Basic Combinatorics - Spring '24 **Instructor: Asaf Shapira**

Home Assignment 4 **Due Date: TBA**

Please submit organized and well written solutions!

Problem 1.

- Show that if $T(n) = T(n/3) + T(2n/3) + n$ then $T(n) = O(n \log n)$.
- Show that if $T(n) = 2T(n/2) + n \log n$ then $T(n) = O(n \log^2 n)$.
- Let $c_1, \dots, c_k$ be k positive reals satisfying $\sum_{i=1}^{k} c_i < 1$. Show that if $T(n) = \sum_{i=1}^{k} T(c_i n) + n$ then $T(n) = O(n)$. The hidden constant in the O-notation may depend on $c_1, \dots, c_k$.

Problem 2. Prove that every tournament has a Hamilton path. Try to find a direct proof (Hint: use induction) as well as an indirect proof relying on a theorem we saw in class.

Problem 3. Prove that every set X of $st + 1$ integers contains one of the following:

- A subset $T = \{x_1, \dots, x_{t+1}\} \subseteq X$ of size $t + 1$ such that x_k divides x_{k+1} for every $1 \leq k \leq t$.
- A subset $S = \{x_1, \dots, x_{s+1}\} \subseteq X$ of $s + 1$ integers such that x_i does not divide x_j for every $x_i, x_j \in S$.

Problem 4. A family of sets $\mathcal{S} = \{S_1, \dots, S_m\}$ is *union-free* if $S_i \cup S_j \neq S_k$ for all $S_i, S_j, S_k \in \mathcal{S}$. Show that every collection $\mathcal{F} = \{S_1, \dots, S_n\}$ of n sets contains a sub-collection $\mathcal{S} \subseteq \mathcal{F}$ of at least $\sqrt{n}$ sets which is union-free.

Hint: Use dual Dilworth's Theorem.

Problem 5. Let P be a finite poset and let x, y be two elements of P that are incomparable under P (that is, $x \not\leq y$ and $x \not\leq y$). Show that P has a linear extension in which $x < y$.

Problem 6. Show that in the setting of Arrow's Theorem, if the individuals have only two options, then they can come up with a non-dictator social choice function.

Hint: Democracy!