Basic Combinatorics - Spring '24 **Instructor: Asaf Shapira**

Home Assignment 2 **Due Date: TBA**

Please submit organized and well written solutions!

Problem 1. Explain why for any $1 \le k \le n$ and $0 < x < 1$ we have

$$\binom{n}{k} x^k \le (1+x)^n \le e^{xn} \ .$$

Use this to show that $\binom{n}{k} \le (\frac{en}{k})^k$. Then modify your proof to show that in fact $\sum_{i=1}^{k} \binom{n}{i} \le (\frac{en}{k})^k$.

Problem 2. Use integration to prove that $n! = \Theta\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right)$

Problem 3. Let $q(n)$ denote the number of **ordered** sets of positive integers whose sum is n.

1. Calculate $q(n)$ using a direct counting argument.
2. How many **ordered** sets of k positive integers are there whose sum is n? Use your answer to calculate $q(n)$ (in a less direct way).

Problem 4. Let $p(n)$ denote the number of **unordered** sets of positive integers whose sum is n. Show that

$$p(n) \ge \max_{1 \le k \le n} \frac{\binom{n-1}{k-1}}{k!}$$

Hint: Recall item (2) from the previous question.
Deduce that there is an absolute constant $c > 0$ for which $p(n) \ge e^{c\sqrt{n}}$.

Problem 5. Let $\pi(m,n)$ denote the set of prime numbers in the interval $[m,n]$.

1. Show that $\prod_{p \in \pi(m+1,2m)} p \le \binom{2m}{m}$.
2. Use the previous item to show that $\prod_{p \in \pi(1,n)} p \le 4^n$.
3. Deduce from the previous item that $|\pi(1,n)| = O(n/\log n)$.

Problem 6. A set of vertices S in a tournament T is *dominated* if there is some vertex $v \in T \setminus S$ that points (that is, sends an edge) to all the vertices of S. Show that every tournament on 2^k vertices contains a set of at most k vertices that is *not* dominated. Show that if $\binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} < 1$, then there is an n-vertex tournament so that every set of k vertices is dominated. Use this to get an explicit estimate for the size of the smallest tournament in which every k-set is dominated.