. We obtain partitions $V(\mathcal{G}) = V_0 \cup \dots \cup V_t$ and \mathcal{P}^{ij} as stated in Proposition 3.4. We will assume henceforth that $n/(2t) > m_0$, where m_0 is the maximum between the values of m_0 claimed in Proposition 3.6 and Corollary 3.8 (with respective parameters). Observe that from the way we chose the parameters above, it follows that the parameters t and ℓ of the partition we obtain through Proposition 3.4 can be bounded from above by functions of \mathcal{F}, k, η . Since the constants c', ϵ we will obtain below will depend only on t, ℓ they will indeed depend only on \mathcal{F}, k, η as asserted in the statement of the lemma.

⁶That is, the smallest R such that every edge-coloring of the complete 3-graph $K_R^{(3)}$ with 3 colors *red*, *blue*, and *green* is guaranteed to contain either a red or a blue $K_k^{(3)}$, or a green $K_f^{(3)}$. see e.g. [2].

Let P_α^{ij} be the collection of bipartite graphs from item (ii) of Proposition 3.4. For each $1 \leq i < j \leq t$ select $\alpha_{ij} \in [\ell]$ independently and uniformly at random, set $P^{ij} = P_{\alpha_{ij}}^{ij}$ and denote by P^{hij} the triad obtained from P^{hi} , P^{hj} and P^{ij} . Then the expected proportion of triads P^{hij} such that \mathcal{H} is δ_3 -regular with respect to P^{hij} , is at least $1 - 10\delta_3$. Thus, there must exist a choice of α_{ij} 's satisfying this property. To simplify the presentation, let us fix one such choice of α_{ij} 's and use P^{ij} to denote the corresponding collection of bipartite graphs defined by this choice. Define the 3-uniform *cluster-hypergraph* \mathcal{R} as follows: let $V(\mathcal{R}) = [t]$ and $\{h, i, j\} \in E(\mathcal{R})$ if \mathcal{H} is δ_3 -regular with respect to the triad P^{hij} . By the above we have $d(\mathcal{R}) \geq 1 - 10\delta_3 > 1 - \binom{s}{3}^{-1}$, so by Claim 3.9 there exists a collection of indices $1 \leq i_1 < \dots < i_s \leq t$ such that all triads $P^{i_p i_q i_r}$ are δ_3 -regular. Without loss of generality, let us assume that $i_1 < \dots < i_s$ are just $1, \dots, s$.

Recapting, what we have at this point is s vertex sets V_1, \dots, V_s of size $(n - |V_0|)/t$ each, and a collection of $\binom{s}{2}$ bipartite graphs $P = \bigcup P^{ij}$, where P^{ij} is $(\delta_2(\ell), 1/\ell)$ -regular (between V_i and V_j). Furthermore, for every $h < i < j$, the 3-graph \mathcal{G} is δ_3 -regular with respect to the triad P^{hij} formed by P^{hi} , P^{hj} and P^{ij} . In other words, we are at Setup A with

$$s, \quad m = (n - |V_0|)/t, \quad P = \bigcup P^{ij}, \quad \mathcal{G} \cap T(P)$$

playing the role of k , m , P and \mathcal{H} , respectively, and δ_2 and δ_3 as defined above.

Now define an edge-coloring Γ of the complete 3-graph on $[s]$ as follows. Color the edge $\{h, i, j\} \in [s]^{(3)}$ *red* if the corresponding triad P^{hij} satisfies $d(\mathcal{G}|P) > 1 - 2\lambda/3$, where $\lambda > 0$ was chosen in (7). Color P *blue* if $d(\mathcal{G}|P) < 2\lambda/3$, and color P *green* otherwise. By our choice of s in (7), one of the following assertions must hold.

- (g) There exist f indices $\{j_1, \dots, j_f\} \subseteq [s]$ such that every $\Gamma(j_p, j_q, j_r)$ is green for all $1 \leq p < q < r \leq f$. Without loss of generality, we may assume that these indices are simply $1, \dots, f$.
- (r) There exist k indices (wlog) $\{1, \dots, k\}$ such that every $\Gamma(p, q, r)$ is red.
- (b) There exist k indices (wlog) $\{1, \dots, k\}$ such that every $\Gamma(p, q, r)$ is blue.

Due to our choice of parameters, we can now invoke the counting lemmas. In Case (g) we apply Corollary 3.8 (note that, by Remark 3.3 and (8), \mathcal{G} is (δ_3, d_{pqr}) -regular with respect to P^{pqr} for some $\lambda/3 \leq d_{pqr} \leq 1 - \lambda/3$) to deduce that \mathcal{G} contains vertices v_1, \dots, v_f such that $v_p \in V_p$ for each $1 \leq p \leq f$ and $\mathcal{G} \cap T(P)$ induces a copy of \mathcal{F} on v_1, \dots, v_f . It remains to show that \mathcal{G} itself also induces a copy of \mathcal{F} on v_1, \dots, v_f .

For a given $1 \leq p < q < r \leq f$, if $\{v_p, v_q, v_r\} \in E(\mathcal{G} \cap T(P)) = E(\mathcal{G}) \cap T(P)$, then clearly also $\{v_p, v_q, v_r\} \in E(\mathcal{G})$. On the other hand, since, by Corollary 3.8, $P[v_1, \dots, v_f]$ is a clique, for each $1 \leq p < q < r \leq f$ we have $\{v_p, v_q, v_r\} \in T(P)$. Thus, if $\{v_p, v_q, v_r\} \notin E(\mathcal{G}) \cap T(P)$, then $\{v_p, v_q, v_r\} \notin E(\mathcal{G})$. We conclude that $\mathcal{G}[v_1, \dots, v_f] = (\mathcal{G} \cap T(P))[v_1, \dots, v_f]$, and thus $\mathcal{G}[v_1, \dots, v_f]$ is indeed an induced copy of \mathcal{F} , a contradiction. So, case (g) cannot occur.

So, suppose that Case (r) holds. Then, applying Proposition 3.5, we obtain that the graph

$$P^* = \bigcup_{1 \leq p < q \leq k} P^{pq} \tag{9}$$

satisfies

$$\mathcal{N}_k(P^*) \geq \frac{1}{2} m^k \left(\frac{1}{\ell}\right)^{\binom{k}{2}}. \quad (10)$$

Furthermore, since by Remark 3.3 and (8), \mathcal{G} is (δ_3, d_{pqr}) -regular with respect to P^{pqr} for some $d_{pqr} \geq 1 - \lambda$, Corollary 3.7 gives

$$\mathcal{N}_k(\mathcal{G} \cap T(P^*)) \geq (1 - \lambda) \cdot (1 - \lambda)^{\binom{k}{3}} \mathcal{N}_k(P^*) = (1 - \eta) \mathcal{N}_k(P^*).$$

Thus, we have found k vertex sets V_1, \dots, V_k of size $m \geq c'n$ and a k -partite graph P^* on these sets, which is (ϵ, η) -dense with respect to $\mathcal{G} \cap T(P^*) \subseteq \mathcal{G}$, where $\epsilon = \frac{1}{2} \left(\frac{1}{\ell}\right)^{\binom{k}{2}}$.

Similarly, in Case (b), considering P^* as in (9) and the 3-graph $\hat{\mathcal{G}} = T(P^*) \setminus \mathcal{G}$ (note that $d(\hat{\mathcal{G}}|P^*) > 1 - 2\lambda/3$) we obtain that (10) holds as before, and, by Corollary 3.7,

$$\mathcal{N}_k(\hat{\mathcal{G}}) \geq (1 - \lambda) \cdot (1 - \lambda)^{\binom{k}{3}} \mathcal{N}_k(P^*) = (1 - \eta) \mathcal{N}_k(P^*).$$

Thus, again, for $\epsilon = \frac{1}{2} \left(\frac{1}{\ell}\right)^{\binom{k}{2}}$ we obtain the sets V_1, \dots, V_k of size $m \geq c'n$ and the graph P^* on these sets, which is (ϵ, η) -dense with respect to $\hat{\mathcal{G}} \subseteq \bar{\mathcal{G}}$. \square

4 Proofs of Theorems 1.2 and 1.3

In this section we prove Proposition 4.1 and deduce Theorems 1.2 and 1.3 as corollaries.

We say that an r -graph \mathcal{F} is *extendable* if any two vertices of \mathcal{F} are contained in some edge and \mathcal{F} contains a copy of $K_{r+1}^{(r)}$ (complete r -graph on $r + 1$ vertices). Note that for $r = 2$ the extendable graphs are precisely the cliques. Now we are ready to state Proposition 4.1.

Proposition 4.1. *For every $r \geq 3$ and an extendable $(r - 1)$ -graph \mathcal{F} there exists an r -graph \mathcal{F}^+ and constant $c_0 = c_0(r, \mathcal{F}) > 0$ and $c_1 = c_1(r, \mathcal{F}) > 0$ such that*

$$R_{r, \mathcal{F}^+}(t) \geq (R_{r-1, \mathcal{F}}(c_0 t))^{c_1 t}.$$

In order to construct the non-universal r -graph in the above proposition from a non-universal $(r - 1)$ -graph we will adapt an idea from [5]. Given an extendable $(r - 1)$ -graph \mathcal{F} , an $(r - 1)$ -graph $\mathcal{G} = (V, E)$ and an edge-labeling $\phi : \binom{[N]}{2} \rightarrow [|V|]$ for some integer $N \geq |V|$, define a red-blue coloring $\chi : \binom{[N]}{r} \rightarrow \{\text{red}, \text{blue}\}$ as follows. For any $1 \leq a_1 < \dots < a_r \leq N$ we define $\chi(\{a_1, \dots, a_r\}) = \text{red}$ if for $i = 2, \dots, r$ all $\phi(a_1, a_i)$ are distinct and $\{\phi(a_1, a_2), \dots, \phi(a_1, a_r)\} \in E(\mathcal{G})$, and otherwise $\chi(\{a_1, \dots, a_r\}) = \text{blue}$. Set $\mathcal{G}^+ = \mathcal{G}^+(\mathcal{G}, \phi)$ to be the r -graph of all red edges in χ .

The first step towards proving Proposition 4.1 is to construct the graph \mathcal{F}^+ .

Lemma 4.2. *Let $r \geq 3$. Then for every extendable $(r - 1)$ -graph \mathcal{F} there exists an r -graph \mathcal{F}^+ satisfying the following. For any induced \mathcal{F} -free $(r - 1)$ -graph \mathcal{G} and any labelling ϕ the r -graph $\mathcal{G}^+ = \mathcal{G}^+(\mathcal{G}, \phi)$ is induced \mathcal{F}^+ -free.*

Proof. To construct \mathcal{F}^+ we proceed as follows. We first define an *ordered* r -graph \mathcal{F}^* such that \mathcal{G}^+ , when viewed as an ordered r -graph under the natural ordering on $[N]$, does not contain an induced

copy of \mathcal{F}^* . Put $f = |V(\mathcal{F})|$, and by choosing an arbitrary ordering suppose that $V(\mathcal{F}) = [f]$. Now, let $V(\mathcal{F}^*) = \{0\} \cup [f]$, and $E(\mathcal{F}^*) = \{\{0\} \cup J : J \in E(\mathcal{F})\} \cup \binom{[f]}{r}$.

Let us now show that \mathcal{G}^+ indeed does not contain an induced copy of \mathcal{F}^* as an ordered subgraph. Suppose for a contradiction that there is an order preserving mapping $\tau : \{0\} \cup [f] \rightarrow [N]$ such that $\{\tau(j_1), \dots, \tau(j_r)\} \in E(\mathcal{G}^+)$ if and only if $\{j_1, \dots, j_r\} \in E(\mathcal{F}^*)$. Denote $\tau(0) = a$ and $\tau(i) = b_i$ for each $1 \leq i \leq f$.

Suppose that $\phi(a, b_i) = \phi(a, b_k)$ for some $1 \leq i < k \leq f$. Then by the definition of χ no edge of \mathcal{G}^+ contains $\{a, b_i, b_k\} = \{\tau(0), \tau(i), \tau(k)\}$. Since τ is an isomorphism, no edge of \mathcal{F}^* contains $\{0, i, k\}$. This in turn implies that $\{i, k\}$ is not contained in any edge of \mathcal{F} , a contradiction to the assumption that \mathcal{F} is extendable. Hence, all labels $\phi(a, b_i)$ for $1 \leq i \leq f$ must be distinct.

Now, take an arbitrary $I = \{j_1, \dots, j_{r-1}\} \subseteq [f]$, and put $a_i = \tau(j_i)$ for each $1 \leq i \leq r-1$. If $I \in E(\mathcal{F})$, then we have $\{0\} \cup I \in E(\mathcal{F}^*)$, thus $\{a, a_1, \dots, a_{r-1}\} = \{\tau(0), \tau(j_1), \dots, \tau(j_{r-1})\} \in E(\mathcal{G}^+)$. Hence, by the definition of χ , we must have $\{\phi(a, a_1), \dots, \phi(a, a_{r-1})\} \in E(\mathcal{G})$. On the other hand, if $I \notin E(\mathcal{F})$, then $\{a, a_1, \dots, a_{r-1}\} \notin E(\mathcal{G}^+)$, which similarly means $\chi(\{a, a_1, \dots, a_{r-1}\}) = \text{blue}$. Since we have established that all $\phi(a, \tau(j_i))$ are distinct, we deduce that $\{\phi(a, a_1), \dots, \phi(a, a_{r-1})\} \notin E(\mathcal{G})$.

Therefore the distinct labels $\{\phi(a, b_1), \dots, \phi(a, b_f)\} \subseteq V(\mathcal{G})$ form a copy of \mathcal{F} as an induced subgraph of \mathcal{G} , in contradiction to the assumption that \mathcal{G} is induced \mathcal{F} -free. We conclude that \mathcal{G}^+ does not contain an ordered induced copy of \mathcal{F}^* .

Now, define \mathcal{F}^+ to be the r -graph such that every ordering of it contains \mathcal{F}^* as an induced ordered subgraph. Such a graph always exists by a theorem of Alon, Pach and Solymosi [1, Theorem 3.5]. Then \mathcal{G}^+ does not contain an induced copy of \mathcal{F}^+ , for otherwise it would contain an ordered induced copy of \mathcal{F}^* . \square

We will later need the following simple observation.

Claim 4.3. *In the natural case when $\mathcal{F} = K_{s-1}^{(r-1)}$ for some $s > r$, one can take $\mathcal{F}^+ = K_s^{(r)}$.*

Proof. \mathcal{F} is extendable, $\mathcal{F}^* = K_s^{(r)}$, and since all orderings of \mathcal{F}^* are indistinguishable, we can take $\mathcal{F}^+ = \mathcal{F}^* = K_s^{(r)}$. \square

Having defined \mathcal{F}^+ we need to find a labelling ϕ , such that if neither \mathcal{G} nor its complement contain a large complete $(r-1)$ -partite subgraph then $\mathcal{G}^+(\mathcal{G}, \phi)$ will also have this property. This is done in the following technical claim whose proof is deferred to the end of the section.

Claim 4.4. *For every integer $r \geq 3$ and constant $\alpha > 1$ there exists a constant $c_1 = c_1(r, \alpha) > 0$ and an integer $t_0 = t_0(r, \alpha)$ as follows. Let $\beta = \frac{2}{1-1/\alpha}$, $c_0 = \frac{1}{\beta r^2}$, and suppose that the integers t, n and N are such that $t \geq t_0$, $n \geq (c_0 t)^\alpha$ and $N = n^{c_1 t}$. Then there exists a labeling $\phi : \binom{[N]}{2} \rightarrow [n]$ with the following property: There do not exist disjoint sets $A_1, \dots, A_r \subseteq [N]$ with $|A_1| = \dots = |A_r| = t/r$ such that for every $a \in A_1$ we have*

$$|\{\phi(a, a_i) : a_i \in A_i\}| < \frac{t}{\beta r}, \quad \text{for some } 2 \leq i \leq r \quad (11)$$

The last ingredient we will need for the proof of Proposition 4.1 is the following claim whose proof is a routine application of the probabilistic deletion method and is thus deferred to the end of the section.

Claim 4.5. *Let \mathcal{F} be an extendable r -graph. Then $R_{r,\mathcal{F}}(t) \geq t^\alpha$ for some $\alpha = \alpha_r > 1$.*

Proof of Proposition 4.1. Given an extendable $(r-1)$ -graph \mathcal{F} , take \mathcal{F}^+ to be the r -graph from lemma 4.2. Let $\alpha = \alpha_{r-1}$ be the constant from Claim 4.5, and suppose that $t \geq t_0$, where $t_0 = t_0(r, \alpha)$ is as defined in Claim 4.4 (due to flexibility in choosing the constants c_0 and c_1 , it suffices to prove the statement of Proposition 4.1 for large t). Let $\beta = \frac{2}{1-1/\alpha}$, $c_0 = \frac{1}{\beta r^2}$, and let $c_1 = c_1(r, \alpha)$ be as in Claim 4.4. Let $n = R_{r-1,\mathcal{F}}(c_0 t)$, and note that, since \mathcal{F} is extendable, Claim 4.5 gives $n \geq (c_0 t)^\alpha$. Let \mathcal{G} to be an $(r-1)$ -graph on n vertices which is induced \mathcal{F} -free and such that \mathcal{G} and $\overline{\mathcal{G}}$ do not contain a $K_{c_0 t, \dots, c_0 t}^{(r-1)}$. Finally, let $N = n^{c_1 t}$, let $\phi : \binom{[N]}{2} \rightarrow [n]$ be a labelling as guaranteed by Claim 4.4, and consider the r -graph $\mathcal{G}^+ = \mathcal{G}^+(\mathcal{G}, \phi)$.

We now claim that $\overline{\mathcal{G}^+}$ contains no copy of $K_{t, \dots, t}^{(r)}$; the fact that \mathcal{G}^+ does not contain a $K_{t, \dots, t}^{(r)}$ can be deduced analogously. So, suppose for a contradiction that $\overline{\mathcal{G}^+}$ contains a copy of $K_{t, \dots, t}^{(r)}$ on vertex classes $Q_1, \dots, Q_r \subseteq V(\mathcal{G}^+)$. Let A' be the first t vertices of $\bigcup_{i=1}^r Q_i$ in the natural ordering of $[N]$. By the pigeonhole principle, some t/r of them will be in the same class, say in Q_1 ; let $A_1 \subseteq A'$ be the set of these t/r vertices. By our choice of A' , for $i = 2, \dots, r$ there will be disjoint sets $A_i \subseteq Q_i$ with $|A_i| = t/r$ such that for every $a_1 \in A_1$, and $a_i \in A_i$ we have $a < a_i$. Note that we do *not* make any claims regarding the order of vertices inside $A_2 \cup \dots \cup A_r$.

By the definition of ϕ and Claim 4.4 there must exist some $a \in A$ such that, with $\Phi(A_i) = \{\phi(a, a_i) : a_i \in A_i\}$, we have $\min\{|\Phi(A_i)| : 2 \leq i \leq r\} \geq \frac{t}{\beta r}$. Now, taking one representative vertex for each color in each $\Phi(A_i)$, we find sets $U_i \subseteq A_i$ for $2 \leq i \leq r$ with $|U_i| = \frac{t}{\beta r^2} = c_0 t$ such that all labels in $\bigcup_{i=2}^r \bigcup_{u \in U_i} \phi(a, u)$ are distinct. However, since for each $(u_2, \dots, u_r) \in U_2 \times \dots \times U_r$ we have $\chi(\{a, u_2, \dots, u_r\}) = \text{blue}$, we obtain that $\overline{\mathcal{G}}[\Phi(U_2), \dots, \Phi(U_r)]$ forms a complete $(r-1)$ -partite graph with parts of size $c_0 t$, in contradiction to the assumption. Hence, $\overline{\mathcal{G}^+}$ contains no copy of $K_{t, \dots, t}^{(r)}$, as claimed.

To summarize, the r -graph \mathcal{G}^+ has $N = n^{c_1 t}$ vertices, by Lemma 4.2 it is induced \mathcal{F}^+ -free, and neither \mathcal{G}^+ nor its complement contain a copy of $K_{t, \dots, t}^{(r)}$. Therefore,

$$R_{r,\mathcal{F}^+}(t) \geq N = (R_{r-1,\mathcal{F}}(c_0 t))^{c_1 t},$$

as needed. □

As corollaries of Proposition 4.1 we obtain Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Apply Proposition 4.1 for $r = 3$ and \mathcal{F} being the triangle, that is, the complete 2-graph on 3 vertices; note that \mathcal{F} is extendable. By Claim 4.3 we can take $\mathcal{F}^+ = K_4^{(3)}$. Hence, by Proposition 4.1, we have $R_{3,\mathcal{F}^+}(t) \geq (c_0 t)^{\alpha c_1 t} = t^{\Omega(t)}$, as claimed. □

Proof of Theorem 1.3. By adding to \mathcal{F} a set of $r+1$ new vertices of full degree, we obtain an extendable graph \mathcal{F}' . Applying Proposition 4.1 to \mathcal{F}' we obtain a graph \mathcal{F}^+ and the constants $c_0(r, \mathcal{F}), c_1(r, \mathcal{F}) > 0$ as stated therein. Then, taking $c(r, \mathcal{F}) = \min\{c_0, c_1\} > 0$, we obtain

$$R_{r,\mathcal{F}^+}(t) \geq (R_{r-1,\mathcal{F}'}(c_0 t))^{c_1 t} \geq (R_{r-1,\mathcal{F}}(ct))^{ct},$$

where the second inequality relies on the fact that \mathcal{F}' contains an induced copy of \mathcal{F} . □

We end this section with the proofs of Claims 4.4 and 4.5.

Proof of Claim 4.4. Set

$$\gamma = \frac{1}{2}\left(1 - \frac{1}{\alpha} - \frac{1}{\beta}\right) > 0, \quad c_1 = \frac{\gamma}{r^2} = \frac{1}{2r^2}\left(1 - \frac{1}{\alpha} - \frac{1}{\beta}\right) \quad \text{and} \quad t_0 = c_1^{-\frac{1+\alpha\gamma}{\alpha\gamma}}.$$

and suppose that t, n and N are as in the statement of the lemma. Define $\phi : \binom{[N]}{2} \rightarrow [n]$ to be an edge-labeling, where each element of $\binom{[N]}{2}$ is independently assigned a color between 1 and n uniformly at random. For disjoint sets $A_1, \dots, A_r \subseteq [N]$ of size t/r each, let X_{A_1, \dots, A_r} be the event that every $a \in A_1$ satisfies (11). Consider some fixed disjoint A_1, \dots, A_r of size t/r . Since for different $a \in A_1$ the edges between a and $A_2 \cup \dots \cup A_r \subseteq [N] \setminus A_1$ are assigned the values of ϕ independently, we obtain

$$\mathbb{P}(X_{A_1, \dots, A_r}) \leq \left[\sum_{i=2}^r \binom{n}{\frac{t}{\beta r}} \left(\frac{\frac{t}{\beta r}}{n} \right)^{|A_i|} \right]^{|A_1|} \leq \left[r \binom{n}{\frac{t}{\beta r}} \left(\frac{t}{n} \right)^{\frac{t}{r}} \right]^{\frac{t}{r}}.$$

By the union bound, the probability that some event X_{A_1, \dots, A_r} holds is at most

$$\begin{aligned} \sum_{A_1, \dots, A_r} \mathbb{P}(X_{A_1, \dots, A_r}) &\leq N^t \left[r \binom{n}{\frac{t}{\beta r}} \left(\frac{t}{n} \right)^{\frac{t}{r}} \right]^{\frac{t}{r}} \leq \left(N^r n^{\frac{t}{\beta r}} \left(c_0^{-1} n^{\frac{1}{\alpha} - 1} \right)^{\frac{t}{r}} \right)^{\frac{t}{r}} \\ &= \left(c_0^{-\frac{t}{r}} n^{c_1 r t + \frac{t}{\beta r} + (\frac{1}{\alpha} - 1)\frac{t}{r}} \right)^{\frac{t}{r}} = \left(c_0^{-\frac{t}{r}} n^{\frac{t}{r} (c_1 r^2 + \frac{1}{\beta} + \frac{1}{\alpha} - 1)} \right)^{\frac{t}{r}} \\ &= (n^{-\gamma} c_0^{-1})^{\left(\frac{t}{r}\right)^2} < (c_0^{1+\alpha\gamma} t^{\alpha\gamma})^{-\left(\frac{t}{r}\right)^2} < 1. \end{aligned}$$

We infer that with positive probability none of the events X_{A_1, \dots, A_r} hold, implying that the required labeling exists. \square

Proof of Claim 4.5. Since $R_{r, \mathcal{F}}(t) \geq R_{r, K_{r+1}^{(r)}}(t)$, it suffices to consider $K_{r+1}^{(r)}$. We will prove the equivalent statement, that for every sufficiently large n there exists a $K_{r+1}^{(r)}$ -free r -graph on $\Theta(n)$ vertices, which does not contain a complete or empty r -partite subgraph of polynomial size.

To this end, set $p = n^{-\frac{r-1/2}{r+1}} = n^{-1 + \frac{3}{2(r+1)}}$ and let \mathcal{G} be a random r -graph $\mathcal{G}_{n,p}^{(r)}$ on n vertices, where (as in random 2-graphs) each r -tuple is selected as an edge independently with probability p . Then the expected number of copies of $K_{r+1}^{(r)}$ in \mathcal{G} will be $\binom{n}{r} p^{r+1} = \Theta(n^{1/2})$. Hence, by Markov's inequality the probability that \mathcal{G} contains more than $n^{3/4}$ copies of $K_{r+1}^{(r)}$ is $o(1)$.

Next, for an integer t , the expected number of copies of the complete r -partite graph $K_{t, \dots, t}^{(r)}$ in $\mathcal{G}_{n,p}$ is

$$\binom{n}{rt} \frac{1}{r!} \left[\prod_{k=r}^1 \binom{kt}{t} \right] p^{t^r} \leq n^{rt} p^{t^r} = (n^r p^{t^r-1})^t,$$

and a straightforward calculation shows that the latter expression is $o(1)$ when $t \geq 3$. Hence, by Markov's inequality, with high probability \mathcal{G} will not contain a copy of $K_{3, \dots, 3}^{(r)}$, let alone of $K_{t, \dots, t}^{(r)}$ for larger values of t . Lastly, for $t = n^{\frac{3}{2r}}$ the expected number of copies of $K_{t, \dots, t}^{(r)}$ in \mathcal{G} is

$$\binom{n}{rt} \frac{1}{r!} \left[\prod_{k=r}^1 \binom{kt}{t} \right] (1-p)^{t^r} \leq n^{rt} e^{-pt^r} = (n^r e^{-pt^r-1})^t. \quad (12)$$

Again, a straightforward calculation shows that with this choice of t the right hand side of (12) is $o(1)$, hence, again by Markov's inequality, with high probability there will be no copy of $K_{t,\dots,t}^{(r)}$ in $\overline{\mathcal{G}}$.

We deduce that for $t = n^{\frac{3}{2r}}$ with positive probability $\mathcal{G} \sim \mathcal{G}_{n,p}^{(r)}$ satisfies all three properties discussed above. Removing a vertex from each $r+1$ -clique thus results in a $K_{r+1}^{(r)}$ -free r -graph on at least $n/2$ vertices satisfying the assertion of the lemma. \square

5 Concluding Remarks

As we have mentioned in Section 1, the best known bound for Ramsey numbers of 3-graphs imply that every 3-graph contains a homogenous set of size $\Omega(\log \log n)$. An intriguing open problem of Conlon, Fox and Sudakov [6] is if for every 3-graph \mathcal{F} , every induced \mathcal{F} -free \mathcal{G} contains a homogeneous set of size $\omega(\log \log n)$. As a (minor) step towards resolving this problem one would first like to find general conditions guaranteeing that certain \mathcal{F} satisfy this condition. For example, it follows from the Erdős–Rado [9] bound that this is the case if \mathcal{F} is a complete 3-graph. Let us now sketch an argument showing that a much broader family of graphs \mathcal{F} have this property. We say that a 3-graph \mathcal{F} on f vertices is *nice* if there is an ordering $\{1, \dots, f\}$ of $V(\mathcal{F})$ and an ordered 2-graph F on $\{1, \dots, f\}$ so that for every $1 \leq i < j < k \leq f$ the triple $(i, j, k) \in E(\mathcal{G})$ if and only if $(i, j) \in E(F)$.

Proposition 5.1. *If \mathcal{F} is nice, then every induced \mathcal{F} -free 3-graph \mathcal{G} on n vertices has a homogeneous set of size $2^{\Omega(\sqrt{\log \log n})}$.*

Proof (sketch): Since \mathcal{F} is nice there is a graph F satisfying the condition in the above paragraph. Pick an arbitrary ordering of $V(F)$ and let F' be a graph so that every ordering of $V(F')$ has an ordered induced copy of F . Such an F' exists by [17].

We recall that the Erdős–Rado [9] scheme of bounding Ramsey numbers proceeds by gradually building an ordered set of vertices $\{1, \dots, p\}$, where $p = \Omega(\sqrt{\log n})$, along with an ordered graph G on $[p]$ so that for every $1 \leq i < j < k \leq p$ the triple $(i, j, k) \in E(\mathcal{G})$ if and only if $(i, j) \in E(G)$. Observe that G is induced F' -free, as otherwise the definition of G , F' and F would imply that \mathcal{G} has an induced copy of \mathcal{F} . By the Erdős–Hajnal theorem [8] mentioned in Section 1 we deduce that G has a homogeneous set of size $2^{\Omega(\sqrt{\log \log n})}$, which implies (using the definition of G) that \mathcal{G} has a homogeneous set of the same size. \square

Lastly, we state and prove Proposition 5.2 which extends Rödl's [15] example mentioned in Section 1 to arbitrary uniformities $r \geq 3$. We would like to reiterate our belief that the following upper bound is tight for all $r \geq 3$. We remind the reader that we use $d(\mathcal{G}) = e(\mathcal{G}) / \binom{|\mathcal{G}|}{r}$ for the *edge-density* of an r -graph \mathcal{G} .

Proposition 5.2. *For every integer $r \geq 3$ there is an r -graph \mathcal{F} with the following properties: (i) for any $\epsilon > 0$ there exists a constant $C = C(r, \epsilon)$ such that for arbitrarily large n there exists an induced \mathcal{F} -free r -graph \mathcal{G} on n vertices, such that for every vertex set $W \subset V(\mathcal{G})$ with $|W| = C(\log n)^{1/(r-2)}$ we have*

$$1/2 - \epsilon \leq d(\mathcal{G}[W]) \leq 1/2 + \epsilon. \quad (13)$$

(ii) there exists a constant $C = C(r)$ such that for every large enough n there is an induced \mathcal{F} -free r -graph \mathcal{G} on n vertices, that does not contain a copy of $K_{t,\dots,t}^{(r)}$ with $t = C(\log n)^{1/(r-2)}$.

Proof. We prove only item (i), since the proof of (ii) is identical. We make use of the parity construction from [4]. Let $V(\mathcal{G}) = [n]$, and consider an $(r - 1)$ -graph $\mathcal{H} \sim \mathcal{G}_{n,1/2}^{(r-1)}$, that is, each $(r - 1)$ -edge is selected randomly and independently with probability $1/2$. Then define \mathcal{G} as follows: let $\{i_1, \dots, i_r\} \in E(\mathcal{G})$ if and only if $e(\mathcal{H}[\{i_1, \dots, i_r\}]) \equiv 0 \pmod{2}$.

It is easy to see that for each set R of $r + 1$ vertices in \mathcal{G} we have $e(\mathcal{G}[R]) \equiv r + 1 \pmod{2}$. Hence, \mathcal{G} will not contain an induced copy of \mathcal{F} , where \mathcal{F} is any r -graph with $e(\mathcal{F}) \equiv r \pmod{2}$.⁷ Hence, to prove the lemma, it suffices to show that (13) holds with positive probability; this is achieved by a standard application of Azuma’s inequality.

Consider a subset $A \subseteq [n]$, with $|A| = a$, and define the random variable $X = e(\mathcal{G}[A])$. Note that each subset $\{i_1, \dots, i_r\} \subseteq A$ forms an edge of \mathcal{G} with probability $1/2$ – this is a consequence of the basic identity

$$\sum_{j=0}^r (-1)^j \binom{r}{j} = (1 - 1)^r = 0.$$

By linearity of expectation, this implies $\mathbb{E}(X) = \frac{1}{2} \binom{a}{r}$.

Now, choose an arbitrary ordering of the subsets of A of size $r - 1$ and consider the edge-exposure martingale $(X_t)_{t=0}^N$, where $N = \binom{a}{r-1}$, and X_t is the expected value of X , after the first t edges have been exposed. In particular, $X_0 = \mathbb{E}(X)$ and $X_N = X$. Observe that exposing a new $(r - 1)$ -edge can change at most a edges of \mathcal{G} . Hence, Azuma’s inequality yields

$$\mathbb{P}(|X_N - X_0| \geq \epsilon a^r) \leq 2e^{-\frac{\epsilon^2 a^{2r}}{2Na^2}} \leq e^{-\epsilon^2 a^{r-1}}.$$

Taking the union over all subsets of $[n]$ of size a , we obtain that \mathcal{G} satisfies (13) with positive probability for all vertex sets of size a as long as

$$n^a e^{-\epsilon^2 a^{r-1}} \leq 1,$$

which is if and only if $a \geq C(\log n)^{1/(r-1)}$ for a constant $C = C(r, \epsilon)$. □

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⁷For even values of r this argument can in fact be applied to *any* r -graph \mathcal{F} by considering either \mathcal{G} or its complement

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