Basic Combinatorics - Spring '24 **Instructor: Asaf Shapira**

Home Assignment 3 **Due Date: TBA**

Please submit organized and well written solutions!

Problem 1. Show that there is an integer n_0 such that for all $n \geq n_0$, in every 9-coloring of the integers $\{1, 2, \ldots, n\}$, one of the 9 color classes contains 4 integers a, b, c, d satisfying $a + b + c = d$.

Problem 2. Show that every tournament on n vertices, contains a transitive tournament on $\lfloor \log_2 n \rfloor$ vertices. Also, show that there exists a tournament on n vertices that does not contain a transitive tournament on $2\log_2 n + 2$ vertices.

Problem 3. Show that if an n-vertex graph $G = (V, E)$ has no copy of $K_{2,t}$ then

$$|E| \leq \frac{1}{2}(\sqrt{t-1}\; n^{3/2} + n)\ .$$

Problem 4. Suppose $S_1, \ldots, S_n \subseteq [n]$ are such that $|S_i \cap S_j| \leq 1$ for all $1 \leq i < j \leq n$. Show that in this case

$$\frac{1}{n}\sum_{i=1}^{n} |S_i| = O(\sqrt{n})$$

Problem 5. Turán's Theorem states that if $G = (V, E)$ has no copy of K_{t+1} then $|E| \leq (1 - \frac{1}{t})\frac{n^2}{2}$. Prove Turán's Theorem using the "weight shifting" method we used to prove Mantel's Theorem.