Problem 1. Explain why for any $1 \leq k \leq n$ and $0 < x < 1$ we have
\[
\binom{n}{k} x^k \leq (1 + x)^n \leq e^{xn}.
\]
Use this to show that $\binom{n}{k} \leq \left(\frac{e}{k}\right)^k$. Then modify your proof to show that in fact $\sum_{i=1}^{k} \binom{n}{i} \leq \left(\frac{en}{k}\right)^k$.

Problem 2. Use integration to prove that $n! = \Theta\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right)$

Problem 3. Let $q(n)$ denote the number of ordered sets of positive integers whose sum is $n$.

1. Calculate $q(n)$ using a direct counting argument.
2. How many ordered sets of $k$ positive integers are there whose sum is $n$? Use your answer to calculate $q(n)$ (in a less direct way).

Problem 4. Let $p(n)$ denote the number of unordered sets of positive integers whose sum is $n$. Show that
\[
p(n) \geq \max_{1 \leq k \leq n} \frac{\binom{n-1}{k-1}}{k!}
\]
Hint: Recall item (2) from the previous question.
Deduce that there is an absolute constant $c > 0$ for which $p(n) \geq e^{c\sqrt{n}}$.

Problem 5. Let $\pi(m, n)$ denote the set of prime numbers in the interval $[m, n]$.

1. Show that $\prod_{p \in \pi(m+1, 2m)} p \leq \binom{2m}{m}$.
2. Use the previous item to show that $\prod_{p \in \pi(1, n)} p \leq 4^n$.
3. Deduce from the previous item that $|\pi(1, n)| = O(n/\log n)$.

Problem 6. A set of vertices $S$ in a tournament $T$ is dominated if there is some vertex $v \in T \setminus S$ that points (that is, sends an edge) to all the vertices of $S$. Show that every tournament on $2^k$ vertices contains a set of at most $k$ vertices that is not dominated. Show that if $\binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k} < 1$, then there is an $n$-vertex tournament so that every set of $k$ vertices is dominated. Use this to get an explicit estimate for the size of the smallest tournament in which every $k$-set is dominated.