Extremal Graph Theory - Fall $'23$

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Home Assignment 2

Due date: TBA

Please submit organized and well written solutions!

Problem 1. Show that $\log(D(n)) = \Theta\left(\frac{\log n}{\log\log n}\right)$, where $D(n)$ is the divisor function.

Problem 2. Find a graph H satisfying $\chi(H) = 3$ and $ex(n, H) \geq \frac{1}{4}n^2 + n^{1.99}$ for all $n \geq n_0$.

Problem 3. Suppose $\chi(H) = r$. Show that if $ex(n, H) = ex(n, K_r)$ then there is an edge $e \in H$ so that $\chi(H \setminus e) < \chi(H)$.

Problem 4. We saw in class that every unweighed graph has a $(2k-1)$-spanner with $O(n^{1+1/k})$ edges. Show that this also holds when the graph's edges are assigned non-negative weights.

Problem 5. Let $b_n(r, \epsilon)$ denote the largest integer b so that any graph with $(1 - \frac{1}{r} + \epsilon)\frac{n^2}{2}$ edges contains a b-blowup of K_{r+1}. Show that $b_n(1, \epsilon) = \Theta\left(\frac{\log n}{\log(1/\epsilon)}\right)$ and that $b_n(2, \epsilon) = O\left(\frac{\log n}{\log(1/\epsilon)}\right)$.

Problem 6. Let $K^3_{t,t,t}$ be the complete 3-partite 3-uniform hypergraph with t vertices in each part. Show that $cn^{3-3/(t^2+t+1)} \leq ex(n, K^3_{t,t,t}) \leq 5n^{3-1/t^2}$.

Problem 7. Show that if G is a bipartite graph on vertex sets A, B with $|A| = n$, $|B| = 2\epsilon \log n$, and $e(A, B) \geq \epsilon|A||B|$, then G contains a $K_{t,t}$ with $t = \epsilon^2 \log n$. You may assume $n \geq n_0(\epsilon)$.

Problem 8. Let G be a graph containing ϵn^3 triangles and suppose $\epsilon < \epsilon_0$ and $n \geq n_0(\epsilon)$.

1. Deduce from Problem 6 that G has a b-blowup of K_3 where $b = \epsilon\sqrt{\log n}$. Use this, along with a problem from the last home assignment, to derive the Erdős-Stone Theorem for $r = 2$.
2. Show that G has a set E' of at least $\frac{\epsilon}{2}n^2$ edges so that each $e \in E'$ is contained in at least $\frac{\epsilon}{2}n$ triangles.
3. Show that G contains a copy of $K_{t,t}$ whose edges all belong to E', with $t = \epsilon \log n$.
4. Use Problem 7 and item (3) to improve the bound in item (1) to $b = C(\epsilon)\log n$. If you feel adventurous, you can try to reprove the full Erdős-Stone Theorem.