Basic Combinatorics - Spring '25 **Instructor: Asaf Shapira**

Home Assignment 5 **Due Date: TBA**

Please submit organized and well written solutions!

Problem 1. Prove that the number of surjective (i.e. onto) mappings from $[n]$ to $[k]$ is given by $\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$. Use this to deduce that:

- $\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^n = n!$.
- $\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = 0$ when $k > n$.
- $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$, where $S(n,k)$ are the Stirling numbers of the second kind.

Problem 2. Consider the number of ways of coloring the integers $\{1, .., 2n\}$ using the colors red/blue in such a way that if i is colored red then so is $i-1$. Deduce the identity

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}2^{2n-2k} = 2n+1$$

Problem 3. Show that the number of subsets of size k of $\{1, \ldots, n\}$ which contain no pair of consecutive integers is given by $\binom{n-k+1}{k}$.

Problem 4. Let $A_1, \ldots, A_n$ be a family of n sets. Show that

$$\left|\bigcup_{i=1}^{n} A_i\right| \geq \sum_{1\leq i\leq n}|A_i| - \sum_{1\leq i<j\leq n}|A_i \cap A_j|$$

and

$$\left|\bigcup_{i=1}^{n} A_i\right| \leq \sum_{1\leq i\leq n}|A_i| - \sum_{1\leq i<j\leq n}|A_i \cap A_j| + \sum_{1\leq i<j<k\leq n}|A_i \cap A_j \cap A_k|$$