Problem 1. Give a (non-probabilistic) proof of the fact that any graph has an independent set of size at least
\[ \sum_{v \in V(G)} \frac{1}{1 + d(v)}. \]
Deduce Turán’s Theorem from the fact that any graph has an independent set as above.
**Hint:** Run the “obvious” greedy algorithm.

Problem 2. Prove Turán’s Theorem using the “weight shifting” method we used to prove Mantel’s Theorem.

Problem 3. Show that for every \( \epsilon > 0 \) and \( k \geq 1 \) there is \( c = c(\epsilon, k) > 0 \) so that any graph with at least \( \left( 1 - \frac{1}{k} + \epsilon \right) \frac{n^2}{2} \) edges contains at least \( cn^{k+1} \) copies of \( K_{k+1} \).

Problem 4. Show that if \( S \) is a set of \( n \) points in the plane, with the property that no two points are at distance greater than 1, then \( S \) has at most \( \lfloor n^2/3 \rfloor \) pairs of points at distance greater than \( 1/\sqrt{2} \). Also, show that the \( \lfloor n^2/3 \rfloor \) bound is tight.

Problem 5. Show that if \( G \) is a triangle-free non-bipartite graph, then
\[ e(G) \leq \left( \left\lfloor \frac{n + 1}{2} \right\rfloor - 1 \right) \left( n - \left\lfloor \frac{n + 1}{2} \right\rfloor \right) + 1. \]
Construct examples showing that this bound is tight.
**Hint:** Try to adapt one of the proofs we saw in class.

Problem 6. Show that a graph with at least \( \lfloor n^2/4 \rfloor + 1 \) edges contains at least \( \lfloor n/2 \rfloor \) triangles, and that this bound is tight.
**Hint:** Use Mantel’s Theorem and induction on \( n \).