Problem 1. Prove that the number of surjective (i.e. onto) mappings from $[n]$ to $[k]$ is given by \[ \sum_{i=0}^{k} (-1)^i \binom{n}{i} (k - i)^n. \] Use this to deduce that:

- \[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)^n = n!. \]
- \[ \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k - i)^n = 0 \text{ when } k > n. \]
- \[ S(n, k) = \frac{1}{n!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k - i)^n, \]
  where $S(n, k)$ are the Stirling numbers of the second kind.

Problem 2. Consider the number of ways of coloring the integers $\{1, \ldots, 2n\}$ using the colors red/blue in such a way that if $i$ is colored red then so is $i - 1$. Deduce the identity
\[ \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} 2^{2n-2k} = 2^n + 1 \]

Problem 3. Show that the number of subsets of size $k$ of $\{1, \ldots, n\}$ which contain no pair of consecutive integers is given by $\binom{n-k+1}{k}$.

Problem 4. Let $A_1, \ldots, A_n$ be a family of $n$ sets. Show that
\[ |\bigcup_{i=1}^{n} A_i| \geq \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \]
and
\[ |\bigcup_{i=1}^{n} A_i| \leq \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \]