Extremal Graph Theory - Fall $'23$

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Home Assignment 1

Due date: TBA

Please submit organized and well written solutions!

Problem 1. Give a (non-probabilistic) proof of the fact that any graph has an independent set of size at least
$$\sum_{v\in V(G)} \frac{1}{1+d(v)} \ .$$
Deduce Turán's Theorem from the fact that any graph has an independent set as above.
Hint: Run the "obvious" greedy algorithm.

Problem 2. Prove Turán's Theorem using the "weight shifting" method we used to prove Mantel's Theorem.

Problem 3. Show that for every $\epsilon > 0$ and $k \geq 1$ there is $c = c(\epsilon, k) > 0$ so that any graph with at least $(1 - \frac{1}{k} + \epsilon)\frac{n^2}{2}$ edges contains at least cn^{k+1} copies of K_{k+1}.

Problem 4. Show that if S is a set of n points in the plane, with the property that no two points are at distance greater than 1, then S has at most $\lfloor n^2/3 \rfloor$ pairs of points at distance greater than $1/\sqrt{2}$. Also, show that the $\lfloor n^2/3 \rfloor$ bound is tight.

Problem 5. Show that if G is a triangle-free non-bipartite graph, then
$$e(G) \leq \left(\left\lfloor \frac{n+1}{2} \right\rfloor - 1\right)\left(n - \left\lfloor \frac{n+1}{2} \right\rfloor\right) + 1 \ .$$
Construct examples showing that this bound is tight.
Hint: Try to adapt one of the proofs we saw in class.

Problem 6. Show that a graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles, and that this bound is tight.
Hint: Use Mantel's Theorem and induction on n.