Extremal Graph Theory - Fall '23 **Instructor: Asaf Shapira**

Home Assignment 3 **Due Date: TBA**

Please submit organized and well written solutions!

Problem 1. Suppose $\mathcal{P} = \{V_1, \ldots, V_k, U_1, \ldots, U_t\}$ is a partition of $V(G)$ with $|V_1| = \cdots = |V_k|$ and $\sum_{i=1}^{t} |U_i| \leq \epsilon n$. Show how to turn $\mathcal{P}$ into an equipartition $\mathcal{P}'$ of order k satisfying $q(\mathcal{P}') \geq q(\mathcal{P}) - 8\epsilon$.

Problem 2. Show that the statement of the regularity lemma remains valid even if instead of asking for an equipartition $\{V_1, \ldots, V_k\}$ in which the number of irregular pairs is bounded by ϵk^2, we ask that for every i there would be at most ϵk indices j, for which (V_i, V_j) is irregular.
Hint: Markov's Inequality.

Problem 3. Suppose H is a 4-uniform hypergraph on n vertices that does not contain 9 vertices spanning more than 2 edges. Show that H has $o(n^2)$ edges.
Hint: Give two proofs, one via the graph removal lemma for K_4 and one via the $(6,3)$-Problem.

Problem 4. Let $T(n)$ denote the number of triangle-free graphs on n (labeled) vertices. Show that $T(n) = 2^{(\frac{1}{4}+o(1))n^2}$.

Problem 5. Show that for every $\epsilon > 0$ there is $n_0 = n_0(\epsilon)$, so that if G is a graph on $n \geq n_0$ vertices and $\delta(G) \geq n/2$ then G contains $(1-\epsilon)n/4$ vertex-disjoint copies of C_4.

Problem 6. Let E be a homogenous linear equation $\sum_{i=1}^{k} a_i x_i = 0$ (with $k \geq 3$ and $a_i \in \mathbb{Z}$) and denote by $R_E(n)$ the size of the largest subset of $[n]$ containing no solution to E with all x_i being distinct.

- Show that if the coefficients of E satisfy $\sum_{i=1}^{k} a_i \neq 0$ then $R_E(n) = \Omega(n)$.
  Hint: Start by solving the problem when the equation is $x + y = z$.
- Show that if the coefficients of E satisfy $\sum_{i=1}^{k} a_i = 0$ then $R_E(n) = o(n)$.
  Hint: Apply Szemerédi's Theorem or the removal lemma for digraphs (preferably both).
- Show that if E is of the form $\sum_{i=1}^{k} a_i x_i = \left(\sum_{i=1}^{k} a_i\right) x_{k+1}$, with all $a_i > 0$, then we have $R_E(n) \geq n/C^{\sqrt{\log n}}$ for some constant C that may depend on $a_1, \ldots, a_k$. Actually, find a subset of $[n]$ of this size where the only solution to E is when $x_1 = x_2 = \ldots = x_{k+1}$.