Basic Combinatorics - Spring '24 **Instructor: Asaf Shapira**

Home Assignment 1 **Due Date: TBA**

Please submit organized and well written solutions!

Problem 1. Suppose $n = 1 \pmod 8$. Show that the number of subsets of an n-element set, whose size is $0 \pmod 4$ is $2^{n-2} + 2^{(n-3)/2}$.

Problem 2. Give a "combinatorial" proof of the identity

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n} .$$

Problem 3. Give a "combinatorial" proof of the identity

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \ldots + \binom{n}{r} = \binom{n+1}{r+1} .$$

Use this identity to show that

$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6} .$$

Explain how you could (in principle) derive explicit formulas for $\sum_{k=1}^{n} k^t$ for any fixed t.

Problem 4. Fermat's Little Theorem (FLT) states that $b^p = b \pmod p$ for any prime p and $b \in \mathbb{F}_p$

- Explain why $(1+x)^p = 1 + x^p \pmod p$. Use this to prove FLT (Hint: induction on b).
- Explain why $\binom{p}{k_1,\ldots,k_b} = 0 \pmod p$ if all $k_i < p$ and $\binom{p}{k_1,\ldots,k_b} = 1 \pmod p$ when some $k_i = p$. Use this to prove FLT by considering b^p as $(1 + \ldots + 1)^p$ and using multinomial expansion.
- We wish to color p chairs arranged on a round carrousel using b colors. Two colorings are considered identical if one can be obtained from the other by rotating the carrousel. Compute the number of (different) ways to color the chairs, and use your answer to deduce FLT.

Problem 5. Prove that if $a_1, \ldots, a_n$ are n (not necessarily distinct) integers, then there always exist $1 \le i \le j \le n$ such that $\sum_{k=i}^{j} a_k$ is divisible by n.